OVERLAPPING GENERATIONS (OLG) MODEL

DIAMOND (1965)

Motivation

1) People are born, get old & can't work, then die

2) "Life-cycle saving" (Modigliani & Brumberg, 1954)
People save while young to be able to consume when they're old & can't work.

Assumptions

Production

As in Solow model, For simplicity forget depreciation

$Y_t = F(K_t, A_t L_t)$ ← CRS

A grows at rate g

$A_t = (1+g) A_{t-1}$

L grows at rate n

$L_t = (1+n) L_{t-1}$

} Discrete time

$$\frac{Y_t}{A_t L_t} = F\left(\frac{K_t}{A_t L_t}, 1\right)$$ ← implication of CRS

$y_t = f(k_t) \qquad MPK = f'(k_t)$

Specific example: Cobb-Douglas $y = k^\alpha$

OLG Model

Assumptions

Production (cont.)

As in Solow model, in a LRSS

$$sf(k) = (n+g)k$$

Golden Rule:

The LRSS that throws off the most output as consumption

$$c = f(k) - sf(k)$$

in any LRSS

$$c = f(k) - (n+g)k$$

To max LRSS c,

$$\frac{\partial c}{\partial k} = f'(k) - (n+g)$$

$$\Rightarrow f'(k) = n+g$$

Saving as a share of national income Y

The corresponding savings rate:

In a LRSS

$$sf(k) = (n+g)k \Rightarrow s = \frac{(n+g)k}{f(k)}$$

At G.R.,

$$n+g = f'(k)$$

$$\Rightarrow s = \frac{f'(k)k}{f(k)} = \frac{f'(k)\frac{K}{AL}}{Y/AL} = \frac{f'(k)K}{Y}$$

Capital's share of income, if capital is paid it's MP

OLG Model

Assumptions (cont.)

Overlapping generations of people

An individual lives two periods

In first period of life, provide one unit of labor

In second period, can't work.

At time t,

L_t young working people

$L_{t-1} = \frac{1}{1+n}$ old retired people

C_{1t} Consumption of a young person in period t

C_{2t} Consumption of an old person in period t

Life history of generation born in period t:

	t	$t+1$
Labor	1	0
Consumption	C_{1t}	C_{2t+1}

OLG Model

Assumptions (cont.)

Market structure etc.

Firms maximize profit
Labor & capital markets competitive
Capital owned by people

Hence, for each unit of capital I buy in period t, by giving up one unit of consumption

I receive $(1+r_{t+1})$ units of stuff in $t+1$

MPK at time $t+1$ $\quad f'(k_{t+1})$

In return for labor in period t, I receive

$w_t A_t$ $\quad$ where $w_t = f(k_t) - k_t f'(k_t)$

real wage paid to an efficiency-unit of labor

From "economic profit is zero"

$$0 = \underbrace{Y}_{\text{sales revenue}} - \underbrace{wAL - rK}_{\text{total cost}}$$

$$wAL = Y - rK$$

$$w = \frac{Y}{AL} - r\frac{K}{AL} = f(k) - f'(k)k$$

OLG Model

Assumptions (cont.)

Preferences

An individual born in period t maximizes

felicity

$$U = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$

lifetime utility

where $u'(c) > 0$, $u''(c) < 0$

diminishing marginal utility

Specific example:

"Constant-relative-risk-aversion" utility

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \quad \text{where } \theta > 0$$

see $u'(c) = c^{-\theta} = \frac{1}{c^{\theta}} > 0$

$$u''(c) = -\theta c^{-\theta-1} = -\theta \frac{1}{c^{\theta+1}} < 0$$

Even more specific example:

"Log utility"

$$u(c) = \ln(c)$$

see $u'(c) = \frac{1}{c} > 0$

$$u''(c) = -\frac{1}{c^2} < 0$$

Equivalent to CRRA as $\theta \to 1$

OLG Model

What would a social planner do?

Tricky: how to weight utility across generations?
But he *certainly* wouldn't choose a LRSS
with $f'(k) < n+g$

$$s > \frac{f'(k)\,k}{f(k)}$$

$$k > k^{GR}$$

Suppose we're in LRSS with $k^* > k^{GR}$.
He could have *current* population eat up $(k^* - k^{GR})$,
which gives higher u to them,
and maintain $k = k^{GR}$ hereafter,
which gives higher u to all future generations.

Makes everyone better off: Pareto improvement

Any LRSS with $k > k^{GR}$ is *not* Pareto optimal.

What will the competitive economy generate?

OLG Model (cont.)

