RAMSEY-CASS-KOOPMANS MODEL

Production

Same as Solow

$$Y_t = F(K_t, A_t L_t) \text{ with CRS} \qquad (2.1)$$

where $A_t = A_0 e^{gt}$ growth rates

$L_t = L_0 e^{nt}$

but no depreciation

Households

H fixed number of households

$\frac{L}{H}$ People per household, grows at rate n

Household lives forever, maximizes

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C_t) \frac{L_t}{H} \, dt \qquad (2.2)$$

Lifetime utility

per capita consumption

"Felicity" or "flow utility" or "instantaneous utility"

Note: the more people the better. "Benthamite" utility.

Romer sticks to an example with CRRA utility:

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}$$

A necessary condition on parameters:

$$\rho - n - (1-\theta) g > 0 \qquad (2.3)$$

or $\rho > n + (1-\theta) g$

What's this about?

R C K (cont.)

Condition on parameters: infinity problem

Why $\rho > n + (1-\theta)g$?

LRSS of this model looks like Solow.

So C grows at rate g

$$U = \int_{t=0}^{\infty} e^{-\rho t} \, u(C_t) \, \frac{L_t}{H} \, dt$$

grows at g   grows at n

The math can't deal with infinity, so we need something that pulls down on U, keeps it finite. That's ρ! ρ must be big enough relative to n, g, & $(1-\theta)$ in felicity function (which determines how much growth in C increases u).

Hence $\rho > n + (1-\theta)g$

Perfect competition in all markets

so

real wage $W_t = MPL = A_t[f(k_t) - k_t f'(k_t)]$ (2.5)

real interest rate (return to holding capital)

$r_t = MPK = f'(k_t)$ (2.4)

so that one unit of investment (output not consumed) turned into capital at time 0 & reinvested until time t (turned back into consumption) gives, at t,

$e^{R(t)}$ where $R(t) = \int_{\tau=0}^{t} r_\tau \, d\tau$

(or, to get one unit of output at time t you must invest $e^{-R(t)}$ at time 0)

The household's maximization problem

As always, choose values of control variable, here c_t, to maximize utility

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(c_t) \frac{L_t}{H} \, dt$$

(note: household does not control L)

subject to budget constraint

$$\underbrace{\int_{t=0}^{\infty} e^{-R(t)} c_t \frac{L_t}{H} \, dt}_{\text{present discounted value of consumption}} \leq \underbrace{\underset{\substack{\uparrow \\ \text{initial wealth}}}{\frac{K_0}{H}} + \int_{t=0}^{\infty} e^{-R(t)} W_t \frac{L_t}{H} \, dt}_{\text{present discounted value of assets and labor income}} \quad (2.7)$$

Understanding the budget constraint: "no Ponzi game"

If lifetime were finite $\left(\int_{t=0}^{S} \cdots \right)$, budget constraint would mean "you can't be in debt when you die" (so that your creditors are left holding the bag).

But this household lives forever!

Why can't he borrow some today, carry that debt forever, never pay it off?

With 2.7), we've assumed he can't.

It's called the no-Ponzi-game condition.

Move on this....

"no Ponzi game" (cont.)

In the math, it's hard to deal with ∞, easier to deal with $\int_{t=0}^{S \to \text{death}} \ldots$ & $S \to \infty$.

We'll put the budget constraint in a form that says "you can't die in debt, even if the date of death is ∞":

$$\lim_{S \to \infty} e^{-R(S)} \frac{K_S}{H} \geq 0 \qquad (2.11)$$

assets at time S

Rearrange (2.7): $$\frac{K_0}{H} + \int_{t=0}^{\infty} e^{-R(t)} [W-C]_t \frac{L_t}{H} dt \qquad (2.8)$$

As a limit: $$\lim_{S \to \infty} \left[\frac{K_0}{H} + \int_{t=0}^{S} e^{-R(t)} [W-C]_t \frac{L_t}{H} dt \right] \geq 0 \qquad (2.9)$$

Now shift reference point to time S:

$$\frac{K_S}{H} = e^{R(S)} \frac{K_0}{H} + \int_{t=0}^{S} e^{R(S)-R(t)} [W-C]_t \frac{L_t}{H} dt \qquad (2.10)$$

$$\frac{K_S}{H} = e^{R(S)} \left[\frac{K_0}{H} + \int_{t=0}^{S} e^{-R(t)} [W-C]_t \frac{L_t}{H} dt \right]$$

(note: $e^{R(S)-R(t)} = e^{R(S)} \cdot e^{-R(t)}$)

so $$e^{-R(S)} \frac{K^S}{H} = \underbrace{\frac{K_0}{H} + \int_{t=0}^{S} e^{-R(t)} [W-C]_t \frac{L_t}{H} dt}_{\text{this is what's in (2.9), which is } \geq 0}$$

so $$\lim_{S \to \infty} e^{-R(S)} \frac{K_S}{H} \geq 0 \qquad (2.11)$$

R C K (cont.)

