| ANSWERS |

Economics 613, Macroeconomics, Christopher Hanes Fall 2022
First midterm examination.

Look over the entire examination before you begin.

**1)** Consider the Cobb-Douglas production function $Y = K^{\alpha}(AL)^{1-\alpha}$ *where* $0<\alpha<1$.
**a)** Show that for this production function the elasticity of substitution between capital and labor is equal to negative one.
**b)** Assume the economy is closed so that total real income is equal to output *Y*, and markets are perfectly competitive so that the real wage is equal to the marginal product of labor. Show that under these conditions the share of national income going to labor is fixed (is not affected by changes in *A*, *K* or *L*).

**2)** Consider an economy that can be described by the Solow model with a fixed savings rate *s,* a rate of population growth *n*, a rate of "technological improvement" $g$, and a rate of depreciation $\delta$.

**a)** Suppose the rate of population growth increases from $n_0$ to a higher value $n_1$.
**i)** Depict this event on a graph with *k* (capital per efficiency-unit of labor) on the horizontal axis and *y* (output per efficiency-unit of labor) on the vertical axis. Referring to the graph, explain in words what happens to *k* and *y*.
**ii)** Depict this event in another graph, one that has time on the horizontal axis and, on the vertical axis, the *log* of output *per worker*. Use $\hat{t}$ to denote the point in time at which the rate of population growth increases.

**b)** Suppose there is an increase in the rate of technological improvement *g* from $g_0$ to a higher value $g_1$.
**i)** Depict this event on a graph with *k* (capital per efficiency-unit of labor) on the horizontal axis and *y* (output per efficiency-unit of labor) on the vertical axis. Referring to the graph, explain in words what happens to *k*, *y* and *c*.
**ii)** Depict this event in another graph, one that has time on the horizontal axis and, on the vertical axis, the *log* of output *per worker*. Use $\hat{t}$ to denote the point in time at which the rate of growth of *A* increases.

**3)** Again consider an economy that can be described by the Solow model with a fixed savings rate *s,* a rate of population growth *n*, a rate of "technological improvement" $g$, and a rate of depreciation $\delta$. The aggregate production function is $Y = K^{\alpha}(AL)^{1-\alpha}$ *where* $0<\alpha<1$.
**a)** Taking the savings rate *s* as given, derive expressions showing the long-run steady-state values of:
- *k* (capital per efficiency-unit of labor).
- *w* (real wage per efficiency-unit of labor)
- the real wage *per worker*.

as a function of *s* and exogenous parameters.
b) Now consider the "Golden rule" long-run steady-state. Derive the Golden rule values of:
- *k* (capital per efficiency-unit of labor).
- *w* (real wage per efficiency-unit of labor)
- the real wage per worker.

as a function of exogenous parameters.

**4)** Consider the Diamond OLG model.
**a)** Make a graph where the horizontal axis is capital per efficiency unit of labor at time *t,* and the vertical axis is capital per efficient unit of labor at time *t+1*. On the graph draw a line depicting the function $k_{t+1}=G(k_t)$ for an economy that has just one LRSS, and that LRSS is *unstable*.
**b)** Suppose an economy like the one you depicted in a) started out in its LRSS. Then the economy was hit by floods that destroyed some of the capital stock. What would happen to the economy in the long run? Explain, referring to your graph in a).

**5)** Consider an economy that can be described by the Diamond OLG model. There is no depreciation and no population growth (*n=0*). "Technology" is not improving (*g=0*). A person has one unit of labor that he provides to firms in exchange for a real wage *w*. The real rental rate on a unit of capital is *r*.

A person's lifetime utility function $U = \left(C^{\beta}{}_1 + C^{\beta}{}_2\right)^{1/\beta}$ $\; where \; 0 < \beta < 1$

His budget constraint is $C_{2t+1} = (1 + r_{t+1})(w_t - C_{1t})$

**a)** Write down the *Lagrangian* that describes the utility-maximization problem of a young person in period *t*. (A young person is a person in the first period of life.)

**b)** Using your Lagrangian from a), derive an expression that gives $(C_2 / C_1)$, that is the ratio of consumption in the second period of life to consumption in the first period of life.

**c)** Using your answer from (b) and the intertemporal budget constraint (or the condition that $\partial \mathcal{L} / \partial \lambda = 0$) derive first-period consumption $C_1$ as a function of the things a person takes as given.

**d)** Using your answer to c), derive *s* as defined for the OLG model,that is the fraction of a young person's income (labor income) devoted to saving as a function of the things a person takes as given.

**e)** Now assume that the aggregate production function is Cobb-Douglas $Y = K^{\alpha} L^{1-\alpha}$ $\; where \; 0 < \alpha < 1$. Using this and your answer to d), write down an equation that gives $k_{t+1}$ as a function of $k_t$ and $r_{t+1}$.