What an individual chooses

$$\text{Max } U = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$

where $c_{2t+1} = (1+r_{t+1})(A_t w_t - c_{1t})$

savings, purchase of capital

or $0 = A_t w_t - c_{1t} - \frac{1}{1+r_{t+1}} c_{2t+1}$

Resulting capital stock next period

$$K_{t+1} = L_t \cdot (A_t w_t - c_{1t})$$

or first-period income

If we define s_t to be fraction of labor income that young person saves,

$$s_t = \frac{A_t w_t - c_{1t}}{A_t w_t} = 1 - \frac{c_{1t}}{A_t w_t}$$

then $c_{1t} = (1-s_t) A_t w_t$

$$K_{t+1} = L_t s_t A_t w_t$$

divide both sides by $A_{t+1} L_{t+1}$

$$\frac{K_{t+1}}{A_{t+1} L_{t+1}} = \frac{L_t}{L_{t+1}} \frac{A_t}{A_{t+1}} s_t w_t$$

$$k_{t+1} = \frac{1}{1+n} \frac{1}{1+g} s_t w_t$$

OLG Model

What an individual chooses

$$\text{Max } u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1})$$

where $c_{2t+1} = (1+r_{t+1})(A_t w_t - c_{1t})$ ← budget constraint

Two ways to solve it.

1) Substitute budget constraint into utility fn, take F.O.C.

2) Set up & solve Lagrangian

1) $$U = u(c_{1t}) + \frac{1}{1+\rho} u\big(\overbrace{(1+r_{t+1})(A_t w_t - c_{1t})}^{c_{2t+1}}\big)$$

$$\frac{\partial U}{\partial c_{1t}} = 0 = u'(c_{1t}) + \frac{1}{1+\rho} u'\big((1+r_{t+1})(A_t w_t - c_{1t})\big)(1+r_{t+1})(-1)$$

$$0 = u'(c_{1t}) - \frac{1+r_{t+1}}{1+\rho} u'\big((1+r_{t+1})(A_t w_t - c_{1t})\big)$$

Note

— This implies $$\frac{u'(c_{1t})}{u'(c_{2t+1})} = \frac{1+r_{t+1}}{1+\rho}$$

— It defines c_{1t}, hence s_t and k_{t+1}

as a function of w_t, r_{t+1} ← $r_{t+1} = f'(k_{t+1}$

OLG Model

What an individual chooses

2) Set up & solve Lagrangian

$$\mathcal{L} = u(c_{1t}) + \frac{1}{1+\rho} u(c_{2t+1}) + \lambda \left[A_t w_t - c_{1t} - \frac{1}{1+r_{t+1}} c_{2t+1} \right]$$

$$\frac{\partial \mathcal{L}}{\partial c_{1t}} = 0 = u'(c_{1t}) + \lambda(-1)$$

implies $u'(c_{1t}) = \lambda$

$$\frac{\partial \mathcal{L}}{\partial c_{2t}} = \frac{1}{1+\rho} u'(c_{2t+1}) + \lambda \frac{1}{1+r_{t+1}}(-1)$$

implies $\dfrac{1+r_{t+1}}{1+\rho} u'(c_{2t+1}) = \lambda$

note this implies $\dfrac{u'(c_{1t})}{u'(c_{2t+1})} = \dfrac{1+r_{t+1}}{1+\rho}$

and using budget constraint (which gives $c_{2t+1} = \ldots c_{1t}$)

it defines c_{1t} etc. as fns of w_t, r_{t+1}.

OLG Model

What an individual chooses

Specific example: CRRA utility

$$u(c) = \frac{c^{1-\theta}}{1-\theta} \qquad u'(c) = \frac{1}{c^{\theta}}$$

gives
$$\frac{c_{2t+1}^{\theta}}{c_{1t}^{\theta}} = \frac{1+r_{t+1}}{1+\rho} \qquad (2.48)$$

$$\frac{c_{2t+1}}{c_{1t}} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} \qquad (2.49)$$

"Euler equation"; relationship between c_t & c_{t+1} if utility maximized

$$c_{2t+1} = \left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} c_{1t}$$

substitute into budget constraint

$$c_{1t} + \frac{1}{1+r_{t+1}}\left(\frac{1+r_{t+1}}{1+\rho}\right)^{1/\theta} c_{1t} = A_t w_t$$

$$c_{1t} + (1+r_{t+1})^{1/\theta - 1}(1+\rho)^{-1/\theta} c_{1t} = A_t w_t$$

$$\frac{1}{\theta} - 1 = \frac{1}{\theta} - \frac{\theta}{\theta} = \frac{1-\theta}{\theta} \quad \text{so}$$

$$c_{1t} + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta}} c_{1t} = A_t w_t \qquad (2.54)$$

$$\left(1 + \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta}}\right) c_{1t} = A_t w_t$$

OLG Model

What an individual chooses

Specific example (cont.)

$$\left(\frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta}} + \ldots\ldots\right) C_{1t} = A_t w_t$$

$$C_{1t} = \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} A_t w_t \quad (2.55)$$

and $$s_t = 1 - \frac{C_{1t}}{A_t w_t} = 1 - \frac{(1+\rho)^{1/\theta}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}}$$

$$= \frac{(1+r_{t+1})^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta} + (1+r_{t+1})^{\frac{1-\theta}{\theta}}} \quad (2.56)$$

