OLG MODEL

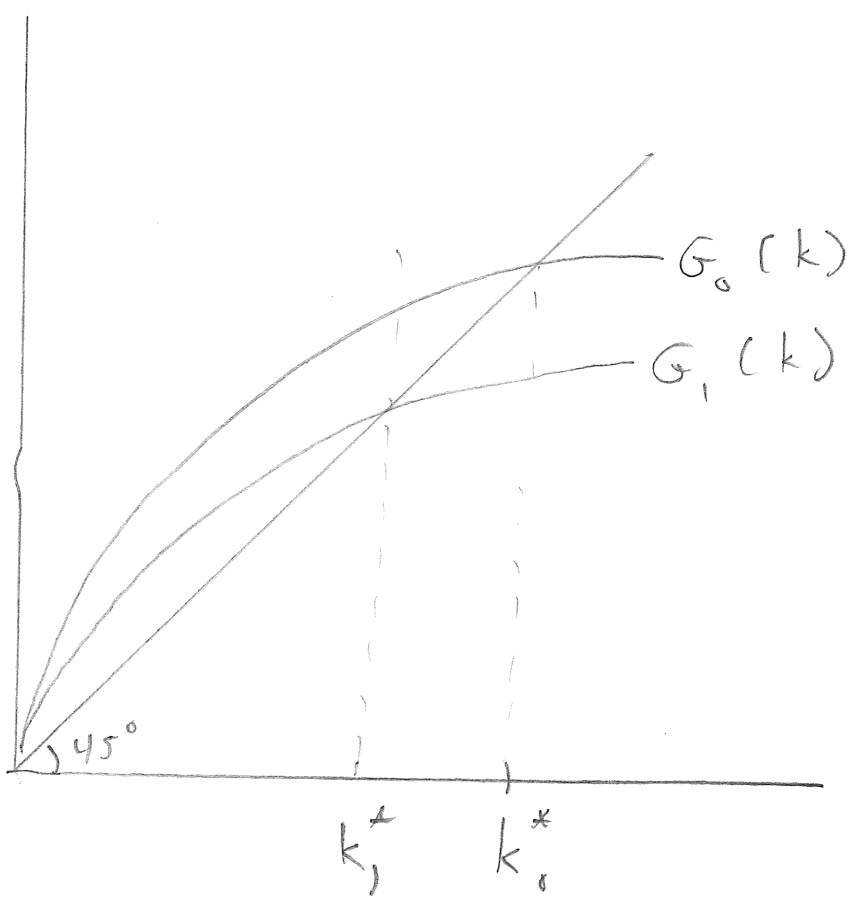
2.14) Diamond model with log utility & Cobb-Douglas

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha)k_t^{\alpha}$$

$G(k_t)$ is concave.

a) $n \uparrow$

At any k_t, k_{t+1} is <u>smaller</u>.



OLG MODEL

b) $f(k)$ is Bk^{α} & B falls.

Where does B fit in?

