AGGREGATE PRODUCTION FUNCTIONS

Reality

Society uses	to make	to be
Labor (hours, effort) Natural resources Skills Tools Buildings etc.	Various goods, services skills (education)	– Consumed immediately for pleasure (consumption) – Used up immediately as inputs to production (materials) – Used repeatedly to produce over & over (capital)

wears out eventually (depreciation) → Used repeatedly to produce over & over (capital)

1) A choice: how much productive resources to devote to immediate consumption versus creation of new capital (investment)?

2) What is optimal, assuming that our goal is maximum consumption?

3) "Knowledge," "technology," "institutions":
what societies can make, how much output/inputs, varies across societies & over time
(invention of agriculture, steam engines, etc.)

AGGREGATE .... (cont.)

Model: aggregate output & capital

We make one good-service Y which can be be eaten immediately as consumption C or nailed to the ground, becoming one kind of capital K to be used in production.

$$Y = C + I$$ (investment, new capital

A fraction δ of existing capital wears out each period so

$$K_{t+1} = (1-\delta) K_t + I_t = (1-\delta) K_t + (Y_t - C_t)$$

Taking tiny unit of time ∂t
& defining $\dot{X} = \partial X / \partial t$,

$$\dot{K}(t) = [Y(t) - C(t)] - \delta K(t) \qquad (1.15)$$

Y is society's "income" which is divided between consumption & "savings" (= investment).

(Later, we'll think about government, taxes, trade with other societies....)

Note $\frac{\dot{X}}{X}$ is rate of growth of X

($100 \cdot \frac{\dot{X}}{X}$ is percent rate of growth)

AGGREGATE ... (cont.)

Model: aggregate production function

$$Y = F(\overset{+}{K}, \overset{+}{L}, \text{other stuff})$$

L: Labor input (hours or workers or...)

Other stuff: natural resources or skills or...

In this course, we'll <u>ignore</u> <u>natural</u> <u>resources</u>

(but see Romer 1.8 if you want).

To account for knowledge, technology, etc.

we'll define that as other stuff A

$$Y(t) = F(\overset{+}{K}(t), \overset{+}{L}(t), \overset{+}{A}(t))$$

$A\uparrow$ means more Y from same K & L, but it could "augment" productivity of K versus L differently

$Y(t) = A(t)\,F(K(t), L(t))$ Hicks-neutral

$Y(t) = F(A(t)K(t), L(t))$ Capital-augmenting

$Y(t) = F(K(t), A(t)L(t))$ Labor-augmenting or Harrod-neutral (1.1)

AGGREGATE...

Model: aggregate... (cont.)

Returns to scale

Think about bigger vs. smaller societies

Society A has K, L

" B " cK, cL — $c > 1$: B is bigger; $c < 1$: B is smaller

If bigger societies are more productive,

$$F(cK, cL) > cF(K, L) \text{ for } c > 1$$
$$< \qquad \text{ for } c < 1$$

"Increasing returns to scale"

" " " " less productive,

$$F(cK, cL) < cF(K, L) \text{ for } c > 1$$
$$> \qquad \text{ for } c < 1$$

"Decreasing returns to scale"

If productivity unrelated to scale,

$$F(cK, cL) = cF(K, L) \qquad (1.2)$$

Note: this is about whole societies not individual establishments or industries, which can have production functions different from aggregate prodn. fn.

AGGREGATE PRODN. ... (cont.)

Marginal products

MPL: $\partial F(K, L, \ldots)/\partial L$ (or $F_K(K, L, \ldots)$) > 0

MPK: $\partial F(K, L, \ldots)/\partial K$ (or $F_L(K, L, \ldots)$) > 0

"Diminishing" MPX (K or L) means

$$\frac{\partial^2 F(K, L, \ldots)}{\partial X^2} \text{ (or } F_{XX}(K, L, \ldots)) < 0$$

Euler's theorem (applied here)

If prodn. Fn. has CRS, then

$$Y = MPK \cdot K + MPL \cdot L$$

Recall if CRS $Y = cF(K, L..) = F(cK, cL, \ldots)$

$$\frac{\partial Y}{\partial c} = F(K, L\ldots) = F_K(cK, cL, \ldots)\frac{\partial cK}{\partial K} + F_L(\ )\frac{\partial cL}{\partial L}$$

$$Y = \underbrace{F_K(\ )}_{MPK} \cdot K + \underbrace{F_L(\ )}_{MPL} \cdot L$$

or

$$1 = \frac{F_K(\ ) \cdot K}{Y} + \frac{F_L(\ ) \cdot L}{Y}$$

AGGREGATE PRODN.... (cont.)