**f)** Your answer to e) implicitly defines the function $k_{t+1} = G(k_t)$. Explain how.

**6)** Consider an economy that can be described by a Ramsey-Cass-Koopmans model. The production function is $Y = K^{\alpha}(AL)^{1-\alpha}$ $\; where \; 0 < \alpha < 1$ and *A* grows at rate *g*. The population is fixed.

Hint: make your graphs **BIG**.

**a)** Suppose the economy is in its long-run steady state. Then, at time $t_0$, there is an unexpected increase in the subjective rate of time-discount $\rho$.

**i)** Using the "phase diagram" graph with *k* (capital per efficiency-unit of labor) on the horizontal axis and *c* (consumption per efficiency-unit of labor) on the vertical axis, show how this change affects the long-run equilibrium levels of *c* and *k*. Let $c_0^*$ and $k_0^*$ denote the old LRSS, $c_1^*$ and $k_1^*$ denote the new one.

**ii)** On the same graph, indicate the path the economy follows over time starting at $t_0$. Use *i* to mark the point on the graph that describes the economy just before $t_0$. Use *ii* to mark the point immediately after $t_0$. Use *iii* to mark a point some time after $t_0$ but before the economy reaches its new long-run steady state.

**iii)** Referring to your graph, describe in words what happens to *c* over time.

**b)** Now suppose that at time $t_0$ there is no immediate change in the rate of subjective time-discount $\rho$. Instead, at time $t_0$ households become aware that the rate of subjective time-discount $\rho$ will increase at a *future* time $t_1$.

**i)** On another graph, different from the graph you used for part a), indicate the path the economy follows over time starting at $t_0$. Use *i* to mark the point on the graph that describes the economy just before $t_0$. Use *ii* to mark the point immediately after $t_0$. Use *iii* to mark a point some time after $t_0$ but before $t_1$. Use *iv* to mark the point at time $t_1$.

**ii)** Referring to your graph, describe in words what happens to *c* over time.

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**1)** Consider the Cobb-Douglas production function $Y = K^{\alpha}(AL)^{1-\alpha}$ *where* $0 < \alpha < 1$.

4pts **a)** Show that for this production function the elasticity of substitution between capital and labor is equal to one.

This is straight from the notes.

1) Get (K/L) as function of relative factor cost

Factor cost = MP of factor (from profit maximization)

$$\text{Capital cost} = MPK = A\alpha K^{\alpha-1} L^{1-\alpha}$$

$$\text{Labor cost} = MPL = A K^{\alpha}(1-\alpha) L^{-\alpha}$$

$$\text{hence } \frac{\text{capital cost}}{\text{Labor cost}} = \text{-----} = \frac{\alpha}{1-\alpha}\left(\frac{K}{L}\right)^{-1}$$

Look! there's K/L! So

$$K/L = \frac{\alpha}{1-\alpha}\left(\frac{\text{capital cost}}{\text{Labor cost}}\right)^{-1}$$

2) Take derivative $\partial(K/L)/\partial(\text{relative cost})$

$$= \frac{\alpha}{1-\alpha}(-1)(\text{relative cost})^{-2}$$

3) Convert to elasticity

Elasticity is $\frac{\partial Y/Y}{\partial X/X} = \frac{\partial Y}{\partial X}\frac{X}{Y}$ so we want

$$\frac{\partial(K/L)}{\partial(\text{capital cost}/\text{labor cost})} \cdot \frac{(\text{capital cost})/(\text{labor cost})}{K/L}$$

$$= \frac{\alpha}{1-\alpha}(-1)\left(\frac{\text{capital cost}}{\text{labor cost}}\right)^{-2} \cdot \left(\frac{\text{capital cost}}{\text{labor cost}}\right) \cdot \frac{1}{\frac{\alpha}{1-\alpha}\left(\frac{\text{capital cost}}{\text{labor cost}}\right)^{-1}}$$

$$= -1$$

4pts

**b)** Assume the economy is closed so that total real income is equal to output $Y$, and markets are perfectly competitive so that the real wage is equal to the marginal product of labor. Show that under these conditions the share of national income going to labor is fixed (is not affected by changes in $A$, $K$ or $L$).