Now, $f'(k) = B\alpha k^{\alpha-1}$ not $\alpha k^{\alpha-1}$

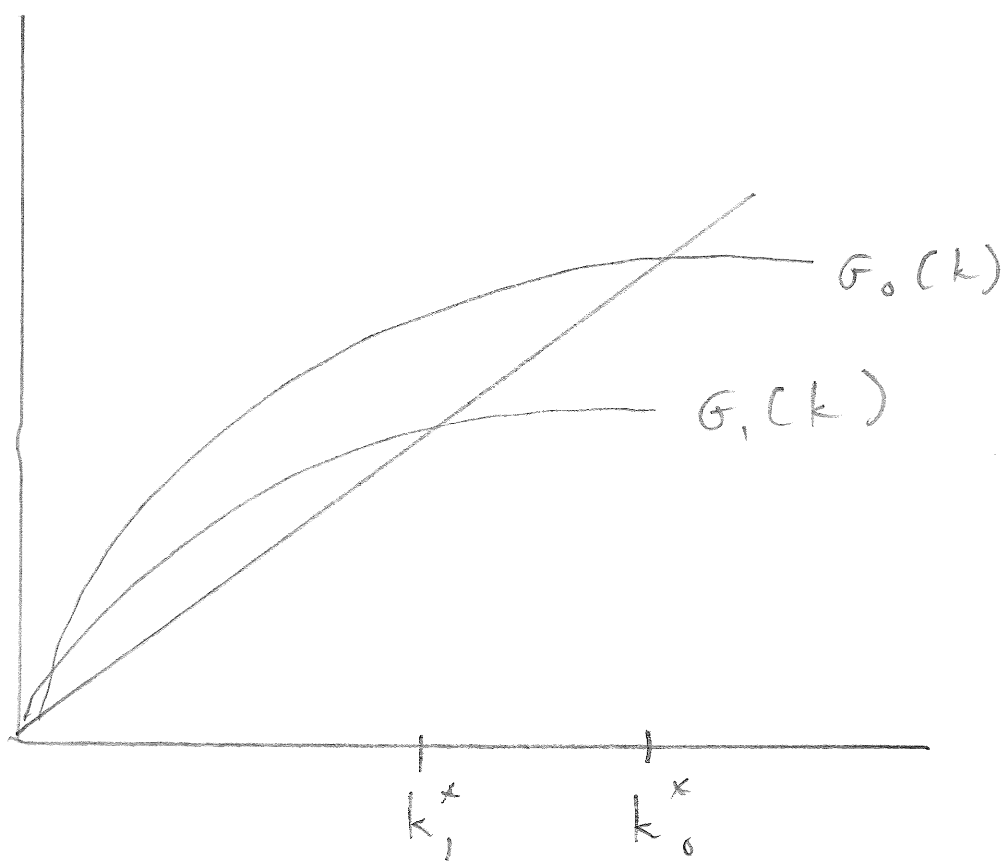
so in

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s\, w_t$$

$$f(k) - f'(k)k = Bk^{\alpha} - B\alpha k^{\alpha-1}k$$
$$= B(1-\alpha)k^{\alpha}$$

so

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} B(1-\alpha)k_t^{\alpha}$$



c) A rise in α

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha}$$

Does $\alpha \uparrow$ increase or decrease k_{t+1} at given k_t?

Look at $\partial k_{t+1} / \partial \alpha$

$$\frac{\partial k_{t+1}}{\partial \alpha} = \frac{1}{(1+n)(1+g)(2+\rho)} \left[-k_t^{\alpha} + (1-\alpha) \underbrace{k_t^{\alpha} \ln k_t}_{\frac{\partial k_t^{\alpha}}{\partial \alpha}} \right]$$

$$= \frac{1}{(1+n)(1+g)(2+\rho} \underbrace{\left[(1-\alpha) \ln k_t - 1 \right]}_{} k_t^{\alpha}$$

sign depends on value of k_t!

$$\frac{\partial k_{t+1}}{\partial \alpha} < 0 \quad \text{if} \quad (1-\alpha) \ln (k_t) - 1 < 0$$

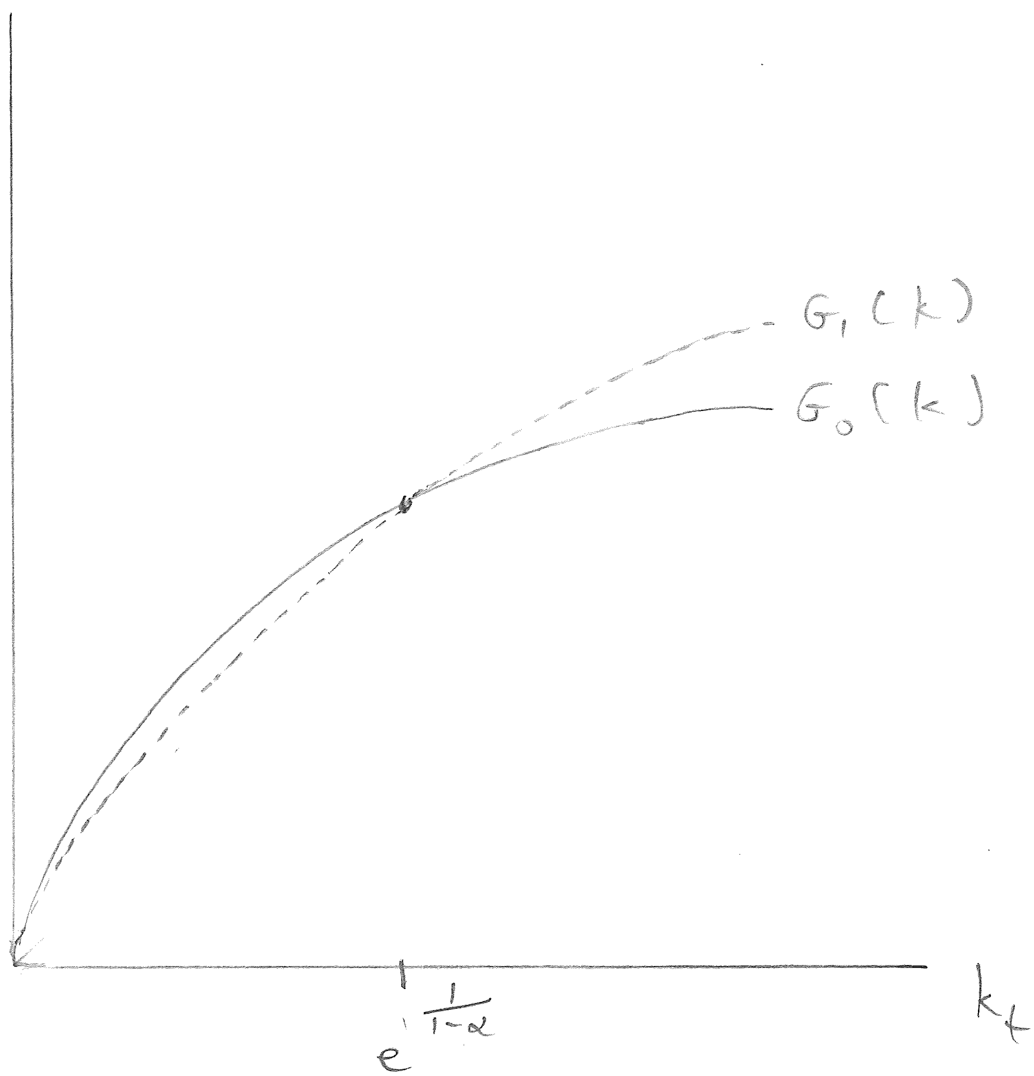
$$\Rightarrow \ln (k_t) < \frac{1}{(1-\alpha)}$$

