©Christopher Hanes Economics 450 Class notes

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II) Loans, bonds, interest rates and yields

A) Intro

1) What we will cover in this section

Here I will tell you about two ways to borrow and lend money: loans and bonds. Probably, you already know something about loans but you don't know about bonds. I will do the usual thing in an economics class: I will start with the simplest possible case and gradually add complications. I will make things only as complicated as needed for you to understand monetary policy, so there are many aspects of bonds and loans that I won't get to.

I will be introducing a lot of *notation*, that is a system of symbols used to describe things. Pay attention to the notation, even though it is boring. We will be using it all through the course.

2) Some things to remember from middle-school math

a) Percents

A percentage is a fraction multipled by 100, so

"Ten percent" or "10%" means a decimal or fraction of $0.10 = \frac{1}{10}$

"Five percent" or "5%" means $0.05 = \frac{1}{20}$

"Half a percent" or "1/2%" means $0.005 = \frac{1}{200}$

b) Powers and exponents

You need to remember how to use powers and exponents. I mean this stuff:

$$x^{1/2} = \sqrt{x} \qquad x^{1/3} = \sqrt[3]{x} \qquad \sqrt{x^2} = x \qquad \left(x^2\right)^{1/2} = x \qquad (x^2)^{1/3} = x^{2/3} \qquad x^{-1} = \frac{1}{x}$$

$$x^{-1/2} = \frac{1}{x^{1/2}} = \frac{1}{\sqrt{x}}$$

If you don't remember this stuff, go to a middle school and ask a seventh-grader about it. The best middle school around here is Vestal Middle School, at 600 South Benita Boulevard, Vestal NY.

B) Definition: borrowers and lenders

I use the word "borrower" to mean a person or firm, or government, or whatever that wants to get money today, and in return will give a promise to pay out a specified amount of money at a specified time in the future.

A "lender" is the opposite: a lender is willing to give money today to acquire a promise of a specified amount of money at a specified future point in time.

Transactions between borrowers and lenders can be structured in many different ways, but there are two main ways: loans, and bonds.

What about shares of stock? When you are buying a share of stock, you are buying a share of a firm's profits. The directors of the firm choose whether to pay those profits to you ("dividends") or reinvest them in the firm ("retained earnings"). In the latter case, you might eventually benefit from getting more dividends in the future or by selling your shares to someone else, but you get no payments in the meantime. Thus, when you buy a share of stock you do *not* receive "a promise of a specified amount of money at a specified future point in time." A sale of stock is not a borrowing/lending transaction, on our definition. We won't talk about stocks much in this course.

C) One-year loans and bonds

1) Introduction

The simplest case to describe is one where the borrower's promise is to make *one* payment of money exactly *one year from today*. So I'll start with that. I should mention, however, that in reality many borrowers actually *do* borrow money for exactly one year. Retailers borrow to buy stocks of goods, promising to pay money back in a year from the proceeds of sales. Manufacturers borrow to buy materials and components, promising to pay money back in a year from the proceeds of selling what they make. Governments borrow money for a year to fund expenditures, promising to pay money back when annual tax payments come in.

2) One-year loans

In a one-year loan, the lender gives the borrower some money today and the borrower gives the lender a promise that the borrower will pay the lender a certain amount of money one year from today. Usually, the amount of money that borrower promises to pay the lender is specified in terms of an *interest rate*, rather than a number of dollars.

Here's an example: you borrow \$100 from your father today, for one year, at an interest rate of 5 percent (5%). That means that today your father gives you \$100. You give him a promise that one year from today you will pay him \$100 plus five percent of \$100. Five percent of \$100 is \$5, so you will pay your father \$105.

Often, the terms of a loan are put into a written contract. Such a contract would name the lender, the borrower, the amount the borrower receives today, and the interest rate.

Here's the notation we'll use to describe one-year loans.

$\$L$ is the amount of money the borrower gives the lender today.

i is the interest rate expressed as a decimal or fraction. I hope you paid attention to that. If I say "the interest rate on the loan is one percent," then i in the formula is 0.01 *not* 1.

$(1+i)\$L = \$L + \$iL$ is the amount of money that the borrower promises to give the lender one year from today.

In the example of the loan from your father, L is 100 and i is 0.05.

Sometimes L is called the "principal" of the loan, and iL is called the "interest." The total $(1+i)\$L$ is "principal plus interest."