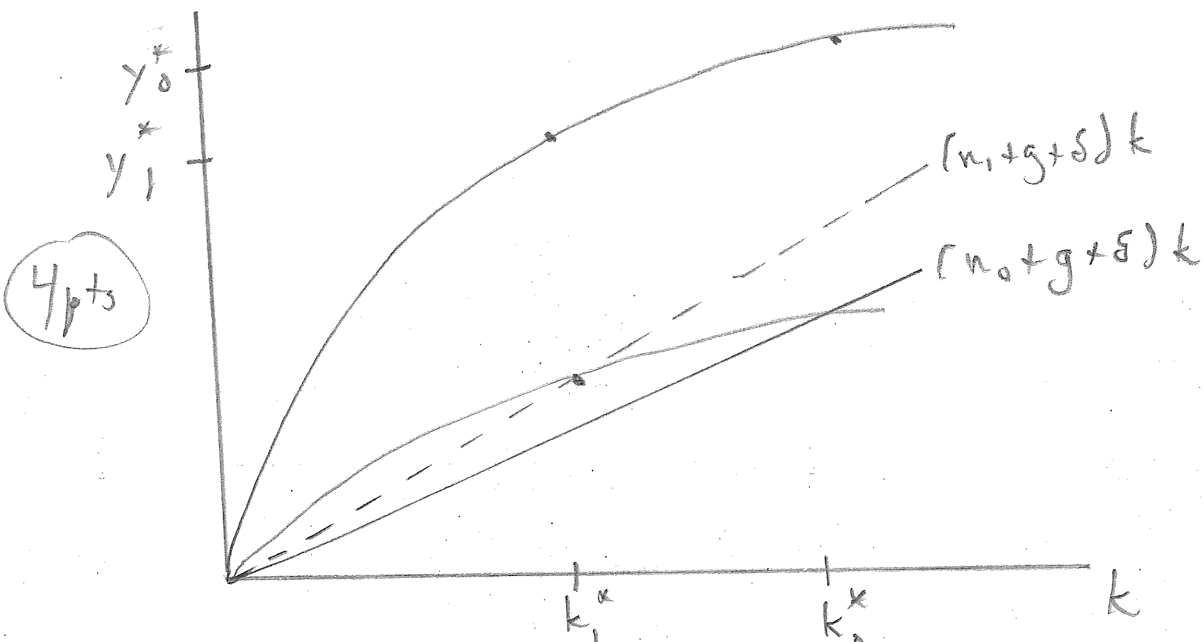
Conditions mean payment to a unit of labor equals MPL, and labor's share of national income is $\frac{MPL \cdot L}{Y}$

so share is $\frac{AK^{\alpha}(1-\alpha)L^{-\alpha} \cdot L}{AK^{\alpha}L^{1-\alpha}} = (1-\alpha)$

**2)** Consider an economy that can be described by the Solow model with a fixed savings rate $s$, a rate of population growth $n$, a rate of "technological improvement" $g$, and a rate of depreciation $\delta$.

**a)** Suppose the rate of population growth increases from $n_0$ to a higher value $n_1$.

**i)** Depict this event on a graph with $k$ (capital per efficiency-unit of labor) on the horizontal axis and $y$ (output per efficiency-unit of labor) on the vertical axis. Referring to the graph, explain in words what happens to $k$, ~~$y$~~ and ~~$c$~~ y ~~(consumption per efficiency-unit of labor)~~.




k falls gradually from $k_0^*$ to $k_1^*$

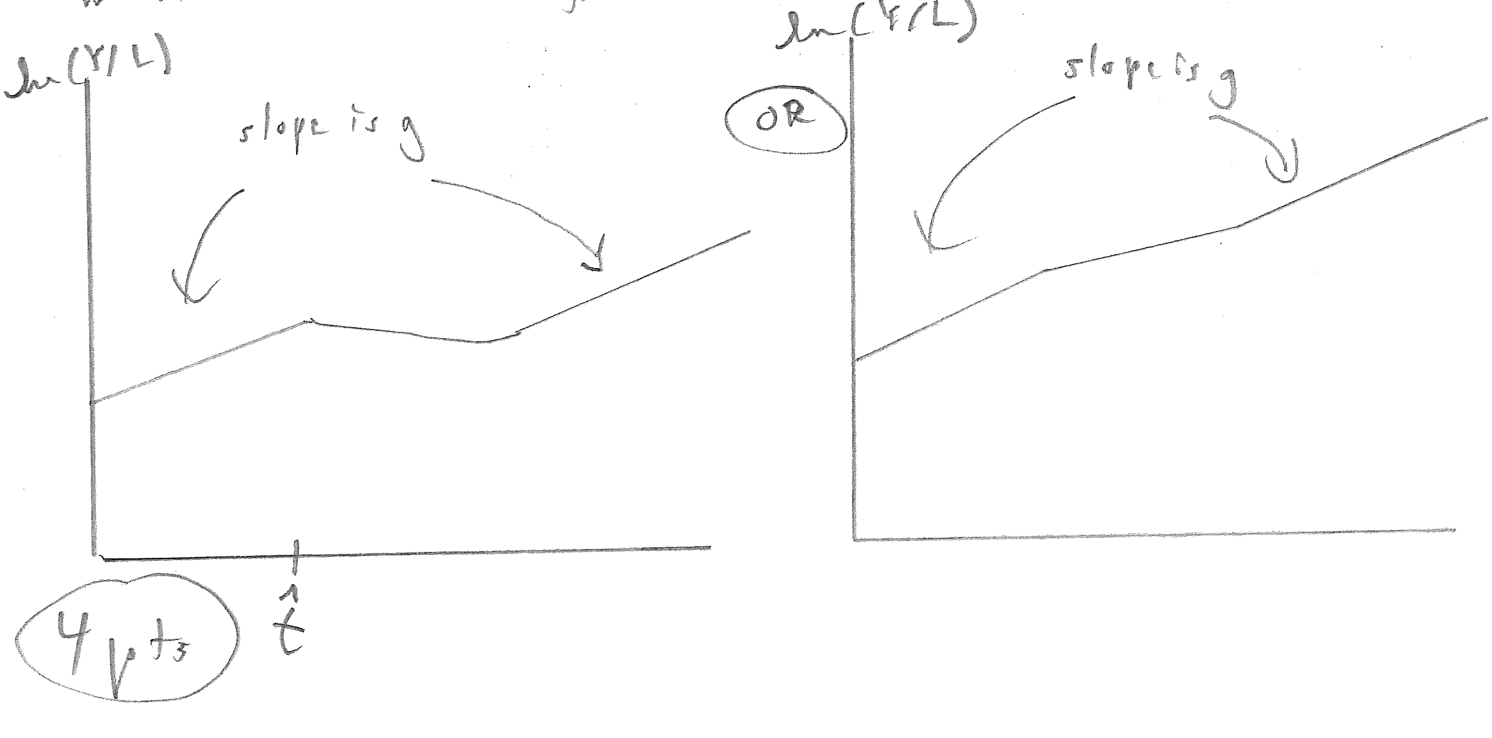
y falls gradually...

