Introduction to the Modeling and Analysis of Complex Systems: Solution Book

Hiroki Sayama

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Chapter 1

Introduction

Exercise 1.1 (omitted)

Exercise 1.2 (omitted)

Exercise 1.3 (omitted)

Chapter 2

Fundamentals of Modeling

Exercise 2.1 In the "wet lawn" scenario discussed above, come up with a few more alternative hypotheses that could explain both the wet lawn and the wet driveway without assuming that it rained. Then think of ways to find out which hypothesis is most likely to be the real cause.

Solution There is no absolutely right or wrong answer to this kind of question. Be creative and imaginative (while remaining logical) to produce your own unique hypotheses. Then share those with others and discuss how those different hypotheses would compare to each other.

Exercise 2.2 Name a couple of scientific models that are extensively used in today's scientific/engineering fields. Then investigate the following:

- How were they developed?

- What made them more useful than earlier models?

- How could they possibly be wrong?

Solution An example would be Newton's classical mechanics. It was developed primarily to describe the motion of celestial bodies. It was much more useful than other theories that existed back then, because it offered the first mathematical explanation of the relationships between positions, velocities, acceleration, and forces. But nowadays we know that its applicability is limited only to the "classical" realm of physics, while it won't apply directly to other cases at quantum or cosmological scales. In this sense, Newton's classical mechanics is a good "model" of how things move, which was well tested only with things at scales seen in our everyday life.

Exercise 2.3 Create a few different models for each of the examples shown above. Discuss how those models differ from each other, and what you should do to determine which model is more appropriate as an explanation of the observed behavior.

Solution This exercise can generate a number of different mathematical models. For example, Eqs. (2.1) and (2.2) are two competing models for the observation seen in Fig. 2.1. They may look identical mathematically, but their behaviors are different, i.e., Eq. (2.1) requires that the value of x must be always C, while Eq. (2.2) simply says that the curve of $x(t)$ should be flat, regardless of the actual value of x. To find out which model is more accurate, you can "poke" the system by adding some perturbation to the value of x at a certain point in time. If the system immediately comes back to the original state C, then Eq. (2.1) is more appropriate, or if the perturbation added to the system is preserved, Eq. (2.2) is more appropriate.

Exercise 2.4 (omitted)

Exercise 2.5 Humanity has created a number of models of the solar system in its history. Some of them are summarized below:

- Ptolemy's geocentric model (which assumes that the Sun and other planets are revolving around the Earth)

- Copernicus' heliocentric model (which assumes that the Earth and other planets are revolving around the Sun in concentric circular orbits)

- Kepler's laws of planetary motion (which assumes that the Earth and other planets are revolving in elliptic orbits, at one of whose foci is the Sun, and that the area swept by a line connecting a planet and the Sun during a unit time period is always the same)

- Newton's law of gravity (which assumes that a gravitational force between two objects is proportional to their masses and inversely proportional to their distance squared)

Investigate these models, and compare them in terms of simplicity, validity and robustness.

Solution Ptolemy's geocentric model was reasonably accurate but very complex, and it was not so robust as a model because it required many intricate mechanisms to explain the movement of celestial bodies. Copernicus' heliocentric model has drastically simplified the model, which is now called the Copernican Revolution in the history of science. Kepler's laws of planetary motion made the model a little more complex (because the orbits are no longer circular but elliptic) but with increased validity. It also added more

robustness to the model as the laws were formulated mathematically and thus could be generalized to many hypothetical cases. Finally, Newton's law of gravity has achieved simplicity, validity and robustness all at once, by discovering the fundamental law that can beautifully explain the observed behaviors of the solar system in just one mathematical formula of gravitational force. This not only applies to our solar system but also to many other systems.

Exercise 2.6 (omitted)

Chapter 3

Basics of Dynamical Systems

Exercise 3.1 (omitted)

Exercise 3.2 What are some appropriate choices for state variables in the following systems?

- population growth

- swinging pendulum

- motions of celestial bodies

- behavior of "rational" individuals playing a negotiation game

Solution

- population growth: number of organisms

- swinging pendulum: angular position and angular velocity

- motions of celestial bodies: positions and velocities of celestial bodies

- behavior of "rational" individuals playing a negotiation game: target prices or contracts (for example)

Exercise 3.3 In the example above, when the ball hits the floor in the vertical tube, its kinetic energy is quickly converted into elastic energy (i.e., deformation of the ball), and then it is converted back into kinetic energy again (i.e., upward velocity). Think about how you can illustrate this process in the phase space. Are the two dimensions enough or do

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you need to introduce an additional dimension? Then try to illustrate the trajectory of the system going through a bouncing event in the phase space.

Solution You will need at least one more dimension to represent the amount of elastic energy stored in the ball. This dimension can be added to the phase space as a z-axis. A typical trajectory of the systm going through bouncing events can be conceptually illustrated as follows:
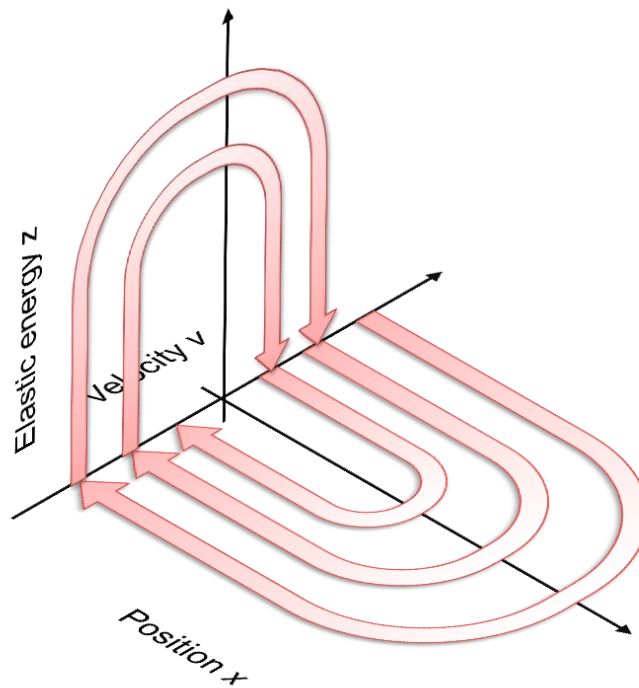
In the above figure, it is assumed that part of the energy is lost at every bouncing event so that the trajectory is spiraling toward the origin.