Consumption & saving are a fraction of first-period income, fraction depends on r_{t+1} only (not w_t)

$$s_t = s(r_{t+1})$$

OLG Model

What an individual chooses

Even more specific example: log utility

$$u(c) = \ln(c) \qquad u'(c) = \frac{1}{c}$$

gives $\dfrac{c_{2t+1}}{c_{1t}} = \dfrac{1+r_{t+1}}{1+\rho}$ Euler equation with log utility

$$c_{2t+1} = \frac{1+r_{t+1}}{1+\rho} c_{1t}$$

substitute into budget constraint

$$c_{1t} + \frac{1}{1+\rho} c_{1t} = A_t w_t$$

$$\left(1 + \frac{1}{1+\rho}\right) c_{1t} = A_t w_t$$

$$\left(\frac{1+\rho}{1+\rho} + \frac{1}{1+\rho}\right) c_{1t} = A_t w_t$$

$$c_{1t} = \frac{1+\rho}{2+\rho} A_t w_t$$

and $s_t = 1 - \dfrac{c_{1t}}{A_t w_t} = 1 - \dfrac{1+\rho}{2+\rho} = \dfrac{1}{2+\rho}$

$$s = \frac{1}{2+\rho}$$

Consumption & saving are a constant fraction of first-period income, don't depend on r_{t+1}.

OLG MODEL

What an individual chooses

Aside: how does r_{t+1} affect saving?

General: $s(r, w) \longrightarrow$ Ambiguous effect

CRRA: $$s_t = \frac{(1+r)^{\frac{1-\theta}{\theta}}}{(1+\rho)^{1/\theta} + (1+r)^{\frac{1-\theta}{\theta}}} \longrightarrow \text{Ambiguous effect}$$

Log utility: $$s_t = \frac{1}{2+\rho} \longrightarrow \text{No effect}$$

How do we know effect is ambiguous if CRRA?

$$\frac{\partial s}{\partial r} = \frac{(1+\rho)^{1/\theta}}{\left((1+\rho)^{1/\theta} + (1+r)^{\frac{1-\theta}{\theta}}\right)^2} \cdot \underbrace{\frac{\partial\left((1+r)^{\frac{1-\theta}{\theta}}\right)}{\partial r}}$$

sign of $\partial s/\partial r$ depends on sign of this

$$\frac{\partial(\quad)}{\partial r} = \frac{1-\theta}{\theta}(1+r)^{\frac{1-2\theta}{\theta}}$$

$$\frac{1-\theta}{\theta} - \frac{\theta}{\theta}$$

if $\theta < 1$, $\frac{\partial(\ )}{\partial r}$ hence $\frac{\partial s}{\partial r} > 0$

$\theta > 1$, ------ $\frac{\partial s}{\partial r} < 0$

$\theta = 1$, ------- $\frac{\partial s}{\partial r} = 0$ log utility

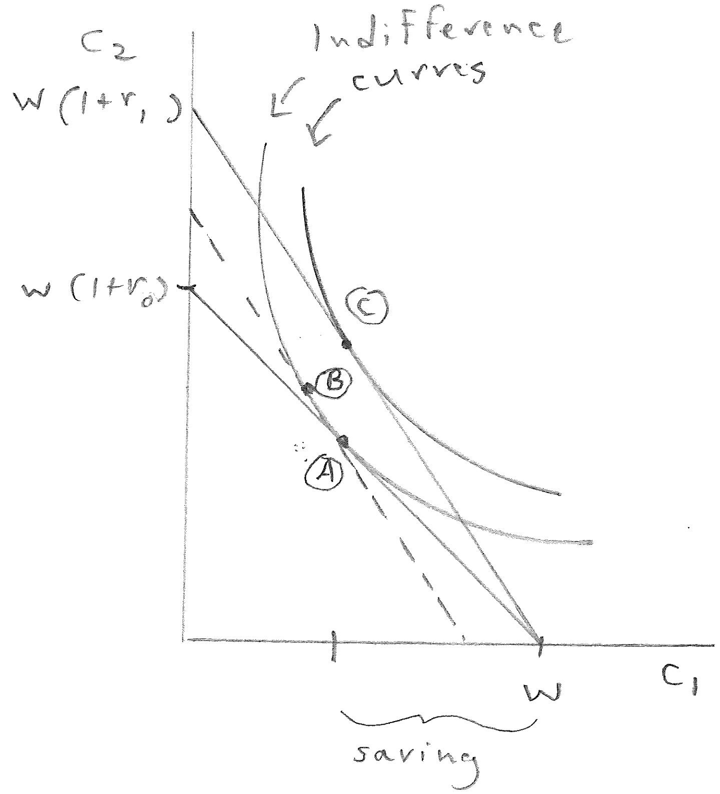
What's going on here?

OLG Model

What an individual chooses

Aside (cont.)

Income vs. substn. effect of Δr on s



When r rises from r_1 to r_2, individual changes preferred combination of c_1 and c_2 from pt. A to pt. C

Saving is $(w - c_1)$

$$s = \frac{(w - c_1)}{w}$$

"Total effect" from A to C is combination of

"Substitution effect" from A to B: reduce c_1, save more

"Income effect" from B to C: raise c_1, save less.

Situation drawn is log utility: substn. effect exactly counters income effect, no change in saving.

If $\theta \neq 1$, indifference curves would be a different shape.