An important ratio to think about, for what we will do later, is: $\dfrac{\text{The future payment promised by the borrower}}{\text{The amount of money the lender pays out today}}$

Using our notation, this ratio is $\dfrac{(1+i)L}{L} = 1+i$

3) One-year bonds

a) What they are

Like a loan contract, a bond is a legally-binding piece of paper. But it is not a contract between a lender and a borrower. It is a sort of one-way contract binding on a borrower. It is a piece of paper on which a borrower promises to pay a certain amount (or amounts) of money, at a specified point (or several points) in time, *to whomever is holding the piece of paper at that point in time.* It names the borrower, the date on which the borrower promises to pay money to the holder of the bond, and the amount of money that the borrower promises to pay on that date.

The borrower prints up the piece of paper, makes it legally binding - in the old days, this was by signing the piece of paper - and *sells* the piece of paper for as much as he can get. This is called "issuing" the bond. The borrower is "issuer" of the bond.

I said a bond is a "piece of paper." That isn't quite right nowadays. All bonds used to be printed (or written) on paper. Nowadays most bonds are just records in a database - virtual pieces of paper. But I don't like electronic things, so I'll go on talking about piece of paper.

There are many ways that a borrower/issuer can sell a bond, but for right now, suppose that the issuer runs an auction, or puts the bond up on Ebay and takes the highest bid. The issuer knows exactly how much money he is promising to pay in the future - he has printed that on the bond. But he does *not* necessarily know what price he will get for the bond when he sells it. That depends on the result of the auction, or the offers he gets on Ebay.

There is no price and no name of a lender written on a bond. The borrower/issuer does not know who will buy the bond, or what price the buyer will pay, until the auction/Ebay sale is complete.

Here's the notation we'll use to describe a one-year bond.

IOU is the amount of money the bond issuer promises to pay to the bondholder one year from today.

P is the price someone pays for the bond today.

For an example, suppose the General Motors Corporation has issued a bond that promises to pay $150,000 one year from today, and somone just bought that bond for $135,000. Then *IOU = 150,000* and *P = 135,000.*

Importantly, the person who buys the bond from the issuer is not necessarily the person to whom the issuer will pay the *IOU* when the bond comes due. The person who buys the bond from the issuer can sell the bond to *another* person. The second person might sell the bond on to a third person, and so on. A bond can change hands many times before it comes due. Whoever is holding the bond when it comes due will get the IOU payment from the borrower/issuer.

In this way, a bond is different from a loan. If a lender makes a one-year loan to someone, the lender must wait a year, the time specified on the loan contract, to get any money back. (I am simplifying things a bit here; I'll return to this later.) If a lender buys a one-year bond, on the other hand, the lender has more options. She can wait a year to get the *IOU*, *or* she can sell the bond right now to someone else. If she sells the bond right now, she does not get the *IOU*: she gets whatever someone else is willing to pay for the bond at the time she sells it. That price might be more, or less, than the price she paid for the bond earlier. I'll return to this point later.

b) Yield: the implicit interest rate on a bond

I haven't mentioned an interest rate. In fact, there is no interest rate written on a bond. There is, however, a sort of *implicit* interest rate defined by the *IOU* and *P*. This implicit interest rate is called a "yield." We will denote it with an *i*, the way we denote interest rates.

It is defined like this:

$$i = \frac{IOU}{P} - 1 \quad \text{or} \quad 1 + i = \frac{IOU}{P}$$

In the example of the General Motors bond,

$$i = \frac{150{,}000}{135{,}000} - 1 = 1.1111 - 1 = 0.1111$$

Like interest rates on loans, bond yields are usually spoken of in terms of percents, not fractions. So one would say that the yield on this bond is "eleven percent" (doing some rounding).

I called the yield an "implicit interest rate." Indeed, the yield on a bond is called the "interest rate" on the bond. Why? How is a yield like an interest rate?

Remember the ratio I mentioned in reference to loans, that is: $\dfrac{\text{The future payment promised by the borrower}}{\text{The amount of money the lender pays out today}}$

For a loan this ratio was $\dfrac{(1+i)L}{L} = 1+i$

For a bond this ratio is $\dfrac{IOU}{P} = 1+i$

Did you see how the yield on a bond is like the interest rate on a loan? Like the interest rate on a loan, a bond's yield expresses the ratio of the future payment made by the borrower to the amount a lender invests today.

If I tell you a bond's *IOU* and its yield, you can figure out what its price is:

$$P = \frac{IOU}{1+i}$$

This is important! In bond markets, people usually don't quote prices of bonds. They quote bond yields, from which infer the prices. A financial website giving information about bonds will quote yields not prices.