Exercise 3.4 Where are the attractor(s) in the phase space of the bouncing ball example created in Exercise 3.3? Assume that every time the ball bounces it loses a bit of its kinetic energy.

Solution The origin of the phase space is the only attractor in this case.


Exercise 3.5 For each attractor obtained in Exercise 3.4 above, identify its basin of attraction.

Solution The entire phase space is the basin of attraction. This means that no matter where the ball starts, it will eventually lose all the kinetic energy and end up resting at the
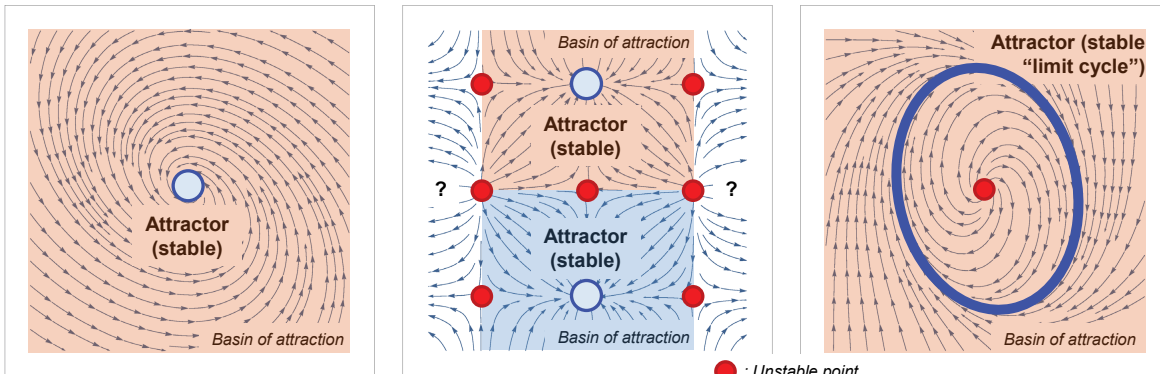
bottom of the tube.

Exercise 3.6 For each of the phase spaces shown below, identify the following:

- attractor(s)

- basin of attraction for each attractor

- stability of the system's state at several locations in the phase space
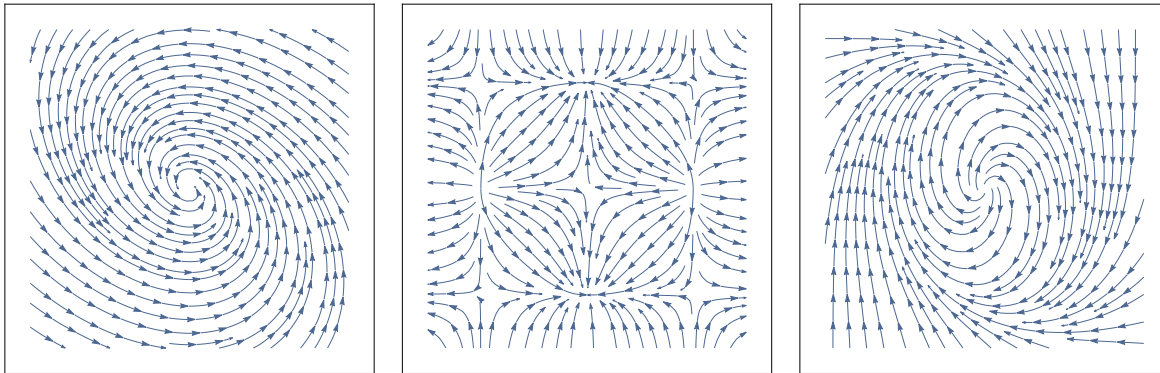
Solution

Exercise 3.7 Consider a market where two equally good products, A and B, are competing with each other for market share. There is a customer review website where users of each product submit their ratings. Since there is no actual difference in the product quality, the average rating scores are about the same between the two products, but the customers can also see the total number of submitted ratings for each product, which shows how popular the product is in the market. Customers tend to adopt a more popular choice. Answer the following questions:

- What would the phase space of this system look like?

- Are there any attractors? Are there any basins of attraction?

- How does the system's fate depend on its initial state?

- If you were in charge of marketing product A, what would you do?

Solution

- There are a few different ways to represent the phase space of this system. If you use the market share of A as a state variable, say x ($0 \leq x \leq 1$), then the phase space is a one-dimensional line segment. Or, if you use the numbers of submitted ratings for the two products as state variables, say m and n ($m \geq 0$, $n \geq 0$), then the phase space is a two-dimensional plane.

- The customers adopt a more popular choice, which makes the chosen product even more popular because some of those customers are likely to submit their ratings. Therefore, there is a "winner-takes-all" type of dynamics going on in this system. In the one-dimensional phase space with the market share of A, the winner-takes-all dynamics are represented by two attractors, i.e., $x = 0$ and $x = 1$. Each attractor means one of the two products completely dominated the market. In the two-dimensional phase space with the numbers of submitted ratings, there are no attractors, but instead the system's state diverges indefinitely toward either $m \to \infty$ or $n \to \infty$.

- The system's fate is determined by which product is more popular at the beginning. The dynamics will simply amplify the difference in initial popularity between the two products.

- You should make a significant marketing campaign for product A at the very beginning of competition, because the system's fate is determined by the initial difference.

Chapter 4

Discrete-Time Models I: Modeling

Exercise 4.1 Write the state transition functions $F_A(s_A, s_B)$ and $F_B(s_A, s_B)$ for this system, where s_A and s_B are the states of A and B, respectively.

Solution To write a state transition function in a mathematical way, you will first need to define how you represent the system's states numerically. One possibility is to use 1 for red and -1 for blue. Then F_A and F_B can be written as simply as follows:

$$s_A(t) = F_A(s_A(t-1), s_B(t-1)) = s_B(t-1)$$
$$s_B(t) = F_B(s_A(t-1), s_B(t-1)) = -s_A(t-1)$$

If you let $x = (s_A \; s_B)^T$, then the model can be further simplified to the following linear form:

$$x_t = Ax_{t-1}$$
$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

Exercise 4.2 Produce a time series of (s_A, s_B) starting with an initial condition with both components in blue, using the model you created. What kind of behavior will arise?