OLG Model

How individual's choice determines k_{t+1}

Recall

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s_t \, w_t$$

$s(r,w)$

$f(k_t) - k_t f'(k_t)$

$f'(k_{t+1})$

(2.59)

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \, s\Big(\underbrace{f'(k_{t+1})}_{r}, \underbrace{f(k_t) - k_t f'(k_t)}_{w}\Big) \underbrace{\big(f(k_t) - k_t f'(k_t)\big)}_{w}$$

Implicitly defines k_{t+1} as a function of k_t.

But you need to specify $f(k)$ and $s(w, r)$.

For log utility $\rightarrow$ $s = \frac{1}{2+\rho}$

Cobb-Douglas $\rightarrow$ $f(k) = k^\alpha$, $f'(k) = \alpha k^{\alpha - 1}$

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} \left(k_t^\alpha - \alpha k_t^{\alpha-1} k_t\right)$$

$$= \text{- - - - - - - -} \; (1-\alpha) k_t^\alpha \qquad (2.61)$$

To be general, define $k_{t+1} = G(k_t)$ "Equation of motion" for k

$k_{t+2} = G(k_{t+1})$

Romer does not name the function $G(k)$

OLG Model

How individual's choice..

Log utility + Cobb-Douglas

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha} \qquad (2.61)$$

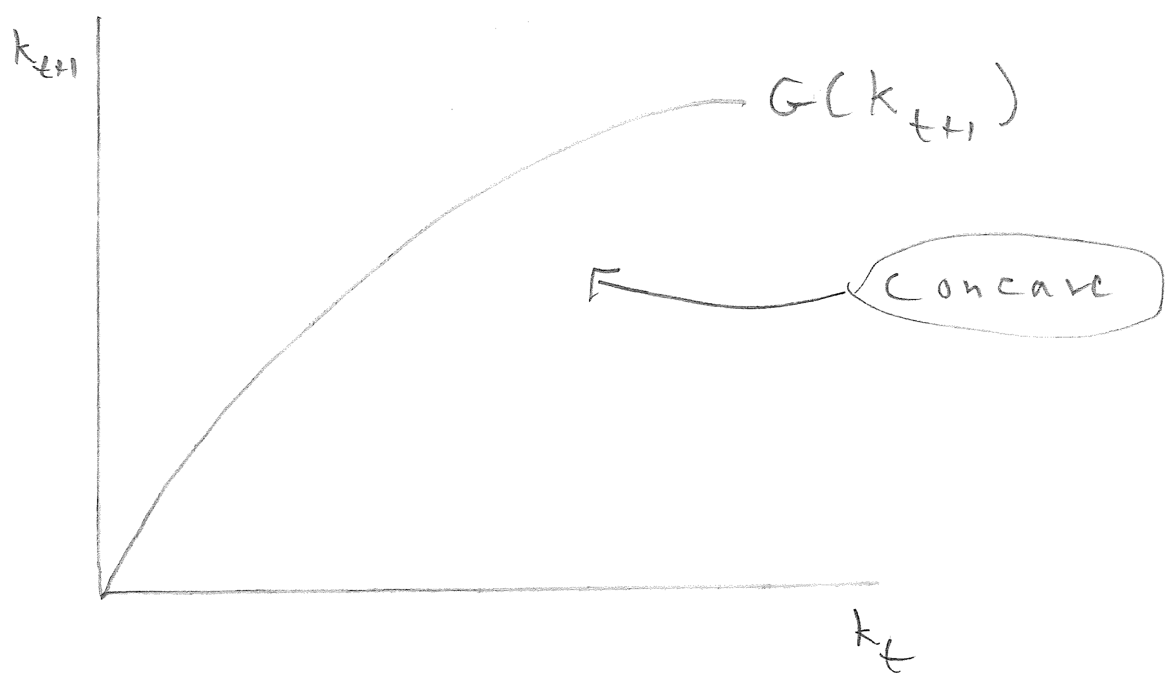
If we graph this, what does it look like?

from $k_t^{\alpha-1}$

$$\frac{\partial k_{t+1}}{\partial k_t} = \text{- - - - - - - - - -} \alpha \frac{1}{k_t^{1-\alpha}} > 0$$

$$\frac{\partial^2 k_{t+1}}{\partial k_t^2} = \text{- - - - - - - - -} \underbrace{\alpha(\alpha-1)}_{\text{less than zero}} \frac{1}{k_t^{2-\alpha}} < 0$$

so it looks like



OLG Model

How individual's choice...

Is $G(k_t)$ generally concave?

No.

$$k_{t+1} = \frac{1}{(1+n)(1+g)} s(f'(k_{t+1})) \frac{f(k_t) - k_t f'(k_t)}{f(k_t)} f(k_t) \qquad (2.68)$$

this is (2.59) rearranged

labor's share of income

For reasonable prodn. & utility fns,

$\dfrac{\partial k_{t+1}}{\partial k_t} > 0$ but $\dfrac{\partial^2 k_{t+1}}{\partial^2 k_t}$ not always < 0

($G(k_t)$ is upward-sloping but not concave)

or even

$\dfrac{\partial k_{t+1}}{\partial k_t} < 0$ sometimes

($G(k_t)$ is backward-sloping for some k_t)

How? One or both of

1) $\dfrac{\partial [\text{labor's share}]}{\partial k_t} > 0$ [not true in Cobb-Douglas, but can be true even if $f'(k) > 0$, $f''(k) < 0$

2) $s'(r) < 0$ [not true for log util, but true for CRRA with $\theta > 1$

$k_t \to k_{t+1}\uparrow \to r\downarrow \to s\uparrow \to k_{t+1}\uparrow$ fights concavity

OLG Model

Recall questions:

1) Is there a LRSS? Stable? More than one?
2) Optimal?

