INTERTEMPORAL OPTIMIZATION

Analogies to maximization of utility at a point in time

Maximization at a point in time

A Apples P_A

B Bananas P_B

I Income

$U(A, B)$ $\quad U_A'(A, B) \equiv \dfrac{\partial U(A, B)}{\partial A}$ $\quad U_B'(A, B) \equiv \dfrac{\partial U(A, B)}{\partial B}$

$$\max_{A, B} U(A, B) \quad \text{s.t. } P_A A + P_B B \leq I$$

F.O.C.:

$$\frac{U_A'(A^*, B^*)}{U_B'(A^*, B^*)} = \frac{P_A}{P_B}$$

units of bananas I get in exchange for one apple

A specific utility function: separable in apples & bananas

$$U = u_A(A) + u_B(B)$$

F.O.C.:

$$\frac{u_A'(A^*)}{u_B'(B^*)} = \frac{P_A}{P_B}$$

This defines a B^* for any given A^*, or an A^* for any given B^*

but most combinations of A^* & B^* that satisfy F.O.C. violate the budget constraint;
there is only one A^* that implies a B^* such that the budget constraint is satisfied with equality.

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Analogies to maximization of utility at a point in time

Two periods

C_1 consumption in first period

C_2 consumption in second period

I_1 Endowment of income, received in first period

r Interest rate. Get $(1+r)$ units of C_2 in exchange for giving up one unit of C_1.

$U(C_1, C_2)$ 'Intertemporal utility function'

An intertemporal utility function separable in C_1 & C_2'

$$U = u(C_1) + \underbrace{\frac{1}{(1+d)}}_{\text{subjective rate of time-discount}} u(C_2) \quad \left[\text{note: } \frac{\partial U}{\partial C_2} = \frac{1}{(1+d)} u'(C_2)\right.$$

$$\underset{C_1, C_2}{\text{Max}} \; u(C_1) + \frac{1}{(1+d)} u(C_2)$$

$$\text{s.t. } C_1 + \frac{1}{(1+r)} C_2 \leq I_1$$

F.O.C:

$$\frac{u'(C_1^*)}{\frac{1}{(1+d)} u'(C_2^*)} = 1+r$$

Euler equation!
This defines a C_2^* for every C_1^*, which is to say for every C_1^* it defines $\Delta C = C_2 - C_1 \, (= \dot{C})$

but there is only one C_1^* that implies a C_2^* that satisfies budget constraint with equality.

Note: willingness to substitute between C_1 & C_2 in response to Δr depends on how rapidly $U'(C)$ diminishes as C increases, i.e. $U''(C)$ or "curvature" of utility function.

INTERTEMPORAL OPTIMIZATION

Analogies to Hamiltonian

Another way to describe two-period optimization

λ_1 Value, measured in utility-units, of a unit of "saving" (not-consumption) in period one

Define a function:

$$H_1 = u(C_1) + \lambda_1 [\underbrace{I_1 - C_1}_{\text{saving}}]$$

Choose C_1 to maximize H_1

F.O.C.:

$$\frac{\partial H_1}{\partial C_1} = 0 = u'(C_1^*) - \lambda_1 \quad \text{hence} \quad u'(C_1^*) = \lambda_1$$

and there is no "saving" in period two: otherwise you wouldn't be satisfying budget constraint with equality.

Sort of like defining

$$H_2 = \left(u(C_2) + \lambda_2 [\underbrace{(1+r)(I_1 - C_1) - C_2}_{\text{saving in period two}}]\right)\frac{1}{1+d}$$

and satisfying budget constraint with equality means

$$\lambda_2 [(1+r)(I_1 - C_1) - C_2] = 0$$

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Analogies to Hamiltonian

Three periods

$$H_1 = u(C_1) + \lambda_1 [I_1 - C_1]$$

$$H_2 = \left(u(C_2) + \lambda_2 [(1+r)(I_1 - C_1) - C_2]\right)\frac{1}{1+d}$$

$$H_3 = \left(u(C_3) + \lambda_3 \underbrace{[(1+r)((1+r)(I_1 - C_1) - C_2) - C_3]}_{\text{satisfying budget constraint means this must equal zero}}\right)\frac{1}{(1+d)^2}$$

F.O.C.'s:

$$\frac{\partial H_1}{\partial C_1} = 0 = u'(C_1^*) - \lambda_1 \quad \text{hence } u'(C_1^*) = \lambda_1$$

$$\frac{\partial H_2}{\partial C_2} = 0 = u'(C_2^*) - \lambda_2 \quad \text{hence } u'(C_2^*) = \lambda_2$$

Recall $\dfrac{u'(C_1^*)}{\frac{1}{(1+d)} u'(C_2^*)} = 1+r$ hence $\dfrac{u'(C_1^*)}{u'(C_2^*)} = \dfrac{1+r}{1+d}$

hence $\dfrac{\lambda_1}{\lambda_2} = \dfrac{1+r}{1+d}$

Define $\Delta\lambda = \lambda_2 - \lambda_1$ then $\dfrac{\lambda_1}{\lambda_1 + \Delta\lambda} = \dfrac{1+r}{1+d}$

solve for $\Delta\lambda$ gives:

another condition that must hold for C_1^*, C_2^*, C_3^* $\Bigg\}$ $\Delta\lambda = \lambda_1\left(\dfrac{d-r}{1+r}\right)$