Solution The result will look like the following cycle:

$$(-1, -1), \quad (-1, 1), \quad (1, 1), \quad (1, -1), \quad (-1, -1), \quad \ldots$$

Exercise 4.3 Decide whether each of the following examples is (1) linear or nonlinear, (2) first-order or higher-order, and (3) autonomous or non-autonomous.

1. $x_t = ax_{t-1} + b$

2. $x_t = ax_{t-1} + bx_{t-2} + cx_{t-3}$

3. $x_t = ax_{t-1}(1 - x_{t-1})$

4. $x_t = ax_{t-1} + bxt - 2^2 + c\sqrt{x_{t-1}x_{t-3}}$

5. $x_t = ax_{t-1}x_{t-2} + bx_{t-3} + \sin t$

6. $x_t = ax_{t-1} + by_{t-1}, \; y_t = cx_{t-1} + dy_{t-1}$

Solution

1. Linear, first-order, autonomous

2. Linear, higher-order, autonomous

3. Nonlinear, first-order, autonomous

4. Nonlinear, higher-order, autonomous

5. Nonlinear, higher-order, non-autonomous

6. Linear, first-order, autonomous

Exercise 4.4 Convert the following difference equations into an autonomous, first-order form.

1. $x_t = x_{t-1}(1 - x_{t-1})\sin t$

2. $x_t = x_{t-1} + x_{t-2} - x_{t-3}$

Solution

1. $x_t = x_{t-1}(1 - x_{t-1})\sin z_{t-1}, \quad z_t = z_{t-1} + 1 \quad (z_0 = 1)$

2. $x_t = x_{t-1} + y_{t-1} - z_{t-1}, \quad y_t = x_{t-1}, \quad z_t = y_{t-1}$

Exercise 4.5 Conduct simulations of this model for various values of parameter a to see what kind of behaviors are possible in this model and how the value of a determines the resulting behavior.

Solution The system shows exponential divergence when $|a| > 1$. If $|a| < 1$ it shows exponential convergence to the origin. In addition, if $a < 0$, the system's state flips its sign in every time step (oscillation with period 2).

Exercise 4.6 Implement a simulation code of the following difference equation:

$$x_t = ax_{t-1} + b, \quad x_0 = 1$$

This equation is still linear, but now it has a constant term in addition to ax_{t-1}. Some real-world examples that can be modeled in this equation include fish population growth with constant removals due to fishing, and growth of credit card balances with constant monthly payments (both with negative b). Change the values of a and b, and see if the system's behaviors are the same as those of the simple exponential growth/decay model.

Solution The essential behavior is the same, but the convergence/divergence in this model can occur relative to a non-zero value, in contrast to the original exponential growth/decay model that always showed exponential behavior with respect to the origin.

Exercise 4.7 Simulate the above two-variable system using several different coefficients in the equations and see what kind of behaviors can arise.

Solution You can see various behaviors coming out from this two-variable system as you change the coefficients. Those behaviors can be classified based on whether the trajectories are converging to or diverging from the origin, and whether the trajectories rotate around the origin. Fig. 5.15 provides useful illustrations of those behaviors.

Exercise 4.8 Simulate the behavior of the following *Fibonacci sequence*. You first need to convert it into a two-variable first-order difference equation, and then implement a simulation code for it.

$$x_t = x_{t-1} + x_{t-2}, \quad x_0 = 1, \quad x_1 = 1$$

Solution Introducing a new memory variable $y_t = x_{t-1}$ makes the following equation:

$$x_t = x_{t-1} + y_{t-1}$$
$$y_t = x_{t-1}$$
$$x_1 = 1$$
$$y_1 = 1$$

You can simulate the behavior of this system by modifying the coefficients in the sample code given in the main text. It should show exponential growth.

Exercise 4.9 Simulate the behavior of the new population growth model for several different values of parameter a and initial condition x_0 to see what kind of behaviors are possible.

$$x_t = \left(-\frac{a-1}{K} x_{t-1} + a \right) x_{t-1}$$

Solution You can revise Code 4.9 to simulate the behavior of this model. The initial condition x_0 should be between 0 and K. For $a < 1$, the population decays exponentially. For $a > 1$, the population grows, and converges to K if a is not too large. But if you let a be sufficiently large, the system may show some weird behavior. This will be discussed further in later chapters.

Exercise 4.10 Create a mathematical model of population growth in which the growth ratio is highest at a certain optimal population size, but it goes down as the population deviates from the optimal size. Then simulate its behavior and see how its behavior differs from that of the logistic growth model.

Solution This model revision can be achieved by using a different mathematical function for $f(x)$ in Eq. (4.21). For example, you can use an upper convex parabola that has its peak at the optimal population size x_{op}. Note that $f(x_{\mathrm{op}})$ needs to be greater than 1 in order to let the population grow.

Exercise 4.11 Test the equations above by assuming extreme values for x and y, and make sure the model behaves as we intended.

Solution

- If $x = 0$: x remains 0 while y shows exponential decay. (OK)

- If $y = 0$: y remains 0 while x shows typical logistic growth. (OK)

- If $x = K$: x has no growth but may decrease because of the predation by y. (OK)

- If y is very large: Most of x's current population disappears immediately due to the predation by y. (OK)

Exercise 4.12 Implement a simulation code for the equations above, and observe the model behavior for various parameter settings.

Solution You can revise Code 4.13 to simulate the behavior of this model.

Exercise 4.13 Try several different parameter settings for r, b, d, and c, and see how the system's behavior changes. In some cases you may find an unrealistic, invalid behavior (e.g., indefinite growth of predators). If so, revise the model to fix the problem.

Solution By changing values of those parameters, you will see several different behaviors, including extinction of predators, extinction of both species, damping oscillation and convergence to stationary population levels, and (as mentioned in the exercise text) divergence of the predator population to infinity. There are various ways to prevent such

divergence. For example, you can include a logistic growth-type restriction to the second equation, e.g.:

$$y_t = y_{t-1} - dy_{t-1} + cx_{t-1}y_{t-1}\left(1 - \frac{y_{t-1}}{K'}\right)$$

Exercise 4.14 Develop a discrete-time mathematical model of two species competing for the same resource, and simulate its behavior.