Is there a LRSS?

LRSS means there is a k^* for which

$$k^* = k_t = k_{t+1} = \dots$$

means $k^* = G(k^*)$

Stable means if $k_t < k^*$,

$$k_t < k_{t+1} \leq k^*$$

k will move toward k^*, and not overshoot

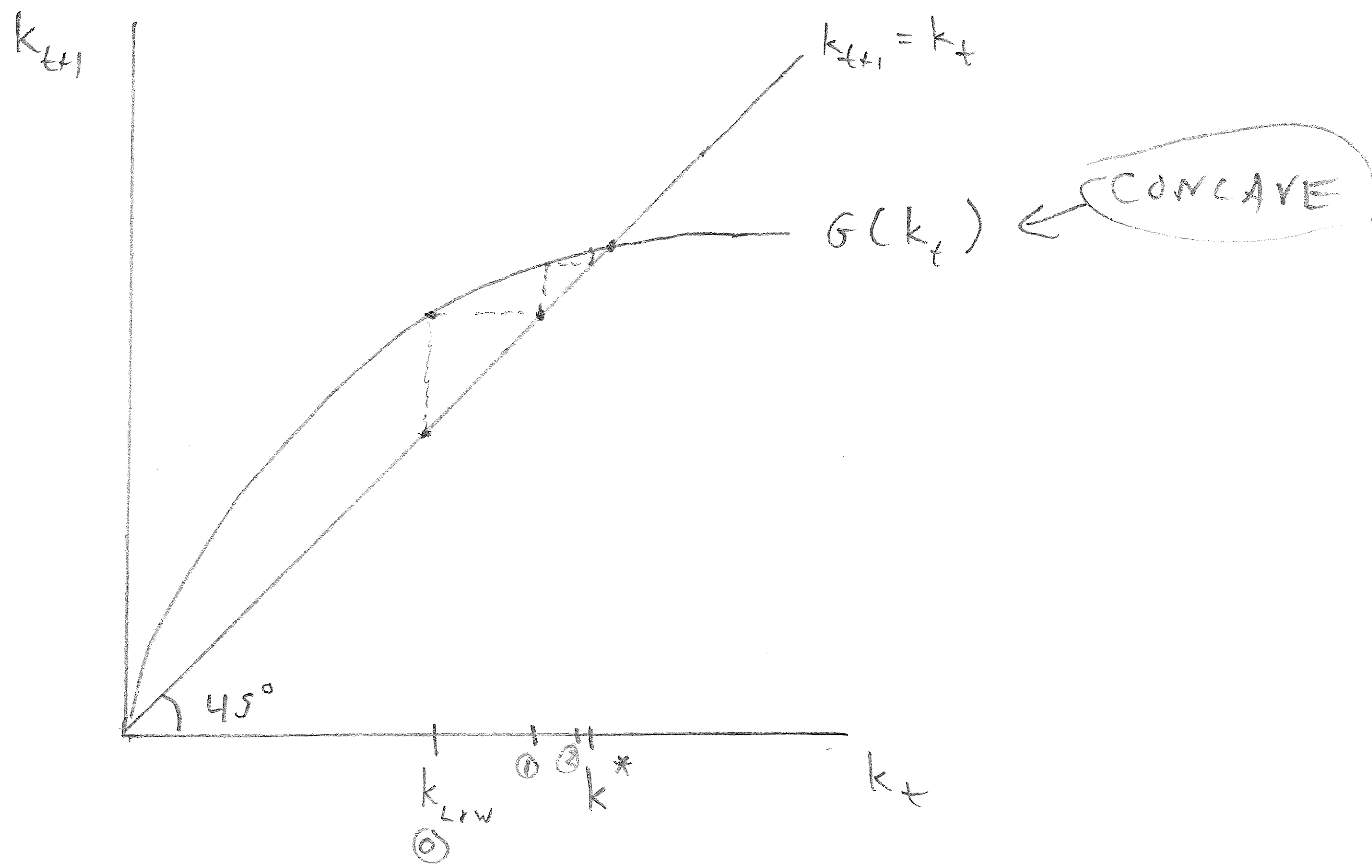
and if $k_t > k^*$, etc.

Only one LRSS means there is only one value of k for which $k^* = G(k)$

Use a graph to talk about LRSS.