$$k_t < e^{\frac{1}{1-\alpha}}$$

$$\frac{\partial k_{t+1}}{\partial \alpha} = 0 \quad \text{if} \quad k_t = e^{\frac{1}{1-\alpha}}$$

$$\frac{\partial k_{t+1}}{\partial \alpha} > 0 \quad \text{if} \quad k_t > e^{\frac{1}{1-\alpha}}$$

OLG MODEL



OLG Model

Answer to 2.17). g is zero. Set $A=1$.

a) Tax each young person T, pay each old person $(1+n)T$.

$$\text{Max } U = \ln C_{1t} + \frac{1}{1+\rho}\ln C_{2t}$$

$$\text{s.t. } C_{2t} = (1+n)T + (1+r)(w_t - T - C_{1t}) = (n-r)T + (1+r)(w_t - C_{1t})$$

or

$$(1+r)(w_t - T - C_{1t}) = C_2 - (1+n)T$$

$$w_t = C_{1t} + T + \frac{1}{1+r}C_{2t+1} - \frac{1+n}{1+r}T$$

You can substitute budget-constraint equation for C_{2t} into utility Fn, and set $\partial U/\partial C_{1t} = 0$.

Or you can set up Lagrangian:

$$\mathcal{L} = \ln C_{1t} + \frac{1}{1+\rho}\ln C_{2t} + \lambda\left[w_t - C_{1t} - T - \frac{1}{1+r}C_{2t+1} + \frac{1+n}{1+r}T\right]$$

and set $\partial\mathcal{L}/\partial C_{1t} = 0$

$\partial\mathcal{L}/\partial C_{2t} = 0$

I'll do it the first way.

$$U = \ln(C_{1t}) + \frac{1}{1+\rho}\ln\left((n-r)T + (1+r)(w_t - C_{1t})\right)$$

OLG Model

Answer to 2.17)

a) (Cont.)

$$\frac{\partial U}{\partial C_{1t}} = 0 = \frac{1}{C_{1t}} + \frac{1}{1+\rho}\,\frac{1}{(n-r)T + (1+r)(w_t - C_{1t})}\,(-(1+r))$$

$$\Rightarrow C_{1t} = \frac{1+\rho}{2+\rho} w_t - \frac{(1+\rho)(r-n)}{(2+\rho)(1+r)} T$$

Saving (not s, but sw)

$$= w - T - C_{1t}$$

$$= \left(1 - \frac{1+\rho}{2+\rho}\right) w_t - \left(1 - \frac{(1+\rho)(r-n)}{(2+\rho)(1+r)}\right) T$$

$$= \frac{1}{2+\rho} w_t - \frac{(2+\rho)(1+r) - (1+\rho)(r-n)}{(2+\rho)(1+r)} T$$

multiply out top of fraction, and rearrange

$$= \frac{1}{2+\rho} w_t - \frac{(1+r) + (1+\rho)(1+n)}{(2+\rho)(1+r)} T$$

which means

$$s = \frac{1}{2+\rho} - \frac{(1+r)+(1+\rho)(1+n)}{(2+\rho)(1+r)}\frac{T}{w} = \frac{1}{2+\rho}\left(1 - \underbrace{\frac{(1+r)+(1+\rho)(1+n)}{1+r}\frac{T}{w}}_{\text{between zero and one, for } T<w}\right)$$

