Problem set on production functions

Consider the aggregate production function $Y = \left(K^{\rho} + L^{\rho}\right)^{(1/\rho)}$ *where* $\rho < 1$.

1) Suppose you add a factor *A* to this production function to take account of progress in knowledge, technology etc.

a) Write down an equation that depicts "Harrod-neutral" progress.

$$Y = (K^{\rho} + (AL)^{\rho})^{1/\rho}$$

b) Write down an equation that depicts "capital-augmenting" progress.

$$Y = ((AK)^{\rho} + L^{\rho})^{1/\rho}$$

c) Write down an equation that depicts "Hick-neutral" progress.

$$Y = A(K^{\rho} + L^{\rho})^{1/\rho}$$

d) Are the three types of progress equivalent in this production function, as they are in the Cobb-Douglas production function? *No.*

2) Does this production function have constant returns to scale? Prove your answer is true. *Yes.*

Does $((cK)^{\rho} + (cL)^{\rho})^{1/\rho} = c(K^{\rho} + L^{\rho})^{1/\rho}$?

$$((cK)^{\rho} + (cL)^{\rho})^{1/\rho} = (c^{\rho}K^{\rho} + c^{\rho}L^{\rho})^{1/\rho} = (c^{\rho}(K^{\rho} + L^{\rho}))^{1/\rho}$$

$$= c(K^{\rho} + L^{\rho})^{1/\rho}$$

3) What is the marginal product of capital and marginal product of labor for this production function?

$$MPK = \tfrac{1}{\rho}(K^\rho + L^\rho)^{1/\rho - 1} \rho K^{\rho-1} = (K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} K^{\rho-1}$$

$$MPL = \qquad = (K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} L^{\rho-1}$$

4) What is the elasticity of substitution between capital and labor?

$$\frac{R/K}{W/P} = \frac{MPK}{MPL} = \frac{(K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} \cdot K^{\rho-1}}{(\qquad)^{\frac{1-\rho}{\rho}} L^{\rho-1}}$$

$$= \frac{K^{\rho-1}}{L^{\rho-1}} = (K/L)^{\rho-1}$$

$$\frac{K}{L} = \left(\frac{R/P}{W/P}\right)^{\frac{1}{\rho-1}}$$

Now take derivative.

$$\frac{\partial(K/L)}{\partial(\quad)} = \text{------} = \frac{1}{\rho-1}\left(\frac{R/P}{W/P}\right)^{\frac{2-\rho}{\rho-1}}$$

Now multiply by...

$$\frac{\partial(K/L)}{\partial(\quad)} \frac{(R/P)/(W/L)}{K/L} = \ldots = \frac{1}{\rho-1}$$

5) Let S denote labor's share of income. Derive an equation that gives S as a function of the capital-labor ratio (K/L).

$$S = \frac{MPL \cdot L}{Y} = \frac{(K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} L^{\rho-1} L}{(K^\rho + L^\rho)^{1/\rho}} = \frac{(K^\rho + L^\rho)^{\frac{1-\rho}{\rho}} L^\rho}{(K^\rho + L^\rho)^{1/\rho}}$$

$$= (K^\rho + L^\rho)^{\frac{1-\rho}{\rho} - \frac{1}{\rho}} L^\rho$$

$$= (K^\rho + L^\rho)^{-1} L^\rho = \frac{L^\rho}{K^\rho + L^\rho}$$ Divide top & bottom by L^ρ

$$S = \frac{1}{\frac{K^\rho}{L^\rho} + 1} = \frac{1}{\left(\frac{K}{L}\right)^\rho + 1} = \left[\left(\frac{K}{L}\right)^\rho + 1\right]^{-1}$$

6) Using your answer to 5), what is $\partial S / \partial (K/L)$?

$$\frac{\partial S}{\partial (K/L)} = -1 \cdot \frac{1}{\left[\left(\frac{K}{L}\right)^\rho + 1\right]^2} \cdot \rho = -\frac{\rho}{\left[\left(\frac{K}{L}\right)^\rho + 1\right]^2}$$

If $\rho > 0$, $\frac{\partial S}{\partial (K/L)} < 0$

$\rho < 0$, > 0

7) Suppose $\rho = 1/2$.

a) Using your answer to 4), what is the elasticity of substitution?

$$\varepsilon = \frac{1}{\frac{1}{2}-1} = \frac{1}{-\frac{1}{2}} = -2$$

b) Using your answers to 5) and 6), what happens to labor's share of income as (K/L) grows - increase, decrease or no effect?

Since $\rho > 0$, $\partial S / \partial(K/R) < 0$, so decrease.

8) Suppose $\rho = -1/2$.

a) Using your answer to 4), what is the elasticity of substitution?

$$\varepsilon = \frac{1}{-\frac{1}{2}-1} = \frac{1}{-\frac{3}{2}} = -\frac{2}{3}$$

b) Using your answers to 5) and 6), what happens to labor's share of income as (K/L) grows - increase, decrease or no effect?

Since $\rho < 0$, $\partial S / \partial(K/R) > 0$, so increase.