Finally, note that if I tell you a bond's price and yield, you can figure out what its *IOU* is:

$$IOU = P(1+i)$$

c) Changes in bond prices and yields

Unlike the interest rate on a loan, the yield on a bond can change from day to day. The yield on a bond is defined by the last price that someone paid for a bond. If the market price of the bond changes - if someone pays a different price for the bond today than someone paid for the bond yesterday - the bond's yield changes.

By definition, if a bond's market price rises, its yield falls, and vice-versa. And remember that bond yields are often called interest rates. So these three headlines which you might see on a financial-news website all mean the same thing:

"Bond prices fall'
"Bond yields rise"
"Interest rates rise"

d) Market prices adjust to equalize bond yields

To a first approximation, market prices of bonds must be such that yields are the same for all bonds that pay off at the same point in time. This isn't *exactly* true, as I will explain later, but for now, for simplicity, assume it is true.

Why would this be true? It is an equilibrium condition. The profit-maximizing actions of traders in bond markets ensures that it holds.

Suppose there are two one-year bonds on the market, bond A and bond B. Bond A's yield is determined by its market price and *IOU:*

$$i_A = \frac{IOU_A}{P_A} - 1$$

Bond B's yield is: $i_B = \dfrac{IOU_B}{P_B} - 1$

Suppose that bond B's price is so high, relative to its IOU, that bond B's yield is *lower* than bond A's: that is $i_B < i_A$. This situation cannot prevail for long! It means that bond B is a *bad deal* for potential bond buyers: bond A gives more return per dollar invested. So people rush to sell bond B and buy bond A. As people sell bond B, its market price falls, which raises its yield, making it a better deal than it was before:

$$i_B \uparrow = \frac{IOU_B}{P_B \downarrow} - 1$$

Meanwhile, as people rush to buy bond A its market price rises, reducing its yield and making it a worse deal than it was before;

$$i_A \downarrow = \frac{IOU_A}{P_A \uparrow} - 1$$

In equilibrium, the two bonds must be equally good deals, so their yields must be the same. The actions of investors in bond markets ensure that the bonds' market prices adjust until their yields are equal.

How long does it take to get to this equilibrium? No more than a few minutes. Bond traders watch market bond prices very carefully. Nowadays bond traders run computer programs that place buy and sell orders whenever bond prices are out of whack.

4) Using interest rates and yields: some examples

a) Inferring a bond's price from its yield

When people talk about a stock's price, they usually talk about the price per share. But when people talk about a bond's price, they do it by talking about the yield. If you went to a financial website to find out about one-year bonds being bought and sold today, you'd see quotes of yields, not prices.

From the yield and the amount of the bond's *IOU*, you can calculate the price. Consider a one-year bond with an *IOU* of \$5,000. If you read that the yield on this bond is 2.5 percent, you know that the market price of the bond is:

$$P = \frac{IOU}{1+i} = \frac{5,000}{1+0.025} = \frac{5,000}{1.025} = 4,878.05$$

b) What price is an issuer likely to get for a newly issued bond?

Suppose a firm wants to borrow money by issuing one-year bonds. It prints up a mess of bonds, each with an *IOU* of \$10,000, planning to auction them off. What price is it likely to receive for each bond?

To figure that out, the firm can look on a financial website and see what the yields have been on one-year bonds sold this morning. Because yields must be the same for all one-year bonds, the price the firm will get when it auctions off its newly-issued one-year bonds will be the price that equates the yield on those bonds to the yields on other one-year bonds being sold this morning.

Suppose yields on one-year bonds sold this morning have been 1/2%. Then the price the firm will receive for each of its bonds is likely to be:

$$P = \frac{IOU}{1+i} = \frac{10{,}000}{1+0.005} = \frac{10{,}000}{1.005} = 9{,}950.25$$

c) Rolling over a series of one-year loans

Suppose that you make a one-year loan. The principal on the loan - the amount of money you give the borrower today - is *$L*. When the borrower pays the money back a year from now, you take all of what she pays you - principal plus interest - and make *another* one-year loan (maybe to the same borrower, maybe to a different borrower). Two years from now, you are paid the principal plus interest on that second one-year loan. Again you take all that money and make *yet another* one-year loan. And so on until, after several years, you cash out.

Making a series of short-term loans like this, always lending out all the principal and interest from the previous loan, is called “rolling over” short-term loans.

How much money will you have when you cash out?