OLG Model

Graph to describe LRSS or not



This economy has a stable LRSS

— if $k_t = k^*$, $k_{t+1} = k^*$

— if $k = k_{Low} < k^*$, $k_t < k_{t+1} \leq k^*$

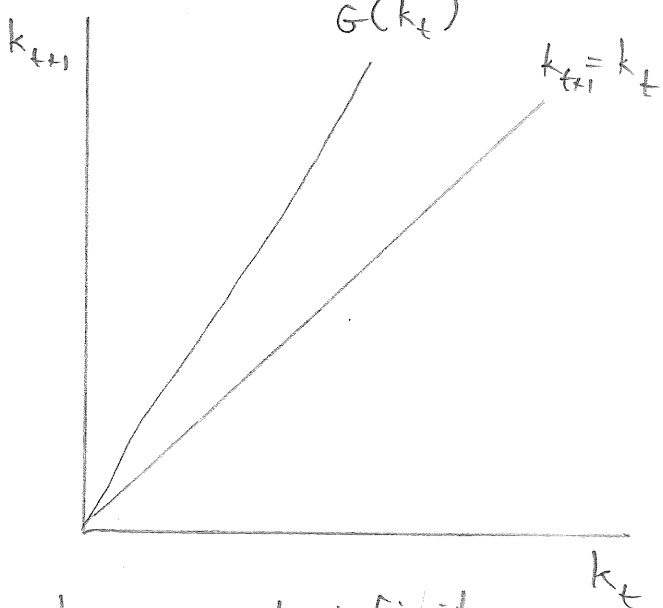
Say $k_0 = k_{Low}$, then $k_1 = 1$, $k_2 = 2$, etc.

— if $k > k^*$, etc.

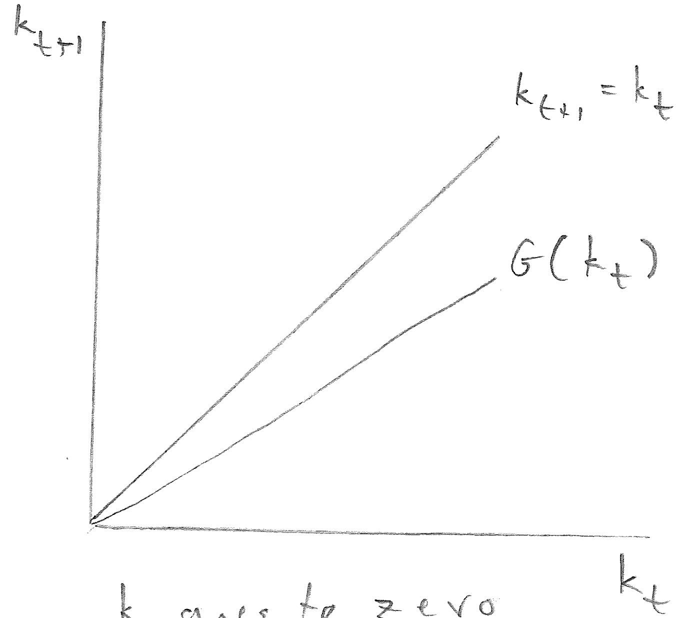
Example: log utility and Cobb-Douglas

OLG Model

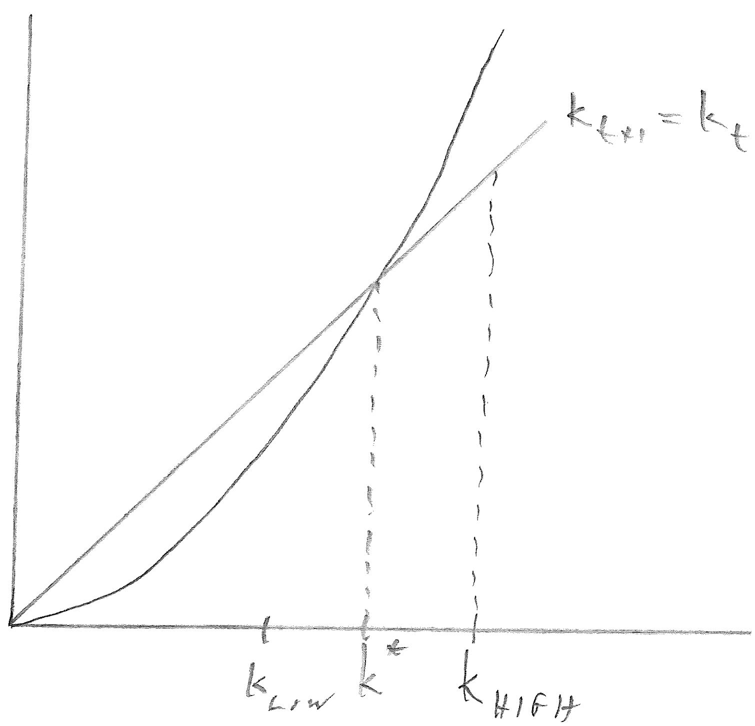
Graph to describe LRSS or not (cont.)