Marginal products in a perfectly competitive economy

If all product & factor markets are perfectly competitive, then every firm:

— takes its product price as given

— pays same cost W for a unit of labor

— pays same "rental rate" R for capital

Thus for each firm i with price p_i & MPL_i, MPK_i, profit-maximization ensures:

$$p_i \cdot MPL_i = W$$

$$p_i \cdot MPK_i = R$$

Averaging across all firms,

$$P \cdot F_L(\quad) = W, \quad P \cdot F_K(\quad) = R$$

price level

From aggregate prodn. fn.

so $F_L(\quad) = \frac{W}{P}$, $F_K(\quad) = \frac{R}{P}$

This implies macroeconomic statistics can tell you things about the aggregate prodn. fn.

[Note that we made a lot of dubious assumptions here!]

AGGREGATE PRODN. (cont.)

Factor shares of income and MP's

National Income & Product Accounts (NIPAs) tell you:

share of national income going to wages, salaries, implied salaries of owner-managers, other forms of employer compensation: "labor's share"

share of national income going to profit, rent, etc.: "capital's share"

$$\underbrace{\frac{\frac{R}{P}\cdot K}{Y}}_{\text{capital's share of real income}} = \underbrace{\frac{R\cdot K}{P\cdot Y}}_{\text{capital's share of nominal income}}$$

Nominal or real, same thing

Under assumptions of previous page,

$$\text{Capital's share} = \frac{MPK\cdot K}{Y}$$

$$\text{Labor's share} = \frac{MPL\cdot L}{Y}$$

and, if aggregate prodn. fn. has CRS, Euler's thm:

$$1 = \frac{F_K(\ )\cdot K}{Y} + \frac{F_L(\ )\cdot L}{Y}$$

shares add up to one.

In U.S. NIPA's, capital's share is $\approx 1/3$.

AGGREGATE PRODN. ... (cont.)

Growth accounting (Romer 1.7, "Growth Accounting")

Suppose $Y = F(K, L, A)$

Then growth in Y (real GDP) over time is due to growth in K, L ("factors of production") or A ("other forces")

Y, K & L are measurable (Y & K in NIPA's)

Assuming prodn. fn. has CRS & factor shares reveal relative MP's (all markets competitive), we can figure out how much growth in Y is due to K & L, & thus how much must be due to growth in A ← "the Solow residual" technology, etc.

You can do this using <u>average</u> growth rates over last hundred years to answer question: "how much growth in Y/L was due to K/L versus technological improvement"

You can do it year to year or ¼ to ¼ to infer short-term fluctuations in A, but this is controversial (we'll get back to this later in the course).

AGGREGATE...

Growth accounting (cont.)

$$\dot{Y}(t) = \underbrace{\frac{\partial Y(t)}{\partial K(t)}}_{MPK} \dot{K}(t) + \underbrace{\frac{\partial Y(t)}{\partial L(t)}}_{MPL} \dot{L}(t) + \frac{\partial Y(t)}{\partial A(t)} \dot{A}(t) \quad (1.34)$$

Divide by $Y(t)$:

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{\partial Y}{\partial K} \frac{\dot{K}(t)}{Y(t)} + \frac{\partial Y}{\partial L} \frac{\dot{L}(t)}{Y(t)} + \frac{\partial Y}{\partial A} \frac{\dot{A}(t)}{Y(t)}$$

then multiply by $\frac{K(t)}{K(t)}$, $\frac{L(t)}{L(t)}$, $\frac{A(t)}{A(t)}$

(1.35)

$$\frac{\dot{Y}(t)}{Y(t)} = \frac{K(t)}{Y(t)} \frac{\partial Y}{\partial K} \frac{\dot{K}(t)}{K(t)} + \frac{L(t)}{Y(t)} \frac{\partial Y}{\partial L} \frac{\dot{L}(t)}{L(t)} + \frac{A(t)}{Y(t)} \frac{\partial Y}{\partial A} \frac{\dot{A}(t)}{A(t)}$$

rates of growth of Y, K, L, which are all observable

$$\frac{K(t)}{Y(t)} \frac{\partial Y}{\partial K} = \frac{MPK \cdot K}{Y}$$ capital's share of income.