To describe this, we must take allow for the possibility that the interest rate you can get on a one-year loans may not be the same next year, or the year after that, as it is today. In fact, it is not just a possibility, it is a probability - interest rates on newly negotiated one-year loans rarely remain exactly the same from one day to the next, even.

Here’s the notation we will use:
i_t is today’s interest rate on one year loans.
i_{t+1} is the interest rate on one year loans that will prevail one year from today.
i_{t+2} is the interest rate on one-year loans that will prevail two years from today.
and so on.

Thus, if you start by making a loan of *$L*,
at the end of the first year, you will have $(1+i_t)L$.
at the end of the second year, you will have $(1+i_{t+1})(1+i_t)L$
(Remember, you took *all* the money you got back at the end of the first year, which was $(1+i_t)L$, and lent all of that out again.)
at the end of the third year, you will have $(1+i_{t+2})(1+i_{t+1})(1+i_t)L$
and so on.

Finally, let’s imagine a case which is unrealistic - it would almost never happen in reality - but which will prove useful for a point I will be making later on. Assume the one-year interest rate remains exactly the same year after year. Let’s denote this unchanging interest rate by $\hat{i}$. So:

$i_t = \hat{i}$

$i_{t+1} = \hat{i}$

$i_{t+2} = \hat{i}$

and so on. Thus,
at the end of the first year, you will have $(1+\hat{i})L$.

at the end of the second year, you will have $(1+\hat{i})(1+\hat{i})L=(1+\hat{i})^2 L$

at the end of the third year, you will have $(1+\hat{i})(1+\hat{i})(1+\hat{i})L=(1+\hat{i})^3 L$

and so on.

Notice that in this special case, because the interest rate is the same in all years, we can write your final cash-out amount in a different way. If we denote the number of years before you cash out by *n*, we can say that the amount of money you will have when you cash out is $(1+\hat{i})^n L$.

d) Rolling over a series of one-year bonds

Let's do the rollover story again but with bonds this time.

You start with *$X* today. You spend all of the *$X* on one-year bonds. Perhaps you buy just one bond with a price that happens to equal *$X* exactly. Perhaps you buy two or more bonds whose prices total to *$X.*

At the end of the year you are paid the *IOU*'s on those bonds. You take all of that money, the total of all the IOU's, and buy another mess of one-year bonds. When those bonds pay off, on a date exactly two years from now, you take the total of the *IOU*'s on *those* bonds and buy *another* mess of one-year bonds.... And so on until, after several years, you cash out. Buying a series of short-term bonds like this, reinvesting all of the *IOU*'s each time, is called "rolling over" short-term bonds.

How much money will you have when you cash out?

To figure this out, recall that, by definition, the *IOU* on a one-year bond is equal to the bond's price times (one plus the yield): $IOU=(1+i)P$, where *i* stands for the yield on one-year bonds. Thus, whatever specific combination of bonds you buy, as long as you spend *$X* in total the total of the IOU's you will be paid at the end of the year is $(1+i)X$. And $(1+i)X$ is the total amount of money you will spend buying your second round of one-year bonds a year from now.

But wait a minute! Like prevailing interest rates on one-year loans, the market yield on one-year bonds can change from day to day. It almost never remains exactly the same from one year to the next. To allow for this, use the same notation I used in the previous section:

i_t is today's yield on one-year bonds.

i_{t+1} is the yield on one-year bonds that will prevail one year from today.

i_{t+2} is the yield on one-year bonds that will prevail two years from today.

and so on.

Thus,

at the end of the first year, you will be paid IOU's that total $(1+i_t)X$.

at the end of the second year, you will be paid IOU's that total $(1+i_{t+1})(1+i_t)X$

at the end of the third year, you will be paid IOU's that total $(1+i_{t+2})(1+i_{t+1})(1+i_t)X$

and so on.

Finally, we can again imagine an unrealistic case in which the yield on one-year bonds does not change, it remains equal to $\hat{i}$ year after year. In that case,

at the end of the first year, you will have $(1+\hat{i})X$.

At the end of the second year, you will have $(1+\hat{i})(1+\hat{i})X = (1+\hat{i})^2 X$

At the end of the third year, you will have $(1+\hat{i})(1+\hat{i})(1+\hat{i})X = (1+\hat{i})^3 X$

and so on.

Let *n* denote the number of years before you cash out. Then we can generally say that the amount of money you will have when you cash out is $(1+\hat{i})^n X$.