Solution This can be modeled in various ways, but here is a simple example. The model assumptions are:

- A finite amount of resource is made available at every time step, and all of them will be fully consumed by the two species for their growth.

- The amount of resource each species can consume is proposal to their relative population size, because the more organisms there are, the faster the resource consumption is.

- The population increase of each species is proportional to the amount of resource it consumes.

Let R be the total amount of resource provided at each time step, and x and y the population sizes of the two competing species. Then, the amount of the resource each species can consume is $R\frac{x}{x+y}$ for x, or $R\frac{y}{x+y}$ for y. The resulting model equations are

$$x_t = x_{t-1} + r_x R\frac{x}{x+y},$$
$$y_t = y_{t-1} + r_y R\frac{y}{x+y},$$

where r_x and r_y are the growth rates, i.e., the numbers of the newborn offspring per unit resource consumption, for species x and y, respectively. You can revise Code 4.13 to simulate the behavior of this model. If the growth rates are different between the two species, the one with the greater growth rate will eventually win. If the rates are exactly the same, the initial population ratio will be maintained.

Exercise 4.15 Consider the dynamics of public opinions about political ideologies. For simplicity, let's assume that there are only three options: conservative, liberal, and neutral. Conservative and liberal are equally attractive (or annoying, maybe) to people, with no fundamental asymmetry between them. The popularities of conservative and liberal ideologies can be represented by two variables, p_c and p_l, respectively ($0 \leq p_c \leq 1$; $0 \leq p_l \leq 1$; $0 \leq p_c + p_l \leq 1$). This implies that $1 - p_c - p_l = p_n$ represents the popularity of neutral.

Assume that at each election poll, people will change their ideological states among the three options according to their relative popularities in the previous poll. For example, the rate of switching from option X to option Y can be considered proportional to $(p_Y - p_X)$ if $p_Y > p_X$, or 0 otherwise. You should consider six different cases of such switching behaviors (conservative to liberal, conservative to neutral, liberal to conservative, liberal to neutral, neutral to conservative, and neutral to liberal) and represent them in dynamical equations.

Complete a discrete-time mathematical model that describes this system, and simulate its behavior. See what the possible final outcomes are after a sufficiently long time period.

Solution This is a rather complicated mathematical model, but you can write down these assumptions as follows:

$$
\begin{aligned}
p_{c,t} &= p_{c,t-1} + r\max(p_{c,t-1} - p_{l,t-1}, 0)p_{l,t-1} + r\max(p_{c,t-1} - p_{n,t-1}, 0)p_{n,t-1} \\
&\quad - r\max(p_{l,t-1} - p_{c,t-1}, 0)p_{c,t-1} - r\max(p_{n,t-1} - p_{c,t-1}, 0)p_{c,t-1} \\
p_{l,t} &= p_{l,t-1} + r\max(p_{l,t-1} - p_{c,t-1}, 0)p_{c,t-1} + r\max(p_{l,t-1} - p_{n,t-1}, 0)p_{n,t-1} \\
&\quad - r\max(p_{c,t-1} - p_{l,t-1}, 0)p_{l,t-1} - r\max(p_{n,t-1} - p_{l,t-1}, 0)p_{l,t-1}
\end{aligned}
$$

Here, $p_{n,t-1} = 1 - p_{c,t-1} - p_{l,t-1}$, and r is a constant that determines the rate of switching. You can revise Code 4.13 to simulate the behavior of this model. You will see that the system will eventually converge to an all-conservative ($p_c = 1, p_l = p_n = 0$), all-liberal ($p_l = 1, p_c = p_n = 0$), or all-neutral ($p_n = 1, p_c = p_l = 0$) state, depending on the initial condition.

Exercise 4.16 Revise the model of public opinion dynamics developed in the previous exercise so that the political parties of the two ideologies (conservative and liberal) run a political campaign to promote voters' switching to their ideologies from their competitions, at a rate *inversely* proportional to their current popularity (i.e., the less popular they are, the more intense their campaign will be). Simulate the behavior of this revised model and see how such political campaigning changes the dynamics of the system.

Solution This new assumption can be implemented in several different ways. For example, you can replace the $\max(\ldots)$ terms in the above equations with reciprocals of p_c and p_l, which represent the intensities of their campaigns that are inversely proportional to the parties' popularity levels, as follows:

$$
\begin{aligned}
p_{c,t} &= p_{c,t-1} + r\frac{1}{p_{c,t-1}}p_{l,t-1} + r\frac{1}{p_{c,t-1}}p_{n,t-1} - r\frac{1}{p_{l,t-1}}p_{c,t-1} \\
p_{l,t} &= p_{l,t-1} + r\frac{1}{p_{l,t-1}}p_{c,t-1} + r\frac{1}{p_{l,t-1}}p_{n,t-1} - r\frac{1}{p_{c,t-1}}p_{l,t-1}
\end{aligned}
$$

Note that there is no term for decrease proportional to $1/p_n$, because there would be no campaign for the neutral. Also note that r should be set to a very small value for the system to show reasonable behavior, because $1/p_c$ or $1/p_l$ could be very large.

If you simulate this revised model, you will see that the system always converge to a tense situation in which the two parties go neck and neck with each other, regardless of the initial condition.

Chapter 5

Discrete-Time Models II: Analysis

Exercise 5.1 Obtain the equilibrium point(s) of the following difference equation:

$$x_t = 2x_{t-1} - x_{t-1}^2$$

Solution

$$x_{\text{eq}} = 2x_{\text{eq}} - x_{\text{eq}}^2$$
$$0 = x_{\text{eq}} - x_{\text{eq}}^2$$
$$= x_{\text{eq}}(1 - x_{\text{eq}})$$
$$x_{\text{eq}} = 0, \quad 1$$

Exercise 5.2 Obtain the equilibrium point(s) of the following two-dimensional difference equation model:

$$x_t = x_{t-1}y_{t-1}$$
$$y_t = y_{t-1}(x_{t-1} - 1)$$

Solution

$$x_{\text{eq}} = x_{\text{eq}}y_{\text{eq}}$$
$$y_{\text{eq}} = y_{\text{eq}}(x_{\text{eq}} - 1)$$

The first equation gives $x_{\text{eq}} = 0$ or $y_{\text{eq}} = 1$. For the former, the second equation gives $y_{\text{eq}} = 0$. For the latter, it gives $x_{\text{eq}} = 2$. Therefore, the equilibrium points are

$$(x_{\text{eq}}, y_{\text{eq}}) = (0, 0), \ (2, 1).$$

Exercise 5.3 Obtain the equilibrium point(s) of the following difference equation:

$$x_t = x_{t-1} - x_{t-2}^2 + 1$$

Note that this is a second-order difference equation, so you will need to first convert it into a first-order form and then find the equilibrium point(s).