Romer calls it "elasticity of output with respect to K"

$$\alpha_K(t) = \frac{\partial Y(t)/Y(t)}{\partial K(t)/K(t)}$$ ← % change in Y / ← % change in K

It's proportional (or percent) change in Y resulting from a proportional change in K.

$\alpha_L(t)$ defined similarly.

AGGREGATE

Growth accounting (cont.)

Using definitions of α_K, α_L:

$$\frac{\dot{Y}(t)}{Y(t)} = \alpha_K(t)\frac{\dot{K}(t)}{K(t)} + \alpha_L(t)\frac{\dot{L}(t)}{L(t)} + R(t) \qquad (1.35)$$

$\alpha_K(t)$ is $\approx 1/3$ in U.S.

$\alpha_L(t)$ is $\approx 2/3 = 1 - 1/3$ (because $\alpha_K + \alpha_L = 1$)

$R(t)$ = "Solow residual," contribution of A to growth in Y over time

$R(t)$ = Rate of growth in real GDP
 $- 1/3 \cdot$ Rate of growth in index of "capital"
 $- 2/3 \cdot$ Rate of growth in labor hours

To put it in terms of output per worker, subtract $\frac{\dot{L}}{L}$ from both sides:

$$\frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = \alpha_K\frac{\dot{K}}{K} + \alpha_L\frac{\dot{L}}{L} - \frac{\dot{L}}{L} + R(t)$$

$$= \alpha_K\frac{\dot{K}}{K} - \underbrace{(1-\alpha_L)}_{\alpha_K}\frac{\dot{L}}{L} + R(t)$$

$$= \alpha_K(t)\left[\frac{\dot{K}(t)}{K(t)} - \frac{\dot{L}(t)}{L(t)}\right] + R(t) \qquad (1.36)$$

Table 1.5. *The sources of labor productivity growth, U.S. private domestic economy, 1800–1989 (sources in percentage points measured across long periods)*

	I. Nineteenth Century			II. Twentieth Century		
	1800–1855	1855–1890	1890–1927	1890–1927	1929–1966	1966–1989
1. Output per manhour	0.39	1.06	2.01	2.00	2.52	1.23
Sources						
2. Capital stock per manhour	0.19	0.69	0.62	0.51	0.43	0.57
3. Crude total factor productivity	0.20	0.37	1.39	1.49	2.09	0.66
4. Labor quality	—	—	0.15	0.15	0.40 (0.30)	0.31 (0.16)
5. Capital quality	—	—	—	—	0.24	0.31
6. Refined total factor productivity	0.20	0.37	1.24	1.34	1.45 (1.55)	0.04 (0.19)
Addenda						
7. Gross factor share weights						
a. Labor	0.65	0.55	0.54	0.58	0.64	0.65
b. Capital	0.35	0.45	0.46	0.42	0.36	0.35

Table 1.6. *The relative importance of crude TFP growth among the sources of labor productivity growth in the U.S. private domestic economy, 1800–1989*

$\frac{K}{L}$

	Percentage of labor productivity growth rate due to			Percentage of interperiod change in labor productivity growth rate due to change in:	
Period	Capital intensity growth rate	Crude TFP growth rate		Capital intensity growth rate	Crude TFP growth rate
I: 1800–1855	49	51			
I: 1855–1890	65	35	1800/1855 to 1855/1890	75	25
I: 1890–1927	31	69	1855/1890 to 1890/1927	−7	107
II: 1890–1927	25	75			
II: 1929–1966	17	83	1890/1927 to 1929/1966	−15	115
II: 1966–1989	46	54	1929/1966 to 1966/1989	−11	111

Note for later: K/L has been rising

Source: Abramowitz & David (2000)

AGGREGATE ... (cont.)

Cobb-Douglas production function

$$Y = K^{\alpha}(AL)^{1-\alpha} \quad \text{where } 0<\alpha<1$$

or

$$Y = AK^{\alpha}L^{1-\alpha}$$

Has CRS. [Try it! Is $(cK)^{\alpha}(AcL)^{1-\alpha}$ equal to $c(K^{\alpha}(AL)^{1-\alpha})$?]