D) More complicated bonds and loans

1) Introduction

Now let's make things more complicated. Bonds and loans can be more complicated in a number of ways. The date on which the borrower promises to make the payment might not be exactly one year from today: it might be two or more years in the future (or less than a year). The borrower might promise to make multiple payments, on multiple dates - that is one payment one year from now, another payment two years from now, and so on. In a case where the borrower promises multiple payments, the first payment might be the same amount of money as the second payment, or more money, or less money, and so on.

I will call bonds and loans in which the borrower promises to make just one payment "single-payment" loans and bonds. Loans and bonds in which the borrower promises to make a series of payments are "multiple-payment."

2) Maturity

The amount of time between now and the specified date on which the borrower will pay off the interest and principal on a loan, or pay the *IOU* on a bond, is called the "maturity" of the loan or bond.

Thus, the bonds and loans I described in the previous section all had a maturity of exactly one year.

A single-payment loan or bond in which the borrower promises to pay the interest and principal on the loan, or pay the IOU on the bond, *two* years from today has a maturity of two years. And so on.

For multiple-payment bonds and loans, "maturity" means the amount of time between now and the date specified for the *final* payment.

Here is something important: the maturity of a bond *changes over time* over the lifetime of the bond. By "lifetime of the bond" I mean the years between its issuance and when the borrower pays the last IOU.

For example, suppose that in September of 2020 the General Motors Corporation issues (prints up and sells) a bond that will make one payment in September of 2030. On the day it is issued, the maturity of that bond is ten years, because its IOU will pay off ten years in the future. As of September 2021, however, one would say that the maturity of the bond is nine years. As of September 2022, the maturity of the bond is eight years. In September 2029, it will be a one-year bond.

The maturity of a bond at the time it is issued is called its ***original maturity***.

(Something you don't need to know, but I'll tell you anyway: sometimes, but not often, people reserve the term "maturity" to mean original maturity and use the term "tenor" to refer to the amount of time between now and when a bond or loan pays off. Anyway, if you see the word "tenor" used in reference to financial markets, that's what it means.)

3) Single-payment loans of maturity greater than one year

In this type of loan the lender gives the borrower some money today and the borrower gives the lender a promise that the borrower will pay the lender a specified amount of money two years, or three years, or four years....from today, as the case may be. As in a one-year loan, the amount of money that borrower promises to pay the lender is specified in terms of an interest rate rather than a number of dollars.

It is customary to define all loans' interest rates on an "annual" or "annualized" basis, even if the maturity of the loan is more or less than one year. What does this mean? It is easiest to understand with examples.

Consider a loan in which the lender gives the borrower *$L* today. The loan contract says that the borrower will make one payment to the lender two years from today and the interest rate - that means annualized interest rate - is five percent. What does this mean? It means that on the date two years from now the borrower will pay the lender an amount of money equal to:

$$(1+0.05)(1+0.05)L=(1+0.05)^2 L$$

Another example. A loan contract says that the borrower will pay three years from today and the interest rate is one percent. This means is that three years from now the borrower will pay the lender an amount of money equal to:

$$(1+0.01)(1+0.01)(1+0.01)L=(1+0.01)^3 L$$

To generalize, if the single loan payment will be made m years from now and the annualized interest rate is i, the amount of the payment will be:

$$(1+i)^m L$$

Notice that this looks like the formula we derived in part IIIC4c). If you think about, you can see that the "annualized interest rate" on a single-payment loan of maturity greater than one year is the answer to the question: what would the interest rate on rolled-over one-year loans have to be in order to give the lender the same final payoff as this longer-maturity loan? Read that again and again until you understand it.

And of course, for $m=1$, this is the same formula we used for one-year loans.

4) Single-payment bonds of maturity greater than one year

a) What they are

This type of bond is a piece of paper on which the issuer/borrower promises to pay an *IOU* to whoever is holding the piece of paper on a date two or more years from today.

b) Yield to maturity

As in the case of a one-year bond, one speaks of the "yield" on the bond, which is defined by the ratio of the *IOU* on the bond relative to the price paid for the bond in the latest sale of the bond. Just as it is conventional to define all loans' interest rates on an annualized basis, it is conventional to define all bonds' yields on an annualized basis. A yield defined in this way is called, specifically, the "yield to maturity." There are other kinds of yields, but you can usually assume that "yield" means "yield to maturity."

What is this "yield to maturity"? I need to introduce some more notation here.

${}_m i$ denotes the yield to maturity on a single-payment bond with a maturity of m years. That is to say, it is a bond on which the issuer is supposed to pay the *IOU* exactly m years from now.