Solution By letting $y_t = x_{t-1}$, we get

$$x_t = x_{t-1} - y_{t-1}^2 + 1$$
$$y_t = x_{t-1}$$

The equilibrium points must satisfy

$$x_{eq} = x_{eq} - y_{eq}^2 + 1$$
$$y_{eq} = x_{eq}$$

The solutions are

$$(x_{eq}, y_{eq}) = (1, 1), \quad (-1, -1).$$

Exercise 5.4 Draw a phase space of the following two-dimensional difference equation model in Python:

$$x_t = x_{t-1} + 0.1(x_{t-1} - x_{t-1}y_{t-1})$$
$$y_t = y_{t-1} + 0.1(y_{t-1} - x_{t-1}y_{t-1})$$
$$(x > 0, \quad y > 0)$$

Solution Code:

Code 5.1:

```python
from pylab import *

def initialize(x0, y0):
    global x, y, xresult, yresult
    x = x0
    y = y0
    xresult = [x]
    yresult = [y]
```

```
def observe():
    global x, y, xresult, yresult
    xresult.append(x)
    yresult.append(y)

def update():
    global x, y, xresult, yresult
    nextx = x + 0.1 * (x - x * y)
    nexty = y + 0.1 * (y - x * y)
    x, y = nextx, nexty

for x0 in arange(0, 2, .1):
    for y0 in arange(0, 2, .1):
        initialize(x0, y0)
        for t in xrange(30):
            update()
            observe()
        plot(xresult, yresult, 'b')

axis([0, 2, 0, 2]) ### added to zoom in the area within which
                   ### initial states were varied
show()
```
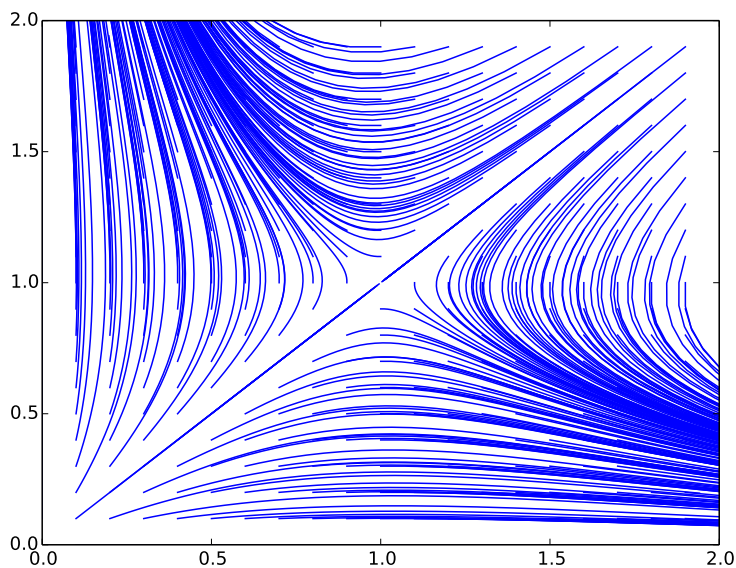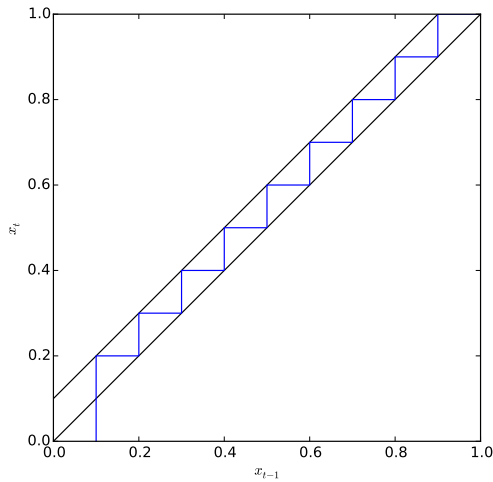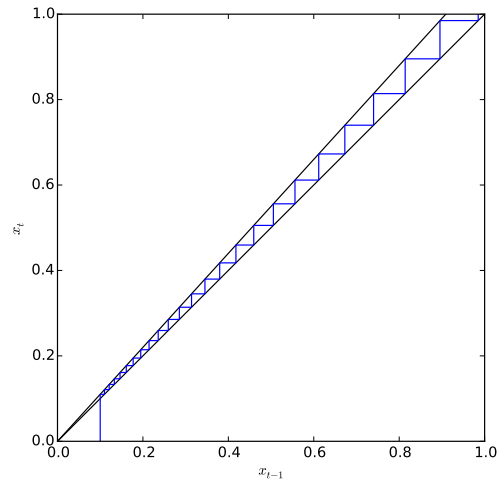
Output:

Exercise 5.5 Draw a cobweb plot for each of the following models:

- $x_t = x_{t-1} + 0.1$, $x_0 = 0.1$

- $x_t = 1.1x_{t-1}$, $x_0 = 0.1$
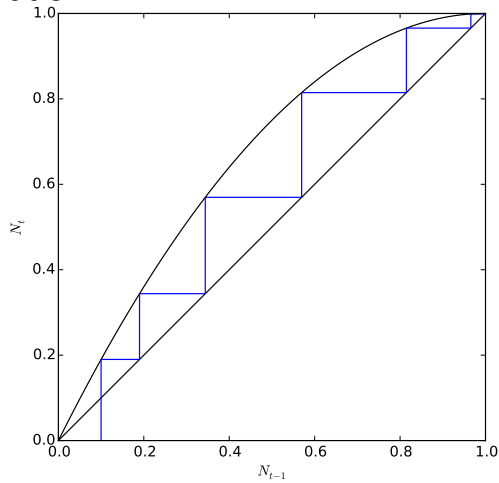
Solution

$$x_t = x_{t-1} + 0.1, \; x_0 = 0.1 \qquad\qquad x_t = 1.1x_{t-1}, \; x_0 = 0.1$$

Exercise 5.6 Draw a cobweb plot of the following logistic growth model with $r = 1$, $K = 1$, $N_0 = 0.1$:

$$N_t = N_{t-1} + rN_{t-1}(1 - N_{t-1}/K)$$

Solution

Exercise 5.7 Using Python, draw a cobweb plot of the logistic growth model with $r = 2.5$, $K = 1$, $N_0 = 0.1$.