Can't distinguish between Hicks-neutral, capital-augmenting, & labor-augmenting A, because it satisfies conditions for all three.

Hicks-neutral: $Y = \underbrace{B}_{A}\underbrace{K^{\alpha}L^{1-\alpha}}_{F(K,L)}$

Capital-augmenting: $Y = (BK)^{\alpha}L^{1-\alpha} = \underbrace{B^{\alpha}}_{A}\underbrace{K^{\alpha}L^{1-\alpha}}_{F(K,L)}$

Labor-augmenting: $Y = K^{\alpha}(BL)^{1-\alpha} = \underbrace{B^{1-\alpha}}_{A}\underbrace{K^{\alpha}L^{1-\alpha}}_{F(K,L)}$

Has diminishing MPK, MPL. Try it!

We use it a <u>lot</u>. Why? Easy & <u>matches</u> <u>fact</u>: "appears to be a good first approximation to actual production functions" (Romer p. 12)

AGGREGATE ...

Cobb-Douglas (cont.)

What is the fact that CD matches?

"The stability of the proportion of the national dividend accruing to labor... is one of the most surprising, yet best-established, facts in the whole range of economic statistics, for both Great Britain and the United States"

(Keynes, EJ 1939)

U.S. NIPAs

	1929	2006
Employee's Compensation	51.1	7448.3
Corporate profits	10.6	1553.7
Net Interest	4.6	598.5
Rental income	5.6	54.5
Depreciation (added back in)	0.5	215.8
Total income	72.4	9870.8
Labor's share of income	71%	75%

(Recently labor's share has fallen, but...)

True even though K/L has been growing,

C.D. + competitive economy reproduces this.

AGGREGATE ...

Cobb-Douglas

C. D. and Factor income shares

$Y = AK^{\alpha}L^{1-\alpha}$ (recall same as $K^{\alpha}(AL)^{1-\alpha}$)

MPK: $\frac{\partial Y}{\partial K} = A\alpha K^{\alpha-1} L^{1-\alpha}$

MPL: $\frac{\partial Y}{\partial L} = AK^{\alpha}(1-\alpha)L^{-\alpha}$

Labor's share $= \dfrac{\frac{W}{P} \cdot L}{Y} = \dfrac{MPL \cdot L}{Y} = \dfrac{\frac{\partial Y}{\partial L} \cdot L}{AK^{\alpha}L^{1-\alpha}} = 1-\alpha$

Capital's share $= \ldots\ldots = \alpha$

Note: if we use CD in a model & want a "realistic" number for α, we'll use $\approx 1/3$.

AGGREGATE...

Elasticity of substitution

Another way too look at changes in labor's share, how labor's share can be constant & why Cobb-Douglas can create a constant labor share.

General definition

When consumers maximize utility, relative demand for two goods X_1 & X_2 changes in response to change in relative price P_1/P_2. "Elasticity of substitution" ε is

$$\frac{\text{Percent change in } (X_1/X_2)}{\text{Percent change in } (P_1/P_2)} = \frac{\partial(X_1/X_2)/(X_1/X_2)}{\partial(P_1/P_2)/(P_1/P_2)}$$

$$= \frac{\partial(X_1/X_2)}{\partial(P_1/P_2)} \frac{(P_1/P_2)}{(X_1/X_2)} = \frac{\partial \ln(X_1/X_2)}{\partial \ln(P_1/P_2)}$$

because $\partial \ln(X) = \frac{\partial X}{X}$

When firms maximize profit, relative demand for K & L changes in response to change in relative cost $\frac{W}{P} / \frac{R}{P} = W/R$. ε is

$$\frac{\partial(K/L)/(K/L)}{\partial(R/W)/(R/W)} = \frac{\partial(K/L)}{\partial(R/W)} \frac{R/W}{K/L}$$

Should be negative.

same in terms of $\frac{R}{P}$, $\frac{W}{P}$

AGGREGATE ....

Elasticity of substn. (cont.)

Elasticity .... & changes in labor's share

Consider $\dfrac{\text{Capital income}}{\text{Labor income}} = \dfrac{\frac{R}{P}\,K}{\frac{W}{P}\,L} = \dfrac{\frac{R}{P}}{\frac{W}{P}}\,\dfrac{K}{L}$

Over history, $\frac{K}{L} \uparrow$. Holding $\frac{R}{W}$ fixed, this lowers labor share.