For example, the yield to maturity on a bond that pays off one year from now is denoted ${}_1i$. The yield to maturity on a bond that pays off four years from now is denoted ${}_4i$. The yield to maturity on a bond that pays off 117 years from now is ${}_{117}i$.

In this notation, here is the formula that defines yield to maturity on a single-payment bond:

$${}_mi = \sqrt[m]{\frac{IOU}{P}} - 1 = \left(\frac{IOU}{P}\right)^{1/m} - 1$$

This formula gives the yield as a fraction or decimal, which must be multipled by 100 to give the yield as a percent. Note that, for *m=1*, this is the same as the definition I gave earlier for the yield on a one-year bond.

Here's an example. A bond will pay an IOU of $20,000 two years from now. Its market price today is $18,000. The yield (yield to maturity) on this bond is:

$$i = \sqrt{\frac{20,000}{18,000}} - 1 = \left(\frac{20,000}{18,000}\right)^{1/2} - 1 = 1.054 - 1 = 0.054 \text{ or } 5.4 \text{ percent.}$$

As with one-year bonds, when people talk about prices of longer-maturity bonds they do it by talking about the yield. If you went to a financial website to find out about longer-maturity bonds being bought and sold today, you'd see quotes of yields (that is, yield to maturity), not prices.

5) Multiple-payment bonds

a) General definition

This means a bond in which the issuer promises to make one payment one year from today, another payment two years from today, another payment three years from today, and so on.

In most *real* multiple-payment bonds, the payments come once a quarter. (A "quarter" means a quarter of a year, three months). For simplicity, we'll just pretend the payments come once a year.

Our notation:
${}_1IOU$ is the first payment
${}_2IOU$ is the second payment
${}_3IOU$ is the third payment
and so on.

There are multiple-payment loans, too, but we don't deal with those much in this course, so I won't introduce notation for them.

Back when bonds were pieces of paper, a multiple-payment bond was several pieces of paper, one for each *IOU*, batched together. You can imagine them as stapled together. (Really, they were printed on one big sheet with perforations, like a sheet of postage stamps.) When the first *IOU* came due, the holder of the bond would detach it from the bunch and present it to the issuer to get paid. When the second *IOU* came due, the holder would detach that *IOU* and present it to the issuer, and so on until all the *IOU*'s were gone.

The issuer sells the *IOU*'s as a bunch, not separately, getting one price for the whole bunch.

But after the bond has been sold by the issuer, anyone holding a multiple-payment bond is free to *pull the IOU's apart and sell them individually* as single-payment bonds. This is important. I'll get back to it later.

b) Yield to maturity on a multiple-payment bond

When I was talking about single-payment bonds, I said people in bond markets usually don't quote prices for bonds directly; they quote yields and leave it to you to calculate what the price must be, and that "yield" means an annualized yield called "yield to maturity." All that is true for multiple-payment bonds as well.

What is the yield to maturity for a multiple-payment bond? Like the yield to maturity of a single-payment bond, it is a number defined by the market price of the bond, the maturity of the bond and its *IOU*s. You get it by setting up and solving an equation. But the equation for a multiple-payment bond is complicated. It is so complicated that you can't solve it algebraically! You have to let a computer or financial calculator solve it for you.

I'll use ${}_{YTM}i$ to denote the yield to maturity on a multiple-payment bond. It is a For a bond that makes two payments, the equation is:

$$P = \frac{{}_1IOU}{(1+{}_{YTM}i)} + \frac{{}_2IOU}{(1+{}_{YTM}i)^2}$$

For a bond that makes four payments,

$$P = \frac{{}_1IOU}{(1+{}_{YTM}i)} + \frac{{}_2IOU}{(1+{}_{YTM}i)^2} + \frac{{}_3IOU}{(1+{}_{YTM}i)^3} + \frac{{}_4IOU}{(1+{}_{YTM}i)^4}$$

and so on.

To generalize, to calculate the yield to maturity on a multiple-payment bond of *n* years maturity, you take as given the *IOU*'s on the bond and the market price of the bond. The yield to maturity is the value of ${}_{YTM}i$ that solves this equation:

$$P = \frac{{}_1IOU}{(1+{}_{YTM}i)} + \frac{{}_2IOU}{(1+{}_{YTM}i)^2} + \ldots\ldots + \frac{{}_nIOU}{(1+{}_{YTM}i)^n}$$

As an example, consider a bond that makes payments for four years. For the first three years the IOU is \$500. For the last year the IOU is \$1000. The market price of the bond is \$1879. This bonds yield to maturity is the value for ${}_{YTM}i$ that solves:

$$1879 = \frac{500}{(1+{}_{YTM}i)} + \frac{500}{(1+{}_{YTM}i)^2} + \frac{500}{(1+{}_{YTM}i)^3} + \frac{1000}{(1+{}_{YTM}i)^4}$$

As I said, this is not a math problem you can solve algebraically. You have to plug the numbers into a financial calculator or computer and let the machine iterate until it finds the solution. For this example, the solution is ${}_{YTM}i = 0.11$ or 11 percent

c) What is the most you would be willing to pay for a multiple-payment bond?