k goes to infinity



k goes to zero

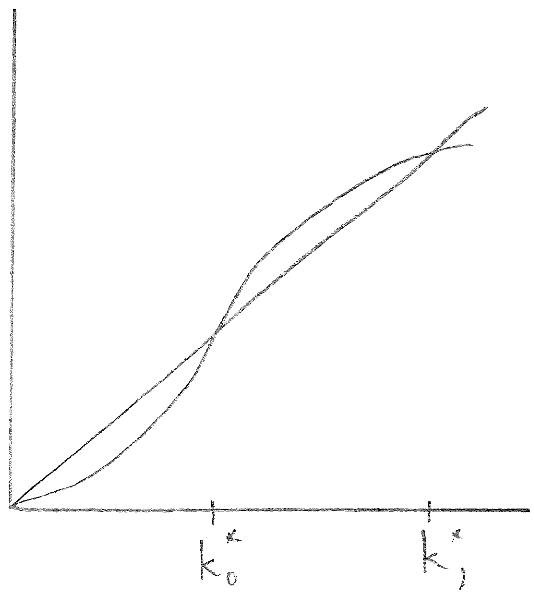


An unstable LRSS: if $k > k^*$, k goes to ∞

if $k < k^*$, k goes to zero

OLG Model

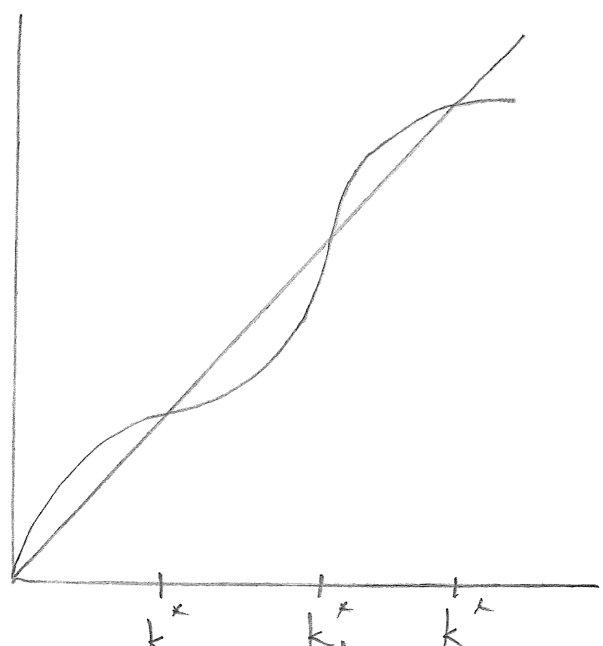
Graph to describe... (cont.)



Two LRSS,

k_0^* is unstable

k_1^* is stable, unless economy is shocked to a pt. below k_1^*



Three LRSS's,

k_0^* and k_2^* are stable,

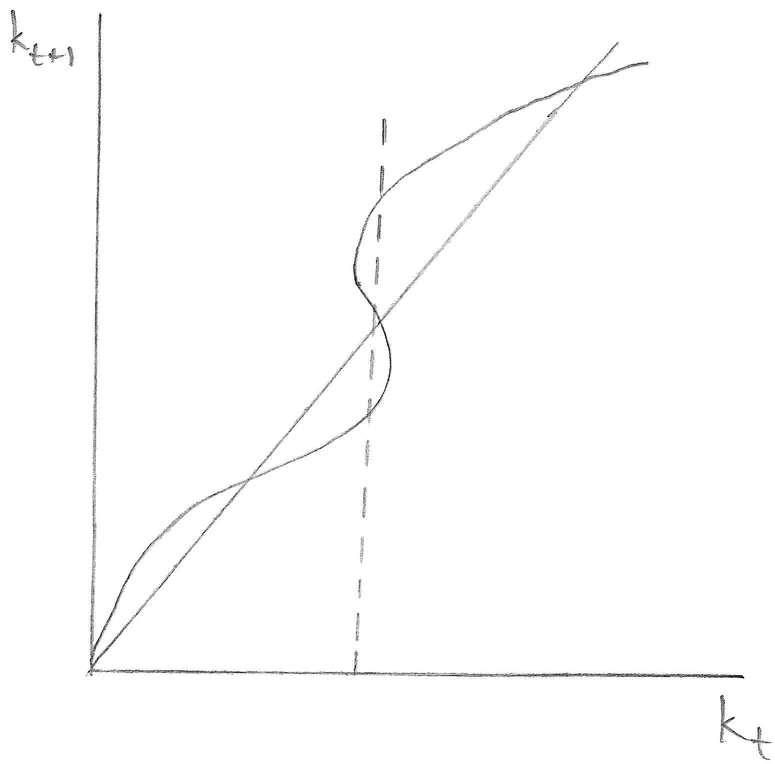
Economy will go to k_0^* if initial $k < k_1^*$,

go to k_1^* if initial $k > k_1^*$.

LRSS is determined by model <u>given</u> initial k.

OLG Model

Graph to describe (cont.)



For some values of k_t, model is consistent with more than one value of k_{t+1}.

k_{t+1} not "uniquely determined"

"indeterminate"

You might go on to assume k_{t+1} determined by

"sunspots" ← something that doesn't affect anything in the model like produ. or prefs., but chooses outcome

"self-fulfilling prophecies"

If people in economy expect big k_{t+1}, it happens

" " " " " small k_{t+1}, it happens

OLG Model

Speed of convergence to k^*

Say we start at time $t=0$ with $k_0 \neq k^*$.