But increase in supply of K relative to L lowers R/W, tending to raise labor's share.

$\%\Delta(X_1 \cdot X_2) = \%\Delta X_1 + \%\Delta X_2$. So:

Labor's share rises (capital's share falls) if

$\left|\%\Delta\left(\frac{R}{P} / \frac{W}{P}\right)\right| > \%\Delta(K/L) \Leftrightarrow$ (means $|\varepsilon| < 1$

falls a lot

Labor's share falls (capital's share rises) if

$\left|\%\Delta\left(\frac{R}{P} / \frac{W}{P}\right)\right| < \%\Delta(K/L) \Leftrightarrow$ (means $|\varepsilon| > 1$

falls just a little

Labor's share constant if

$| \qquad | = \%\Delta(K/L) \Leftrightarrow$ (means $|\varepsilon| = 1$

Confused? Let's reverse it....

AGGREGATE --

... & changes in labor's share

1) $|\varepsilon| < 1$, means $-1 < \varepsilon < 0$

$$\left|\frac{\partial(K/L)/(K/L)}{\partial(R/W)/(R/W)}\right| \text{ small, so } \left|\frac{\partial(R/W)/(R/W)}{\partial(K/L)/(K/L)}\right| \text{ big,}$$

so $(K/L)\uparrow \rightarrow (R/W)\Big\downarrow \rightarrow$ labor's share $\uparrow$

2) $|\varepsilon| > 1$, means $\varepsilon < -1$

$(K/L)\uparrow \rightarrow (R/W)\downarrow \rightarrow$ labor's share $\Downarrow$

3) $|\varepsilon| = 1$, means $\varepsilon = -1$

$(K/L)\uparrow \rightarrow$ exactly countervailing $(R/W)\downarrow$.

Summary: $(K/L)\uparrow$

— increases labor share if $-1 < \varepsilon < 0$

— decreases " " " $\varepsilon < -1$

— no effect if $\varepsilon = -1$

But what does this have to do with the prodn. fn.?

AGGREGATE...

Elasticity of substitution (cont.)

Elasticity ... and aggregate prodn. fn.

Recall (if markets competitive) changes in w/p must be equal to changes in MP's, so constant income shares also means

$$\% \Delta (MPK/MPL) = -\% \Delta (K/L)$$

$$-1 = \frac{\partial (K/L)}{(K/L)} \quad \frac{MPK/MPL}{\partial (MPK/MPL)}$$

As K/L grew over history, MPK/MPL decreased exactly enough.

The way MPK/MPL changes in response to changes in K/L is determined by the form of the prodn. fn.

C.D. is the form for which the above equation holds.

AGGREGATE...

Elasticity of substitution

.... and aggregate prodn. fn. (cont.)

Cobb-Douglas elasticity of substn.

1) Get (K/L) as function of $(R/P)/(W/P)$

2) Get $\partial(K/L)/\partial(\quad)$

3) Multiply by $\dfrac{((R/P)/(W/P))}{K/L}$

1) Get (K/L)...

$$\frac{R/P}{W/P} = \frac{MPK}{MPL} = \frac{A\alpha K^{\alpha-1}L^{1-\alpha}}{AK^{\alpha}(1-\alpha)L^{-\alpha}} = \frac{\alpha}{1-\alpha}\left(\frac{K}{L}\right)^{-1}$$

hence $$\frac{K}{L} = \frac{\alpha}{1-\alpha}\left(\frac{R/P}{W/P}\right)^{-1}$$

2) Get $\partial(K/L)/\partial(\;)$

$$\frac{\partial(K/L)}{\partial(\quad)} = \frac{\alpha}{1-\alpha}(-1)\left(\frac{R/P}{W/P}\right)^{-2}$$

3) Multiply by ...

$$\frac{\alpha}{1-\alpha}(-1)\left(\frac{R/P}{W/P}\right)^{-2}\left[\frac{K}{L}\right]^{-1}\left(\frac{R/P}{W/P}\right)$$

$$= \text{------}\left[\left(\frac{\alpha}{1-\alpha}\frac{R/P}{W/P}\right)^{-1}\right]^{-1}\text{---} = -1$$