Solution Code:

Code 5.2:

```python
from pylab import *

r = 2.5
K = 1.

def initialize():
    global N, result
    N = 0.1
    result = [N]

def observe():
    global N, result
    result.append(N)

def f(N):
    return N + r * N * (1. - N / K)

def update():
    global N, result
    N = f(N)

initialize()
for t in xrange(30):
    update()
    observe()

Nmin, Nmax = 0, 1.4
plot([Nmin, Nmax], [Nmin, Nmax], 'k')

rng = arange(Nmin, Nmax, (Nmax - Nmin) / 100.)
plot(rng, map(f, rng), 'k')

horizontal = [result[0]]
vertical = [result[0]]
```
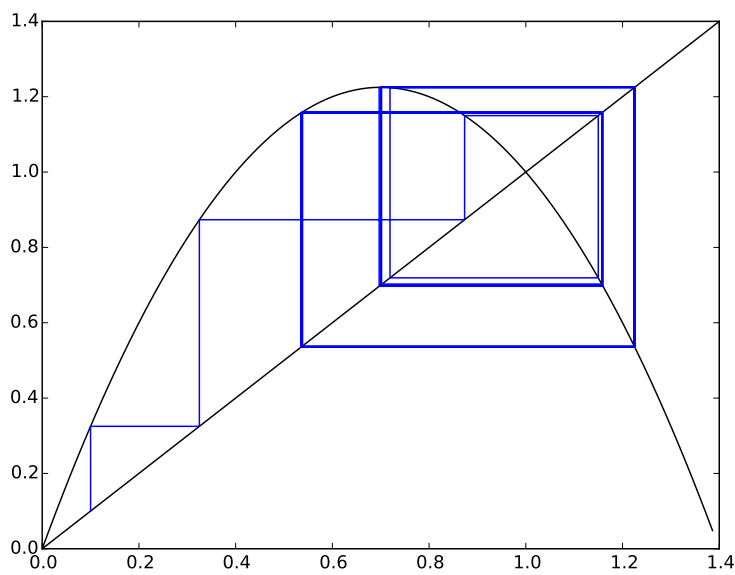
```
for N in result[1:]:
    horizontal.append(vertical[-1])
    vertical.append(N)
    horizontal.append(N)
    vertical.append(N)
plot(horizontal, vertical, 'b')

show()
```

Output:

Exercise 5.8 Draw a phase space of the following difference equation within the range $0 \leq x < 100$ by modifying Code 5.5:

$$x_t = x_{t-1}^{x_{t-1}} \mod 100$$

Solution Code:

Code 5.3:

```
from pylab import *
import networkx as nx

g = nx.DiGraph()
```
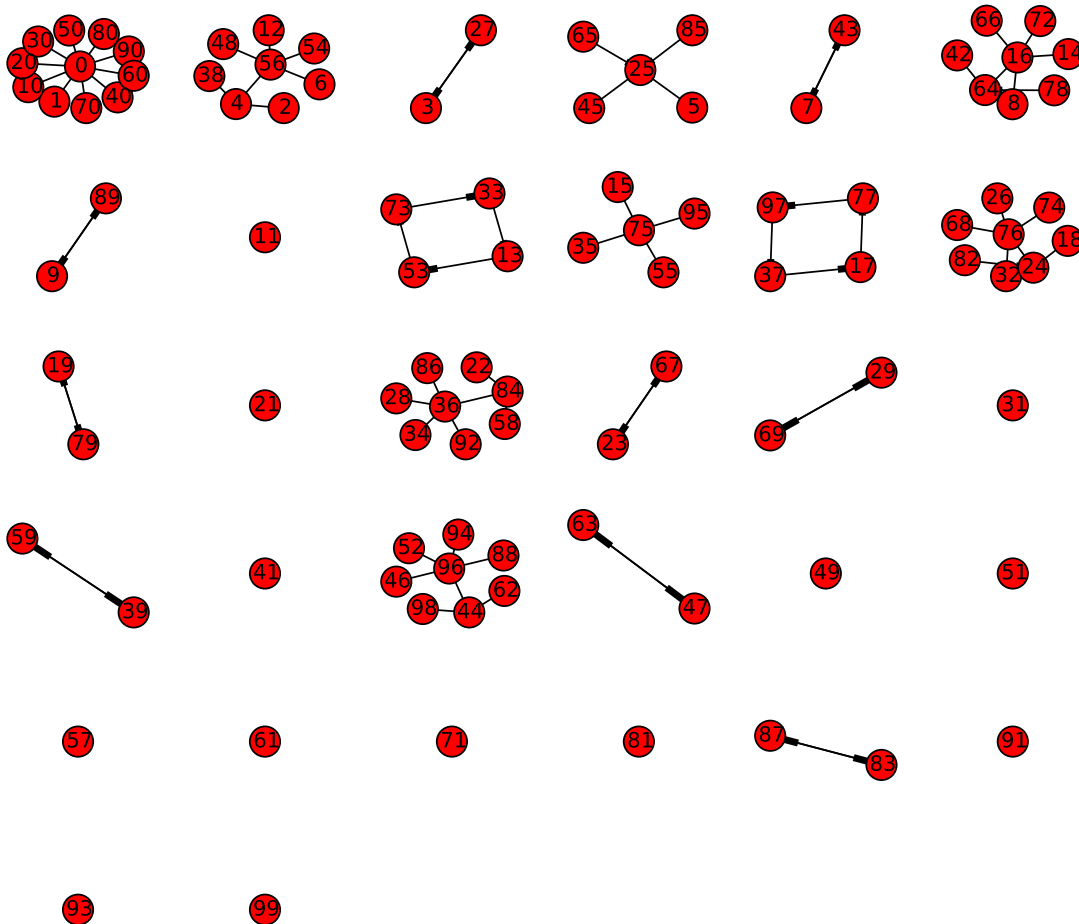
```
for x in range(100):
    g.add_edge(x, x ** x % 100)

ccs = [cc for cc in nx.connected_components(g.to_undirected())]
n = len(ccs)
w = ceil(sqrt(n))
h = ceil(n / w)
for i in xrange(n):
    subplot(h, w, i + 1)
    nx.draw(nx.subgraph(g, ccs[i]), with_labels = True)

show()
```