**ii)** Depict this event in another graph, one that has time on the horizontal axis and, on the vertical axis, the *log* of output *per worker*. Use $\hat{t}$ to denote the point in time at which the rate of population growth increases.

In old LRSS, k is stable, (Y/L) grows at rate g.

In new LRSS, same.

While k is falling, (Y/L) grows at slower rate. So:



**b)** Suppose there is an increase in the rate of technological improvement $g$ from $g_0$ to a higher value $g_1$.

**i)** Depict this event on a graph with $k$ (capital per efficiency-unit of labor) on the horizontal axis and $y$ (output per efficiency-unit of labor) on the vertical axis. Referring to the graph, explain in words what happens to $k$, $y$ ~~and e~~.

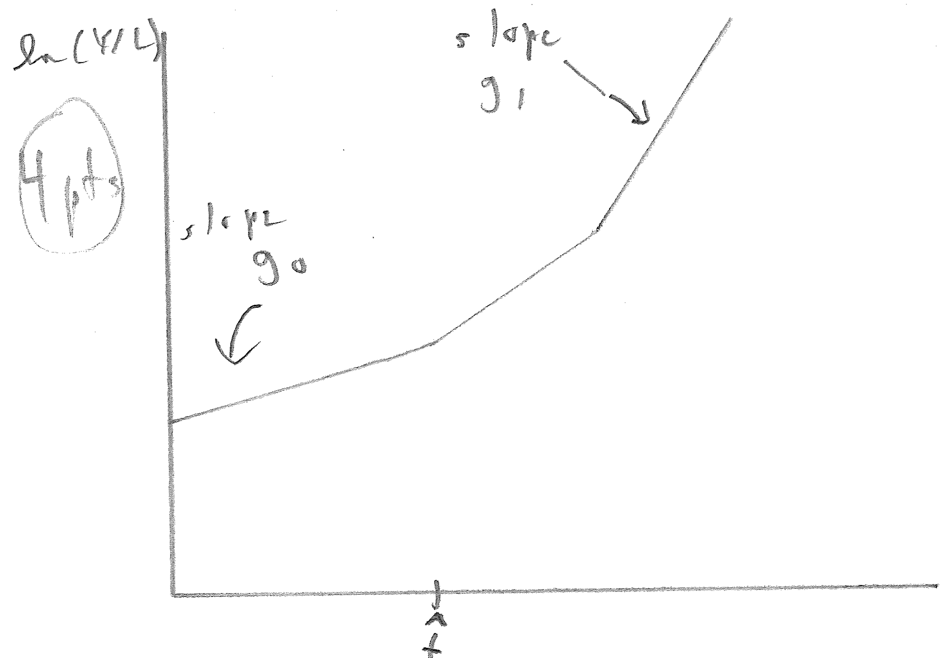
Same as 2) a) i)

4 pts

**ii)** Depict this event in another graph, one that has time on the horizontal axis and, on the vertical axis, the *log* of output *per worker*. Use $\hat{t}$ to denote the point in time at which the rate of growth of $A$ increases.

In old LRSS, k is stable, (Y/L) grows at rate $g_0$.
In new " " " ", (Y/L) grows at rate $g_1$.
While k is falling, (Y/L) grows slower than $g_1$. So:

4 pts




or...

