DSGE Models: introduction

Dynamic stochastic general equilibrium

Most macro models are of this type nowadays.

"General equilibrium" means model is an entire economy (like Solow, OLG, RCK)

"Dynamic" means there are many periods (or continuous time) & variables are related across periods; it's not just a row of single-period models.

"Stochastic" means there are random variables in the model, often denoted with ε's.

Fluctuations in ε's are called "shocks."

Example: $Y_t = K_t^{\alpha}(A_t L_t)^{1-\alpha}$

$$\ln Y_t = \alpha \ln K_t + (1-\alpha)\ln A_t + (1-\alpha)\ln L_t$$

$$\ln A_t = \underbrace{\bar{A} + g_t}_{\text{what we had before}} + \underbrace{\tilde{A}_t}_{\text{new!}}$$

$$\tilde{A}_t = \rho_A \tilde{A}_{t-1} + \varepsilon_{At}$$

this is called an AR(1) process

where $0 < \rho_A < 1$

ε_{At} is "white noise"

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The realized values of the ε's are the exogenous variables in the model.

Solving the model means getting

x_t ← endogenous variable, e.g. L_t, C_t

as a function of ε_t ← vector of all the ε's

ε_{t-1} lagged ε's

ε_{t-2} ...

For most DSGE models you can define a "nonstochastic LRSS." This means, "shut down" the ε's (set them equal to zero) & solve for LRSS as if it were Solow, OLG, RCK.

Why do this? DSGE's hard to solve!

Often, we get an approximate solution using the nonstochastic LRSS & first-order Taylor approximations around it.

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New things we need

1) "Expected utility"

How do people in the model deal with, react to, uncertainty about the future?

e.g. $$U_t = \sum_{\tau=0}^{j} \frac{1}{(1+\rho)^{\tau}} \frac{C_{t+\tau}^{1-\theta}}{1-\theta}$$

How do you choose C_t to max this if you don't know for sure what $C_{t+\tau}$ will be?

We assume people maximize

$$E[U_t] = \frac{C_t^{1-\theta}}{1-\theta} + E_t\left[\sum_{\tau=1}^{j} \frac{1}{(1+\rho)^{\tau}} \frac{C_{t+\tau}^{1-\theta}}{1-\theta}\right]$$

Note! there is a lot of evidence people don't do this.

There are other models (theories) of how people deal with uncertainty.

But....

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New things we need

2) New math tricks

Hamiltonians are for certainty. To solve the problem of maximizing expected utility we'll use "value functions" & "Bellman equations." These will give us first-order conditions & things like TVC's.

3) Graphs of "impulse response functions"

When model is solved you have

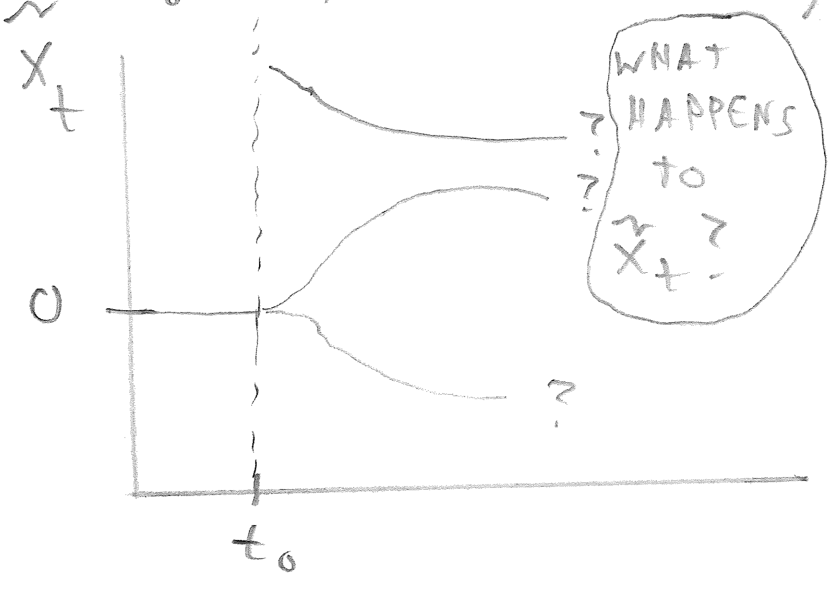
$X_1(\varepsilon_{1t}, \varepsilon_{1t-1}, \ldots \varepsilon_{2t}, \varepsilon_{2t-1}, \ldots$ etc. $)$

$X_2($

etc.