Output:

Exercise 5.9 Simplify the following difference equation by variable rescaling:

$$x_t = \frac{a}{x_{t-1} + b}$$

Solution By letting $x \to \alpha x'$, we obtain:

$$\alpha x'_t = \frac{a}{\alpha x'_{t-1} + b}$$

$$x'_t = \frac{a}{\alpha^2 x'_{t-1} + \alpha b}$$

$$= \frac{1}{(\alpha^2/a)x'_{t-1} + (\alpha b/a)}$$

From here, for example, if we let $\alpha = a/b$ and $c = \alpha^2/a = a/b^2$, the equation becomes simplified as follows:

$$x'_t = \frac{1}{cx'_{t-1} + 1}$$

Exercise 5.10 Simplify the following two-dimensional predator-prey difference equation model by variable rescaling:

$$x_t = x_{t-1} + rx_{t-1}\left(1 - \frac{x_{t-1}}{K}\right) - \left(1 - \frac{1}{by_{t-1} + 1}\right)x_{t-1}$$

$$y_t = y_{t-1} - dy_{t-1} + cx_{t-1}y_{t-1}$$

Solution By letting $x \to \alpha x'$ and $y \to \alpha y'$, we obtain:

$$\alpha x'_t = \alpha x'_{t-1} + \alpha rx'_{t-1}\left(1 - \frac{\alpha x'_{t-1}}{K}\right) - \left(1 - \frac{1}{\beta by'_{t-1} + 1}\right)\alpha x'_{t-1}$$

$$x'_t = x'_{t-1} + rx'_{t-1}\left(1 - \frac{\alpha x'_{t-1}}{K}\right) - \left(1 - \frac{1}{\beta by'_{t-1} + 1}\right)x'_{t-1}$$

$$= rx'_{t-1}\left(1 - \frac{\alpha x'_{t-1}}{K}\right) + \frac{x'_{t-1}}{\beta by'_{t-1} + 1}$$

$$\beta y'_t = \beta y'_{t-1} - \beta dy'_{t-1} + \alpha\beta cx'_{t-1}y'_{t-1}$$

$$y'_t = y'_{t-1} - dy'_{t-1} + \alpha cx'_{t-1}y'_{t-1}$$

From here, for example, if we let $\alpha = K$, $\beta = 1/b$, $p = 1 - d$ and $q = \alpha c = Kc$, the equations become simplified as follows:

$$x'_t = rx'_{t-1}(1 - x'_{t-1}) + \frac{x'_{t-1}}{y'_{t-1} + 1}$$

$$y'_t = py'_{t-1} + qx'_{t-1}y'_{t-1}$$

Exercise 5.11 Visualize the phase space of Eq. (5.48).

Solution You can simply replace the content of the `update` function of Code 5.1 with the following:

Code 5.4:
```
nextx = x + y
nexty = x
```

Exercise 5.12 Study the asymptotic behavior of the following three-dimensional difference equation model by calculating its eigenvalues and eigenvectors:

$$x_t = x_{t-1} - y_{t-1}$$
$$y_t = -x_{t-1} - 3y_{t-1} + z_{t-1}$$
$$z_t = y_{t-1} + z_{t-1}$$

Solution This model can be rewritten by letting $(x_t, y_t, z_t) \Rightarrow x_t$ as follows:

$$x_t = \begin{pmatrix} 1 & -1 & 0 \\ -1 & -3 & 1 \\ 0 & 1 & 1 \end{pmatrix} x_{t-1}$$

The eigenvalues and eigenvectors of the above matrix can be calculated using Python:

Code 5.5:
```
from pylab import *
eig([[1, -1, 0], [-1, -3, 1], [0, 1, 1]])
```

The result is shown below:

Code 5.6:
```
(array([-3.44948974, 1. , 1.44948974]),
 array([[ 2.14186495e-01, -7.07106781e-01, -6.73887339e-01],
        [ 9.53020614e-01, 2.42373658e-16, 3.02905447e-01],
        [ -2.14186495e-01, -7.07106781e-01, 6.73887339e-01]]))
```

The first and third eigenvalues have their absolute values greater than 1, so they represent growing components (while the second eigenvalue is 1, which means that the second component is conserved). The first eigenvalue has the largest absolute value, so it is the dominant one. The system's state will asymptotically align the dominant eigenvector

(the first column vector of the second `array` in the above result), with alternating signs because the dominant eigenvalue is negative.

Exercise 5.13 Consider the dynamics of opinion diffusion among five people sitting in a ring-shaped structure. Each individual is connected to her two nearest neighbors (i.e., left and right). Initially they have random opinions (represented as random real numbers), but at every time step, each individual changes her opinion to the local average in her social neighborhood (i.e, her own opinion plus those of her two neighbors, divided by 3). Write down these dynamics as a linear difference equation with five variables, then study its asymptotic behavior by calculating its eigenvalues and eigenvectors.