E.g. $k_0 < k^*$.

If $G(k_t)$ is such that economy converges to k^*, what is timing? How much ground do we make up in one period, two periods, ...?

We can approximate it using idea of "Taylor approximation":
For a function $Y = F(X)$, and two values X_0 & $\hat{X}$,

$$F(X_0) \approx F(\hat{X}) + F'(\hat{X})(X_0 - \hat{X}) + F''(\hat{X})(X_0 - \hat{X})^2 + \ldots$$

"First-order approximation": first-derivative only.

This is useful if it's easy to figure out $F(\hat{X})$, $F'(\hat{X})$, hard to figure out $F(X_0)$.

Here, we want to know $(k^* - k_0)$, $(k^* - k_1)$, $(k^* - k_2)$, ...

Our approximation: $(k^* - k_1) = \lambda (k^* - k_0)$

$$(k^* - k_2) = \lambda (k^* - k_1) = \lambda \cdot \lambda (k^* - k_0) = \lambda^2 (k^* - k_0)$$

$$(k^* - k_t) = \lambda^t (k^* - k_0)$$

See: if $0 < \lambda < 1$, convergence to k^*.

"Taylor approximation" gives λ, as long as $G(k_t)$ has right shape. Example: Cobb-Douglas, $\ln(C)$.

OLG Model

Speed of convergence (cont.)

Do a first-order Taylor approximation of $G(k)$ around k^* to get $G(k_0)$:

$$k_1 \approx G(k^*) + G'(k^*)(k_0 - k^*)$$

$G(k^*) = k^*$ (that's definition of k^*!) so

$$k_1 \approx k^* + G'(k^*)(k_0 - k^*)$$

$$k_1 - k^* \approx \underbrace{G'(k^*)}(k_0 - k^*)$$

$$k_1 - k^* \approx \lambda \, (k_0 - k^*)$$ so

$$(k^* - k_1) \simeq G'(k^*)(k_0 - k^*)$$

$$(k^* - k_t) \simeq (G'(k^*))^t (k_0 - k^*)$$

This maps out timing of convergence to k^* (approximately).

Now, let's do an example with Cobb-Douglas & log utility.

OLG Model

Speed of convergence (cont.)

Cobb-Douglas & log utility

Recall $k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho}(1-\alpha) k_t^{\alpha}$ G(k) (2.62)

To get k^*, set $k_{t+1} = k_t$, solve:

$$k^* = \left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{1/1-\alpha} \quad (2.63)$$

From (2.62), $\frac{\partial k_{t+1}}{\partial k_t} = \alpha \frac{1-\alpha}{(1+n)(1+g)(2+\rho)} k_t^{\alpha-1}$

For k_t, put in k^* from (2.63) to get:

$$\lambda = \left.\frac{\partial k_{t+1}}{\partial k_t}\right|_{k_t=k^*} = \alpha \frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\left[\frac{1-\alpha}{(1+n)(1+g)(2+\rho)}\right]^{\frac{\alpha-1}{1-\alpha}} \quad (2.67)$$

$= \alpha \Leftarrow$ capital's share of income $\approx 1/3$

So $k_1 = 1/3 (k_0 - k^*)$

in one period, k grew 2/3 of the distance to k^*

$k_2 = (1/3)^2 (k_0 - k^*) = 1/9 (k_0 - k^*)$

after two periods, k is almost at k^*

etc.

OLG Model

Is LRSS optimal?

Assuming there is a LRSS, is it Pareto optimal?

Same as asking, can $k^* > k^{GR}$?

Answer: possibly not P.O., because k^* can be too big!

Example: log utility & Cobb-Douglas, $g = 0$

Recall

$$k_{t+1} = \frac{1}{(1+n)(1+g)} \frac{1}{2+\rho} (1-\alpha) k_t^{\alpha} \qquad (2.61)$$

To find k^*, set $k_{t+1} = k_t = k^*$ and solve.

$$k^* = \left[\frac{1}{1+n} \frac{1}{2+\rho} (1-\alpha)\right]^{\frac{1}{1-\alpha}} \qquad (2.63)$$

$$f'(k^*) = \alpha k^{*\,\alpha-1} = \frac{\alpha}{1-\alpha}(1+n)(2+\rho) \qquad (2.69)$$

Recall $f'(k^{GR}) = n \leftarrow$ For $g = 0$

So for range of values for α and ρ,

$$f'(k^*) < f'(k^{GR})$$

$$k^* > k^{GR}$$

OLG Model
Is LRSS optimal?
What would social planner do?

Suppose economy is in LRSS with $k^* > k^{GR}$

Social planner would create Pareto improvement by

1) Force current young to save less. By itself, this would reduce $C_{2\,t+1}$

2) In $t+1$, transfer income from $t+1$ young to $t+1$ old (young in t)

and so on forever.

Note this would not be possible if economy had a final period, because you couldn't transfer income to final-period old.

Social security

Govt. policy that taxes young, pays old.

If each young guy pays T, each old gets $(1+n)T$

Reduces saving at given r.

If k^* without policy is $> k^{GR}$, can be Pareto improvement! (unless you overdo it).