Fraction of wage income

Note! introduction of T <u>reduces</u> s

OLG Model

Answer to 2.17)

a) (cont.)

i) How does this affect old equation

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} \underbrace{(1-\alpha)k^{\alpha}}_{w} \qquad (2.60)$$

here, zero

$$k_{t+1} = \frac{K_{t+1}}{L_{t+1}} = \frac{L_t \cdot s_t \cdot w}{L_{t+1}} = \frac{1}{1+n} s_t \cdot w$$

$$= \frac{1}{1+n} \frac{1}{2+\rho} \left(1 - \frac{(1+r)+(1+\rho)(1+n)}{1+r} \frac{T}{w}\right)(1-\alpha)k^{\alpha}$$

or alternatively one can write it as

$$= \frac{1}{1+n} \frac{1}{2+\rho} (1-\alpha)k^{\alpha} - \frac{1}{1+n} \frac{(1+r)+(1+\rho)(1+n)}{(2+\rho)(1+r)} T$$

Either way, you can see introduction of T

<u>reduces</u> k_{t+1} for any given k_t.

OLG Model

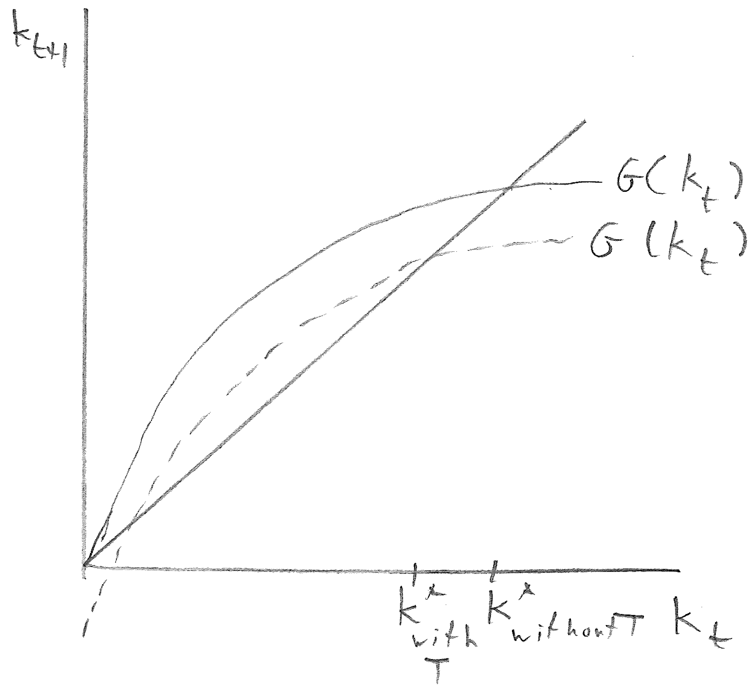
Answer to 2.17

a)

ii) How does it affect k^*?

In a graph:

shifts down $G(\ )$, reduces k^*,



In math:

old $$k^* = \left(\frac{1}{1+n}\frac{1}{2+\rho}(1-\alpha)\right)^{\frac{1}{1-\alpha}} \qquad (2.68)$$

Now, setting $k_{t+1} = k_t = k^*$ gives

$$k^* = \left(\frac{1}{1+n}\frac{1}{2+\rho}(1-\alpha)\left(1 - \frac{(1+r)+(1+\rho)(1+n)}{1+r}\frac{T}{w}\right)\right)^{\frac{1}{1-\alpha}}$$

Smaller than old k^*!

OLG Model

Answer to 2.17

a) iii) Is T a good thing?

Yes, if without T we had $k^* > k^{GR}$ (and T not too big).

No, if without T $k^* \leq k^{GR}$

b) Fully-Funded social security

$$\text{Max } U = \ln C_{1t} + \frac{1}{1+\rho} \ln C_{2t}$$

$$\text{s.t. } C_{2t} = (1+r)T + (1+r)(w_t - T - C_{1t})$$

$$= (1+r)(w_t - C_{1t}) + (1+r)T - (1+r)T$$

$$= (1+r)(w_t - C_{1t})$$

Hey! T drops out of budget constraint!

Choice of C_{1t} is exactly the same as <u>without</u> T.

T reduces "private saving" by exactly same amount, has no effect on economy.