A multiple-payment bond is really just a bunch of single-payment bonds stuck together. You could get exactly the same thing by buying a bunch of single-payment bonds that give the same pattern of *IOU*'s over time. Consider the bond in the example above. You could reproduce this bond by buying a one-year single-payment bond with an *IOU* of $500, along with a two-year single-payment bond with an *IOU* of $500, a three-year bond with an *IOU* of $500, and a four-year bond with a $1000 *IOU* (or two four-year bonds with *IOU*'s of $500 each).

The fact that you can reproduce any multiple-payment bond by buying a set of single-payment bonds establishes a *maximum* that you would be willing to pay for any multiple-payment bond. That maximum is the total cost of buying the set of single-payment bonds that reproduces the multiple-payment bond. As an analogy, suppose that at the grocery store you patronize a jar of mayonnaise costs $2.50, a bottle of ketchup is $3, and a jug of milk is $4. What is the most you would be wiling to pay for a bag containing a jar of mayonnaise, a bottle of ketchup and a jug of milk? $9.50!

Let's apply this logic to the bond in the example above. How much would it cost to reproduce this bond with a batch of single-payment bonds? Remember that bond dealers quote prices for bonds indrectly, by quoting yields. You can look at a financial website that gives you current yields on single-payment bonds. Suppose the website says that current market yields on zero-coupon bonds are as follows:

$${}_1i = 0.01\ (one\ percent)$$
$${}_2i = 0.04\ (four\ percent)$$
$${}_3i = 0.02\ (two\ percent)$$
$${}_4i = 0.03\ (three\ \ percent)$$

Then:

a zero-coupon bond that will pay you $500 in one year costs $P = \dfrac{500}{(1+0.01)} = 495.05$

a zero-coupon bond that pays $500 in two years costs $P = \dfrac{500}{(1+0.04)^2} = 462.28$

a zero-coupon bond paying $500 in three years costs $P = \dfrac{500}{(1+0.02)^3} = 471.16$

and a bond paying $1000 in four years costs $P = \dfrac{1000}{(1+0.03)^4} = 888.49$

The total cost of *all* these bonds would be *495.05 + 462.28 + 471.16 + 888.49 = 2316.98*
So $2316.98 is the most you, or anyone, would be willing to pay for the multiple-payment bond in question.

To generalize, to calculate the most a buyer would be willing to pay for a multiple-payment bond with a maturity of *n* years, you take as given the bond's *IOU*'s and current yields on single-payment bonds, and use this formula to calculate the maximum price anyone would pay:

$$P = \frac{{}_1IOU}{(1+{}_1i)} + \frac{{}_2IOU}{(1+{}_2i)^2} + \ldots\ldots + \frac{{}_mIOU}{(1+{}_mi)^m}$$

WARNING! It is easy for you to confuse this formula with the one I gave above that defines the yield to maturity on a multiple-payment bond. They are not the same. Make sure that you understand how they are different.

d) What is the market price of a multiple-IOU bond?

The price we just talked about in the previous section, that is the highest price anyone would be willing to pay for a multiple-payment bond, is also the *equilibrium market price* of the bond. The profit-maximizing actions of traders and dealers in bond markets ensure that is the case.

To understand why, you must remember something I said earlier: “after the bond has been sold by the issuer, the person who bought the bond can pull all the IOU's apart and sell them individually as single-payment bonds.” And you must consider what would happen if the market were out of equilibrium, that is if the market price were greater than/less than the equibrium price defined in the previous section.