**3)** Again consider an economy that can be described by the Solow model with a fixed savings rate *s,* a rate of population growth *n*, a rate of "technological improvement" $g$, and a rate of depreciation $\delta$. The aggregate production function is $Y = K^{\alpha}(AL)^{1-\alpha}$ *where* $0<\alpha<1$.

**a)** Taking the savings rate *s* as given, derive expressions showing the long-run steady-state values of:

- *k* (capital per efficiency-unit of labor).
- *w* (real wage per efficiency-unit of labor)
- the real wage *per worker.*

as a function of *s* and exogenous parameters.

$$s k^{*\alpha} = (n+g+\delta)k^* \rightarrow k^{*\,1-\alpha} = \frac{s}{n+g+\delta} \rightarrow k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

w is $\frac{\partial Y}{\partial (AL)}$ $= K^{\alpha}(1-\alpha)(AL)^{-\alpha} = (1-\alpha)\left(\frac{K}{AL}\right)^{\alpha}$

MP of efficiency unit

$= (1-\alpha)k^{\alpha}$

so at $k^*$,

$= (1-\alpha)\left(\left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}\right)^{\alpha}$

(6 pts)

$= (1-\alpha)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$

Real wage per worker is

$$Aw = A(1-\alpha)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Or, you could get $\partial Y/\partial L$ (not $\partial Y/\partial(AL)$) in terms of k and substitute in $k^*$ from above.

b) Now consider the "Golden rule" long-run steady-state. Derive the Golden rule values of:
- $k$ (capital per efficiency-unit of labor).
- $w$ (real wage per efficiency-unit of labor)
- the real wage per worker.
as a function of exogenous parameters.

$y = k^{\alpha}$

At Golden rule, $\frac{\partial y}{\partial k} = (n+g+\delta)$ so $\alpha k^{\alpha-1} = n+g+\delta$.

Or you could use, at GR $\frac{\partial Y}{\partial K} = n+g+\delta$.

So $\alpha {k^{*}_{GR}}^{\alpha-1} = n+g+\delta$

6 pts

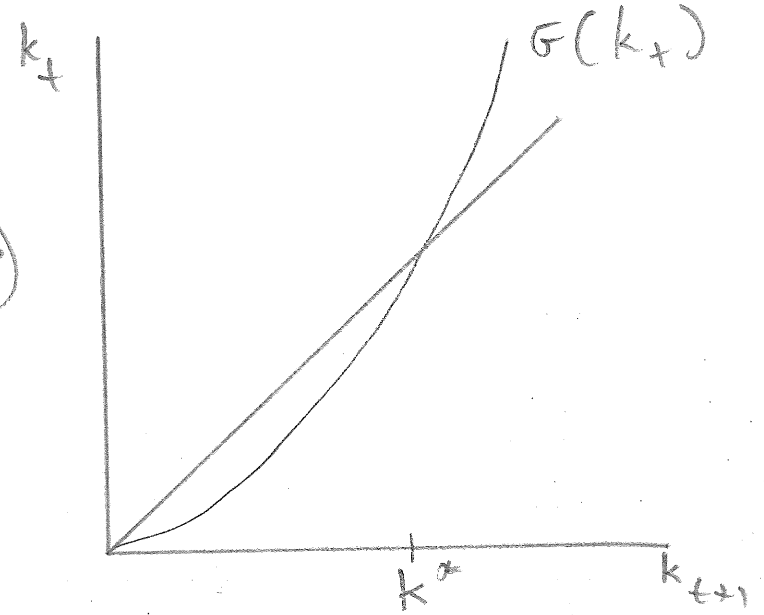
$$k^{*}_{GR} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$w^{*}_{GR} = (1-\alpha)\, {k^{*}_{GR}}^{\alpha} = (1-\alpha)\left(\frac{\alpha}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

Real wage per worker $= A(1-\alpha)\left(\frac{\alpha}{n+g+\alpha}\right)^{\frac{\alpha}{1-\alpha}}$

**4)** Consider the Diamond OLG model.

**a)** Make a graph where the horizontal axis is capital per efficiency unit of labor at time *t,* and the vertical axis is capital per efficient unit of labor at time *t+1*. On the graph draw a line depicting the function $k_{t+1}=G(k_t)$ for an economy that has just one LRSS, and that LRSS is *unstable*.




4pts

**b)** Suppose an economy like the one you depicted in a) started out in its LRSS. Then the economy was hit by floods that destroyed some of the capital stock. What would happen to the economy in the long run? Explain, referring to your graph in a).

If $k < k^*$, $k_{t+1} < k_t$, so k keeps falling to zero.

4pts

5) ~~3~~ Consider an economy that can be described by the Diamond OLG model. The aggregate production function is Cobb-Douglas: $Y = K^{\alpha}(\cancel{A}L)^{1-\alpha}$ *where* $0<\alpha<1$

There is no depreciation. ~~The rate of growth of population is $n$.~~ "Technology" is not improving: the value of the technology parameter $A$ is fixed, equal to *one*. A person has one unit of labor that he provides to firms in exchange for a real wage $w$. The real rental rate on a unit of capital is $r$.