Hard to understand what they're telling you.

To see implications of model, we graph results of "simulations." Say X_t^* ← value of X on nonstochastic BGSS path



$$\tilde{X}_t = \ln X_t - \ln X_t^*$$

At t_0, "shock" hits.

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Example of a model solution

Solow model, no trend growth in A, but shocks. A fluctuates around a fixed trend value

$$A_t = \bar{A} e^{\varepsilon_t}$$ normal, mean zero: "white noise"

Define $y = Y/L$, $k = K/L$

Cobb-Douglas $y = A k^\alpha = \bar{A} e^{\varepsilon_t} k^\alpha$

Define $\tilde{x}_t = \ln x_t - \ln x^*$, means $x_t = x^* e^{\tilde{x}_t}$

hence $\tilde{A}_t = \ln(\bar{A} e^{\varepsilon_t}) - \ln \bar{A} = \varepsilon_t$

Note $\dfrac{y_t}{y^*} = \dfrac{A k_t^{\alpha}}{\bar{A} k^{*\alpha}} = \dfrac{A_t}{\bar{A}} \left(\dfrac{k_t}{k^*}\right)^\alpha$

$$\ln y_t - \ln y^* = \ln A_t - \ln \bar{A} + \alpha(\ln k_t - \ln k^*)$$

$$\tilde{y}_t = \varepsilon_t + \alpha \tilde{k}_t$$

If we choose a variance for ε & draw values of ε_t from the distribution, what would be resulting path for y & k? A "simulated" economy.

ε_t	$\tilde{k}_t$	$\tilde{y}_t$	$\tilde{c}_t$
$\varepsilon_0 = 0$	k^*	y^*	c^*
$\varepsilon_1 \neq 0$	k_1	y_1	c_1
$\varepsilon_2 = 0$	k_2	y_2	c_2
$\varepsilon_3 = 0$	k_3	y_3	c_3

graph each column

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Example... (cont.)

Log-linear approximation

If we assume ε's are small so that variables deviate only a few % from LRSS values (e.g. $\tilde{x}$ about 0.05 at most)

we can use some approximations:

① $e^{\tilde{x}_t} \approx 1 + \tilde{x}_t$

② $e^{\tilde{x}_{1t} + a\tilde{x}_{2t}} \approx 1 + \tilde{x}_{1t} + a\tilde{x}_{2t}$

③ $\tilde{x}_{1t}\tilde{x}_{2t} \approx 0$

hold closely for $\tilde{x} < 0.05$, less closely for $\tilde{x} < 0.10$

Try it!

What we'll do: using these approximations, derive k_{t+1} as fn of k_t & ε_t

Since we know $\tilde{y}_t = \varepsilon_t + \alpha\tilde{k}_t$

$$\tilde{c}_t = (1-s)\tilde{y}_t$$

we can fill out table for simulated economy!

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Example (cont.)

Key equations

Recall $K_{t+1} = (1-\delta) K_t + s Y_t$

$$\frac{K_{t+1}}{L_{t+1}} = \frac{(1-\delta) K_t}{L_{t+1}} + s \frac{Y_t}{L_{t+1}} \qquad L_{t+1} = (1+n) L_t$$

$$= \frac{1}{1+n}\left[(1-\delta)\frac{K_t}{L_t} + s \frac{Y_t}{L_t}\right]$$

$$k_{t+1} = \frac{1}{1+n}\left[(1-\delta) k_t + s\, y_t\right] \qquad y_t = A_t k_t^{\alpha}$$

$$k_{t+1} = \frac{1}{1+n}\left[(1-\delta) k_t + s \bar{A} e^{\varepsilon_t} k_t^{\alpha}\right]$$