Solution This model can be written using a five-dimensional state vector x_t as follows:

$$x_t = \begin{pmatrix} 1/3 & 1/3 & 0 & 0 & 1/3 \\ 1/3 & 1/3 & 1/3 & 0 & 0 \\ 0 & 1/3 & 1/3 & 1/3 & 0 \\ 0 & 0 & 1/3 & 1/3 & 1/3 \\ 1/3 & 0 & 0 & 1/3 & 1/3 \end{pmatrix} x_{t-1}$$

The eigenvalues and eigenvectors of the above matrix can be calculated using Python:

Code 5.7:
```
from pylab import *
a = 1./3.
eig([[a, a, 0, 0, a], [a, a, a, 0, 0], [0, a, a, a, 0], [0, 0, a, a, a],
    [a, 0, 0, a, a]])
```

The result is shown below:

Code 5.8:
```
(array([-0.20601133, 0.53934466, 1. , 0.53934466, -0.20601133]),
 array([[ 0.63245553, -0.63245553, -0.4472136 , 0.1374608 , -0.05004529],
        [-0.51166727, -0.19543951, -0.4472136 , 0.62959976, -0.33009489],
        [ 0.19543951, 0.51166727, -0.4472136 , 0.25165325, 0.58415005],
        [ 0.19543951, 0.51166727, -0.4472136 , -0.4740695 , -0.61507975],
        [-0.51166727, -0.19543951, -0.4472136 , -0.54464431, 0.41106988]]))
```

The third eigenvalue is dominant, which is 1, indicating that the component represented by its corresponding eigenvector is conserved. In the meantime, the absolute values of all the other eigenvalues are less than 1, meaning that those components will shrink and disappear. The dominant eigenvector (the third column vector of the second `array` in the above result) shows a homogeneous state, which means that the opinions of the five people will eventually become identical due to diffusion.

Exercise 5.14 What if a linear system has more than one dominant, real-valued eigenvalue? What does it imply for the relationship between the initial condition and the asymptotic behavior of the system?

Solution Having more than one dominant, real-valued eigenvalue means that the components of the system's state that correspond to those eigenvalues will grow (or shrink) at the same rate. Therefore, their ratios are maintained over time, which means that the asymptotic direction of the system's state *will* depend on the initial condition.

Exercise 5.15 Consider the following iterative map ($a > 0$, $b > 0$):

$$x_t = x_{t-1} + a \sin(bx_{t-1})$$

Conduct linear stability analysis to determine whether this model is stable or not at its equilibrium point $x_{eq} = 0$.

Solution Let $F(x)$ represent the right hand side of the above equation. Then,

$$F'(x) = 1 + ab \cos(bx)$$
$$F'(0) = 1 + ab \qquad\qquad > 1 \quad (\text{because a} > 0, \ \text{b} > 0)$$

Therefore, the equilibrium point $x_{eq} = 0$ is unstable.

Exercise 5.16 Consider the following two-dimensional difference equation model:

$$x_t = x_{t-1} + 2x_{t-1}(1 - x_{t-1}) - x_{t-1}y_{t-1}$$
$$y_t = y_{t-1} + 2y_{t-1}(1 - y_{t-1}) - x_{t-1}y_{t-1}$$

1. Find all of its equilibrium points.

2. Calculate the Jacobian matrix at the equilibrium point where $x > 0$ and $y > 0$.

3. Calculate the eigenvalues of the matrix obtained above.

4. Based on the result, classify the equilibrium point into one of the following: stable point, unstable point, saddle point, stable spiral focus, unstable spiral focus, or neutral center.

Solution

1. By letting $x \to x_{eq}$ and $y \to y_{eq}$, the first equation gives $x_{eq} = 0$ or $y_{eq} = 2(1 - x_{eq})$. Applying $x_{eq} = 0$ to the second equation gives $y_{eq} = 0$ or 1. Applying $y_{eq} = 2(1 - x_{eq})$ to the second equation gives $x_{eq} = 1$ or $2/3$ (which makes $y_{eq} = 0$ or $2/3$, respectively). By combining these results together, the equilibrium points are $(x_{eq}, y_{eq}) = (0,0)$, $(0,1)$, $(1,0)$, $(2/3, 2/3)$.

2. The Jacobian matrix of this model is given by

$$\begin{pmatrix} 3 - 4x - y & -x \\ -y & 3 - 4y - x \end{pmatrix}.$$

There is only one equilibrium point where $x > 0$ and $y > 0$, which is $(x_{\text{eq}}, y_{\text{eq}}) = (2/3, 2/3)$. With these values, the above Jacobian matrix becomes

$$\begin{pmatrix} -1/3 & -2/3 \\ -2/3 & -1/3 \end{pmatrix}.$$

3. The eigenvalues of the above matrix are calculated as follows:

$$\begin{vmatrix} -1/3 - \lambda & -2/3 \\ -2/3 & -1/3 - \lambda \end{vmatrix} = 0$$

$$(-1/3 - \lambda)^2 - (-2/3)^2 = 0$$

$$\lambda^2 + \frac{2}{3}\lambda - \frac{1}{3} = 0$$

$$(\lambda + 1)(\lambda - 1/3) = 0$$

$$\lambda = -1, \ 1/3$$

4. It turns out that this is a subtle case, because the dominant eigenvalue ($\lambda = -1$) has 1 as its absolute value. This does not show linear expotential growth, so it can be considered a (Lyapunov) stable point[1]. The eigenvalues don't have imaginary parts, so it doesn't involve any spiral behaviors.

Exercise 5.17 Consider the following two-dimensional difference equation model:

$$x_t = x_{t-1}y_{t-1}$$
$$y_t = y_{t-1}(x_{t-1} - 1)$$

1. Find all equilibrium points (which you may have done already in Exercise 5.2).

2. Calculate the Jacobian matrix at each of the equilibrium points.

3. Calculate the eigenvalues of each of the matrices obtained above.

4. Based on the results, discuss the stability of each equilibrium point.

---

[1]In this case it is indeed Lyapunov stable, but it would generally require a nonlinear analysis to show that an equilibrium point is truly Lyapunov stable.

Solution

1. $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (0, 0), \ (2, 1)$

2. The Jacobian matrix of this model is given by

$$\begin{pmatrix} y & x \\ y & x - 1 \end{pmatrix}.$$

   At $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (0, 0)$, the above Jacobian matrix becomes

$$\begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}.$$

   At $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (2, 1)$, the above Jacobian matrix becomes

$$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

3. The first Jacobian matrix has eigenvalues $\lambda = 0, \ -1$. The second Jacobian matrix has eigenvalues $\lambda = 1 \pm \sqrt{2}$.

4. Based on these results, $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (0, 0)$ appears to be a (Lyapunov) stable point[2], while $(x_{\mathrm{eq}}, y_{\mathrm{eq}}) = (2, 1)$ is unstable because one of its eigenvalues has its absolute value greater than 1.

---

[2]But again, it would generally require a nonlinear analysis to show that an equilibrium point is truly Lyapunov stable.