A person's lifetime utility function $U = \left(C^{\beta}{}_1 + C^{\beta}{}_2\right)^{1/\beta}$ *where* $0<\beta<1$

**a)** Write down the *Lagrangian* that describes the utility-maximization problem of a young person in period $t$. (A young person is a person in the first period of life.)

$$\mathcal{L} = (C_1^{\beta} + C_2^{\beta})^{1/\beta} + \lambda\left[w - C_1 - \frac{1}{1+r_{t+1}} C_2\right]$$

2 pts

2 pts **b)** Using your Lagrangian from a), derive an expression that gives $(C_2/C_1)$, that is the ratio of consumption in the second period of life to consumption in the first period of life, as a function of the things a person takes as given.

$$\frac{\partial \mathcal{L}}{\partial C_1} = 0 = \frac{1}{\beta}(C_1^{\beta}+C_2^{\beta})^{\frac{1}{\beta}-1}\beta C_1^{\beta-1} - \lambda = (C_1^{\beta}+C_2^{\beta})^{\frac{1-\beta}{\beta}} C_1^{\beta-1} - \lambda$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = 0 = \frac{1}{\beta}(\quad)^{\frac{1}{\beta}-1}\beta C_2^{\beta-1} - \lambda\frac{1}{1+r} = (\quad)^{\frac{1-\beta}{\beta}} C_2^{\beta-1} - \lambda\frac{1}{1+r}$$

From $\frac{\partial \mathcal{L}}{\partial C_1}$, $\lambda = (C_1^{\beta}+C_2^{\beta})^{\frac{1-\beta}{\beta}} C_1^{\beta-1}$

From $\frac{\partial \mathcal{L}}{\partial C_2}$, $\lambda = (\quad)^{\frac{1-\beta}{\beta}} C_2^{\beta-1}(1+r)$

hence $(C_1^{\beta}+C_2^{\beta})^{\frac{1-\beta}{\beta}} C_1^{\beta-1} = (\quad)^{\frac{1-\beta}{\beta}} C_2^{\beta-1}(1+r)$

so $C_1^{\beta-1} = C_2^{\beta-1}(1+r)$

$$\frac{1}{1+r} = \left(\frac{C_2}{C_1}\right)^{\beta-1} \rightarrow \frac{C_2}{C_1} = \left(\frac{1}{1+r}\right)^{\frac{1}{\beta-1}} = (1+r)^{\frac{1}{-1(\beta-1)}} = (1+r)^{\frac{1}{1-\beta}}$$

Sees $(1+r)\uparrow \rightarrow (C_2/C_1)\uparrow$ makes sense

**c)** Using your answer from (b) and the intertemporal budget constraint (or the condition that $\partial \mathcal{L} / \partial \lambda = 0$) derive first-period consumption $C_1$ as a function of the things a person takes as given.

From b), $\frac{C_2}{C_1} = (1+r)^{\frac{1}{1-\beta}}$

2 pts

$$C_2 = (1+r)^{\frac{1}{1-\beta}} C_1$$

$$C_1 = C_2 (1+r)^{-\frac{1}{1-\beta}}$$

From $0 = w - C_1 - \frac{1}{1+r} C_2$, $\quad C_2 = (1+r)(w - C_1)$

$$\text{so } C_1 = (1+r)(w-C_1)(1+r)^{\frac{-1}{1-\beta}} = (1+r)^{1-\frac{1}{1-\beta}} (w - C_1)$$

$$= (1+r)^{\frac{1-\beta-1}{1-\beta}} (w - C_1) = (1+r)^{\frac{-\beta}{1-\beta}} (w - C_1)$$

$$C_1 = (1+r)^{\frac{-\beta}{1-\beta}} w - (1+r)^{\frac{-\beta}{1-\beta}} C_1$$

$$\left(1 + (1+r)^{\frac{-\beta}{1-\beta}}\right) C_1 = (1+r)^{\frac{-\beta}{1-\beta}} w$$

$$C_1 = \frac{(1+r)^{\frac{-\beta}{1-\beta}}}{1 + (1+r)^{\frac{-\beta}{1-\beta}}} w = \frac{1}{1 + \frac{1}{(1+r)^{\frac{-\beta}{1-\beta}}}} w$$

$$= \frac{1}{1 + (1+r)^{\frac{\beta}{1-\beta}}} w$$

See: $(1+r)\uparrow \rightarrow (1+r)^{\frac{\beta}{1-\beta}} \uparrow \rightarrow C_1 \downarrow$ makes sense

**d)** Using your answer to c), derive $s$ as defined for the OLG model,that is the fraction of a young person's income (labor income) devoted to saving, as a function of the things a person takes as given.