Recall $x_t = x^* e^{\tilde{x}_t}$ so

$$k^* e^{\tilde{k}_{t+1}} = \frac{1}{1+n}\left[(1-\delta) k^* e^{\tilde{k}_t} + s \bar{A} e^{\varepsilon_t} (k^* e^{\tilde{k}_t})^{\alpha}\right]$$

$$(1+n) k^* e^{\tilde{k}_{t+1}} = (1-\delta) k^* e^{\tilde{k}_t} + s \bar{A} e^{\varepsilon_t} e^{\alpha \tilde{k}_t} \qquad (1)$$

In LRSS, $\tilde{k} = 0$, $\varepsilon = 0$

$$(1+n) k^* = (1-\delta) k^* + s \bar{A} k^{*\alpha} \qquad (2)$$

$$1 = \frac{1-\delta}{1+n} + \frac{s\bar{A}}{1+n} k^{*\alpha-1}$$

$$1 - \frac{1-\delta}{1+n} = \frac{1+n-(1-\delta)}{1+n} = \frac{n+\delta}{1+n} = \frac{s\bar{A}}{1+n} k^{*\alpha-1} \qquad (3)$$

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Example ... (cont.)

Derivation of $\tilde{k}_{t+1}(\tilde{k}_t, \varepsilon_t)$

In (1), using ①, $e^{\tilde{k}_{t+1}} \approx 1 + \tilde{k}_{t+1}$

$e^{\tilde{k}_t} \approx 1 + \tilde{k}_t \qquad e^{\varepsilon_t} \approx 1 + \varepsilon_t$

using ②, $e^{\alpha \tilde{k}_t} \approx 1 + \alpha \tilde{k}_t$

so (1) becomes approximately:

$$(1+n)k^*(1+\tilde{k}_{t+1}) = (1-\delta)k^*(1+\tilde{k}_t) + s\bar{A}(1+\varepsilon_t)k^{*\alpha}(1+\alpha\tilde{k}_t)$$

$$(1+n)k^* + (1+n)k^*\tilde{k}_{t+1} = (1-\delta)k^* + s\bar{A}k^{*\alpha} + \ldots\ldots$$

From (2), $(1+n)k^* = (1-\delta)k^* + s\bar{A}k^{*\theta}$

subtract from both sides

$$(1+n)k^*\tilde{k}_{t+1} = (1-\delta)k^*\tilde{k}_t + s\bar{A}\varepsilon_t k^{*\alpha} + s\bar{A}k^{*\alpha}\alpha\tilde{k}_t + s\bar{A}k^{*\alpha}\alpha\varepsilon_t\tilde{k}_t$$

using ③, $\varepsilon_t \tilde{k}_t \approx 0$ so above is approximately

$$(1+n)k^*\tilde{k}_{t+1} = (1-\delta)k^*\tilde{k}_t + s\bar{A}\varepsilon_t k^{*\alpha} + s\bar{A}k^{*\alpha}\alpha\tilde{k}_t$$

Look! It's got $\tilde{k}_{t+1}$ & $\tilde{k}_t, \varepsilon_t$!

$$\tilde{k}_{t+1} = \frac{(1-\delta)}{(1+n)}\tilde{k}_t + \underbrace{\frac{s\bar{A}k^{*\alpha-1}}{1+n}}\varepsilon_t + \underbrace{\frac{s\bar{A}k^{*\alpha-1}}{1+n}}\alpha\tilde{k}_t$$

from (3), this is $\frac{n+\delta}{1+n}$

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Example... (cont.)

Derivation of k_{t+1} (cont.)

$$\tilde{k}_{t+1} = B\tilde{k}_t + D\varepsilon_t$$

where $B = \frac{1+\alpha n - \delta(1-\alpha)}{1+n} < 1$

$$D = \frac{n+\delta}{1+n}$$

We know $\alpha \approx 1/3$.

If we also pick values for n, δ & s ("calibrate") we can fill out rest of table.

Note it's <u>important</u> that $B < 1$.

otherwise, economy blows up: LRSS not stable.