What would happen if the market price were less than the equilibrium price, that is if:

$$P < \frac{{}_1IOU}{(1+{}_1i)} + \frac{{}_2IOU}{(1+{}_2i)^2} + \ldots\ldots + \frac{{}_mIOU}{(1+{}_2i)^m}$$

In this case, you could profit money by buying up as many of these particular multiple payment bonds as you can get your hands on. You take each bond, pull all of its *IOU*s apart into separate pieces of paper which are now single-payment bonds. You sell each of these single-payment bonds individually. What is the total amount that you will receive when you sell them individually? Well the price of ${}_1IOU$ will of course be determined by the current yield on one-year single-payment bonds, ${}_1i$. The price of ${}_2IOU$ will be determined by ${}_2i$, and so on. The total amount you will receive is:

$$\frac{{}_1IOU}{(1+{}_1i)} + \frac{{}_2IOU}{(1+{}_2i)^2} + \ldots\ldots + \frac{{}_mIOU}{(1+{}_mi)^m}$$

As long as this number is greater than the multiple-payment bond's price *P,* you will make a profit. This process is called ***stripping*** a bond, because you are stripping apart the components of the multiple-payment bond to make many single-payment bonds.

Traders and dealers in bond markets are quick to take advantage of profitable opportunities. So you can be sure that they will in fact rush to buy these particular multiple-IOU bonds, to strip them. That will drive up the market price of the multiple-payment bond until the profit opportunity - the price disequilibrium - has disappeared.This makes money for you, because the total you will receive by selling the IOUs as single-payment bonds is equal to: the right-hand side of the expression above.

What if the opposite case holds? What would happen if the market price were more than the equilibrium price, like:

$$P > \frac{{}_1IOU}{(1+{}_1i)} + \frac{{}_2IOU}{(1+{}_2i)^2} + \ldots\ldots + \frac{{}_mIOU}{(1+{}_mi)^m}$$

In that case, traders and dealers in bond markets would do the *opposite of stripping*. They would buy a batch of single-payment bonds that reproduce the payment pattern of the overvalued multiple-IOU bond, thus creating more bonds identical to the overvalued bond, and sell those newly-created multiple-payment bonds at a profit. Their actions would, of course, drive *down* the market price of the multiple-payment bond until the price disequilibrium disappears. There is not a special word for this process, but it happens.

E) Names of common bond types

1) Introduction

I have used the word “bond” to refer to all instruments of borrowing and lending that feature *IOU*’s payable to whoever is holding the *IOU* at the time it comes due. I will continue to do that in this course, to keep our terms simple. But in reality, there are different names for different types of "bonds." In this section I will tell you some of those names. Keep in mind that many bonds that do not fall into these common types. Almost any pattern of *IOU*’s you can imagine has been issued as a bond at one time or another.

Finally, remember what the phrase “original maturity” means: the maturity of a bond at the time it is issued.

2) Types of single-payment bonds

a) Bill

A single-payment bond with an original maturity of one year or less is call a “bill.”

A “Treasury bill” is a bill issued by the U.S. Treasury.

“Commercial paper” is a bill issued by a private corporation.

b) Zero-coupon bond

This is a single-payment bond with an original maturity greater than one year. You’ll come to understand why it is called this when you read below about “coupon bonds.”

c) Strip

This is a zero-coupon bond that was originally part of a multiple-payment bond, and was stripped.

2) Types of multiple-payment bonds

a) Fixed-payment bond

A fixed-payment bond is a bond that makes one payment every year for m years - that is, its maturity is m years - and each payment is the same size (same amount of money). Example: a bond that pays $100 once a year for fifty years.

b) Coupon bond

A coupon bond is a bond that makes a small payment once a year for n years, all the same size (same amount of money), and in the final year - the mth year - makes one *big* payment in addition to the small payment. An example is a bond that pays an *IOU* of $50 every year for five years, and in the fifth year pays a $50 *IOU* plus *another* IOU of $1,000.

A coupon bond’s small payments are called “coupons” or “coupon payments.” We will denote them by C. In the example above, $C = \$50$.

The extra, big payment that comes in the last year is called the “par value” or “face value” of the bond. We will denote it by F. In the example above, $F = \$1,000$.

The ***coupon rate*** is *C/F*. This is a fraction, usually spoken of as a percent. In the example, the coupon rate is:

$$\frac{C}{F} = \frac{50}{1000} = 0.05$$ or 5 percent.

c) Consol (*or* perpetuity)

This is a fixed-payment bond that makes a payment once a year forever - the maturity is infinity. Such bonds were issued in the nineteenth century by the governments of Britain and the U.S. They are called "consols" after a British government issue of 1752. At that time the British government had many bonds outstanding, issued at various times in the past. It was a pain for the Treasury to manage the payment of the various *IOU*'s. So the Treasury issued a new set of bonds and used the money from the sale of these bonds to buy up all the outstanding bonds. An operation like this is called "consolidating" debt. The new bonds were perpetuities. So "consol," from "consolidated," became a word for any perpetual bond.