$$s = \frac{w - c_1}{w} = \frac{w - \frac{1}{1+(1+r)^{\frac{\beta}{1-\beta}}} w}{w}$$

2 pts

$$= 1 - \frac{1}{1+(1+r)^{\frac{\beta}{1-\beta}}}$$

$$= \frac{1+(1+r)^{\frac{\beta}{1-\beta}} - 1}{1+(1+r)^{\frac{\beta}{1-\beta}}} = \frac{(1+r)^{\frac{\beta}{1-\beta}}}{1+(1+r)^{\frac{\beta}{1-\beta}}}$$

$$= \frac{1}{1 + \frac{1}{(1+r)^{\frac{\beta}{1-\beta}}}} = \frac{1}{1+(1+r)^{\frac{-\beta}{1-\beta}}}$$

See: $(1+r)\uparrow \rightarrow (1+r)^{\frac{-\beta}{1-\beta}} \downarrow \rightarrow s\uparrow$

**e)** Using your answer to d) and the aggregate production function, write down an equation that gives $k_{t+1}$ as a function of $k_t$ and $r_{t+1}$.

$$k_{t+1} = s_t w_t = \frac{1}{1+(1+r_{t+1})^{\frac{-\beta}{1-\beta}}} w_t$$

$$w = MPL = \frac{\partial Y}{\partial L} = K^{\alpha}(1-\alpha)L^{-\alpha} = (1-\alpha)(K/L)^{\alpha} = (1-\alpha)k^{\alpha}$$

so

$$k_{t+1} = \frac{1}{1+(1+r_{t+1})^{\frac{-\beta}{1-\beta}}}(1-\alpha)k_t^{\alpha}$$

2 pts

**f)** Your answer to e) implicitly defines the function $k_{t+1}=G(k_t)$. Explain how.

$$r = MPK = \frac{\partial Y}{\partial K} = \alpha K^{\alpha-1}L^{1-\alpha} = \alpha(K/L)^{\alpha-1} = \alpha k^{\alpha-1}$$

2 pts

so $r_{t+1} = \alpha k_{t+1}^{\alpha-1}$

so

$$k_{t+1} = \frac{1}{1+(1+\alpha k_{t+1}^{\alpha-1})^{\frac{-\beta}{1-\beta}}}(1-\alpha)k_t^{\alpha}$$

Defines $k_{t+1}$ as function of $k_t$.

6) ~~4)~~ Consider an economy that can be described by a Ramsey-Cass-Koopmans model. The production function is $Y = K^{\alpha}(AL)^{1-\alpha}$ *where* $0 < \alpha < 1$ and $A$ grows at rate $g$. The population is fixed.

**a)** Suppose the economy is in its long-run steady state. Then, at time $t_0$, there is an unexpected increase in the subjective rate of time-discount $\rho$.

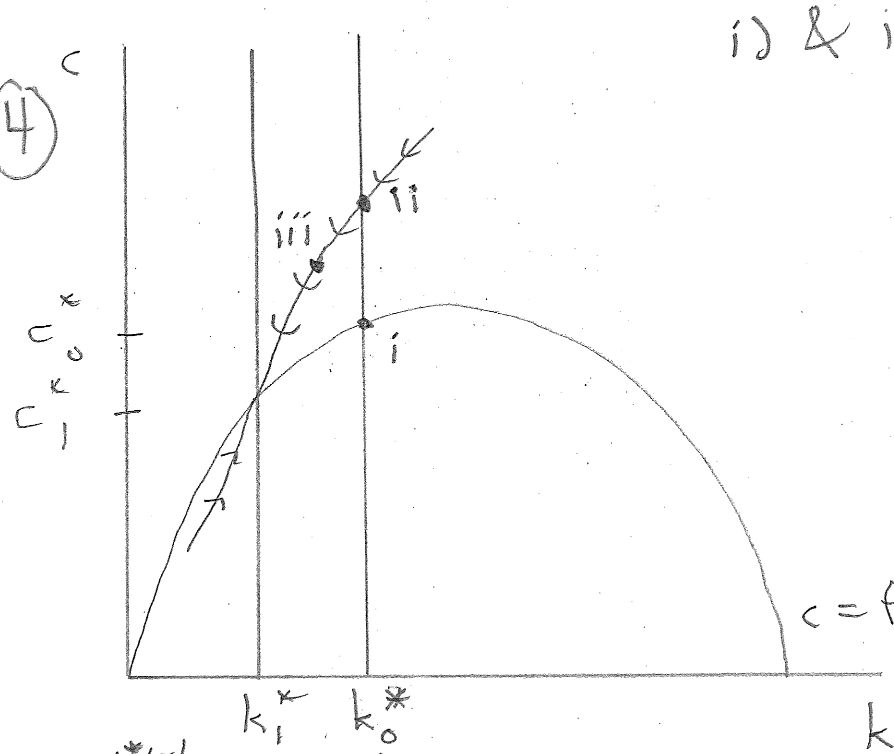
**i)** Using the "phase diagram" graph with $k$ (capital per efficiency-unit of labor) on the horizontal axis and $c$ (consumption per efficiency-unit of labor) on the vertical axis, show how this change affects the long-run equilibrium levels of $c$ and $k$. Let $c_0^*$ and $k_0^*$ denote the old LRSS, $c_1^*$ and $k_1^*$ denote the new one.

**ii)** On the same graph, indicate the path the economy follows over time starting at $t_0$. Use *i* to mark the point on the graph that describes the economy just before $t_0$. Use *ii* to mark the point immediately after $t_0$. Use *iii* to mark a point some time after $t_0$ but before the economy reaches its new long-run steady state.

**iii)** Referring to your graph, describe in words what happens to $c$ over time.

i) & ii) on graph

4




$c = f(k) - gk = k^{\alpha} - gk$

$\alpha k^{*\alpha-1} = f'(k^*) = \rho + \theta g$

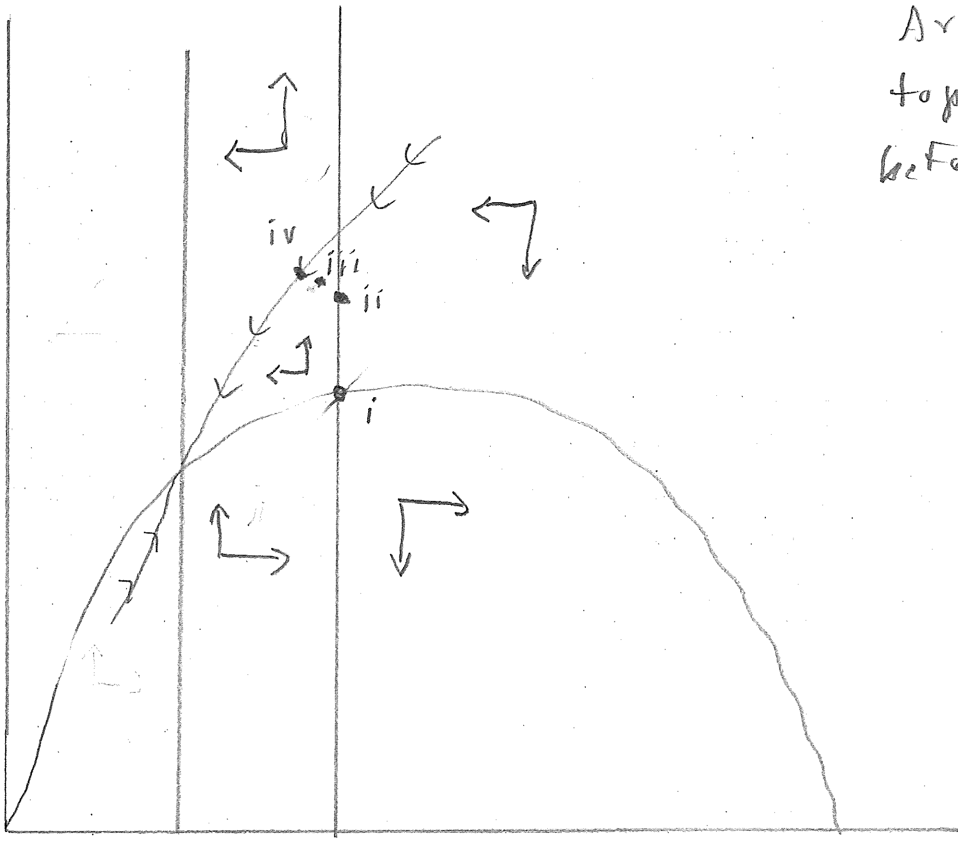
iii) At $t_0$, c jumps up (i to ii), then falls (ii to iii) down to $c_1^*$, ending up lower than it was before $t_0$.

4

**b)** Now suppose that at time $t_0$ there is no immediate change in the rate of subjective time-discount $\rho$. Instead, at time $t_0$ households become aware that the rate of subjective time-discount $\rho$ will increase at a *future* time $t_1$.

**i)** On another graph, different from the graph you used for part a), indicate the path the economy follows over time starting at $t_0$. Use *i* to mark the point on the graph that describes the economy just before $t_0$. Use *ii* to mark the point immediately after $t_0$. Use *iii* to mark a point some time after $t_0$ but before $t_1$. Use *iv* to mark the point at time $t_1$.

**ii)** Referring to your graph, describe in words what happens to *c* over time.




Arrows show topography prevailing before $t_1$.

ii) At $t_0$, c jumps up (i to ii), then keeps increasing gradually (ii to iii) until it reaches new saddle path at $t_1$ (iii to iv) then falls down to $c_1^*$, ending up lower than it was before $t_0$.