Social planner's problem

Maximize

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C_t) \frac{L_t}{H} \, dt$$

subject to society's budget constraint

$$Y_t = F(K_t, A_t L_t)$$

$$\dot{K}_t = F(K_t, A_t L_t) - H \frac{L_t}{H} C_t$$

or, in terms of Solow model,

$$y_t = f(k_t)$$

$$\dot{k}_t = f(k_t) - c_t - (n+g) k_t$$

$\frac{C}{AL}$

where $C_t = H \frac{L_t}{H} A_t \, c_t$

Social planner's problem versus household's

Here, unlike in OLG model,

the "First fundamental theorem of welfare economics"

applies, because

- rationality of agents
- perfect information
- competitive markets
- no externalities
- finite number of interacting agents

So solution to social planner's problem same as solution to household's.

What we'll do differently from Romer

1) Set $H = 1$
2) Solve model by solving social planner's problem
   - solving social planner's problem
   - using a math trick, a "Hamiltonian"
3) Start with simple case of fixed A, then add g
4) Use utility function

$$U = \int_0^\infty e^{-\rho t} u(c_t)\, dt$$

Not Benthamite. (If you set our discount factor equal to ($\rho_{\text{Benthamite}} + n$), this is same as Romer.)

Note this means necessary condition on parameters to avoid infinity-blowing-up-U problem is

$$\rho > (1-\theta) g \quad \text{for CRRA felicity}$$

RAMSEY-CASS-KOOPMANS MODEL

Simple case: A fixed (=1), H = 1, Social Planner

$$Y_t = F(K_t, L_t) = C_t + \dot{K}_t$$

in per capita terms

$$f(k_t) = c_t + \dot{k}_t + n k_t$$

$$U = \int_0^\infty e^{-\rho t} u(c_t)\, dt$$

Social planner's problem:

Choose path for c from t_0 to ∞ so as to maximize U

subject to $\dot{k}_t = f(k_t) - c_t - n k_t$

$$k_0 > 0$$

A method for solving dynamic "control problems" like this one:

"Maximum principle": an extension of method of Lagrange multipliers

Define a "Hamiltonian function" that incorporates objective function & constraint(s).

Path for "control variable" is optimal if certain conditions are satisfied with respect to Hamiltonian.

Applying the method here

Hamiltonian function:

$$H_t = u(c_t) e^{-\rho t} + \mu_t [\overbrace{f(k_t) - c_t - n k_t}^{\dot{k}_t}]$$

μ_t is "costate variable": value as of t_0 of getting another unit of capital at time t

RAMSEY-CASS-KOOPMANS MODEL

Simple case (cont.)

3 Necessary & sufficient conditions for optimum path:

$$(1)\quad \frac{\partial H_t}{\partial C_t} = 0 = u'(C_t) - \mu_t e^{\rho t}$$

$$(2)\quad \dot{\mu}_t = -\frac{\partial H}{\partial k} = -\mu_t\left(f'(k_t) - n\right)$$

and "Transversality condition"

$$(3)\quad \lim_{t\to\infty} k_t \mu_t = 0$$

Useful change in notation

$$\lambda_t = \mu_t e^{\rho t}$$

value as of t_0 of getting capital at time t

value as of t of getting capital at time t

Note $\mu_t = \lambda_t e^{-\rho t}$ hence $\dot{\mu}_t = \dot{\lambda} e^{-\rho t} + (-\rho) e^{-\rho t} \lambda_t$

Hamiltonian & conditions for optimum in new notation

$$H_t = e^{-\rho t}\left[u(c_t) + \lambda_t\left(f(k_t) - c_t - n k_t\right)\right]$$

$(1)\quad 0 = u'(c_t) - \lambda_t$ hence $u'(c_t) = \lambda_t$

$$\dot{u}'(c_t) = \dot{\lambda}_t$$

$$(2)\quad \dot{\lambda} e^{-\rho t} + (-\rho) e^{-\rho t} \lambda_t = \lambda_t e^{-\rho t}\left(f'(k_t) - n\right)$$

hence $\dot{\lambda}_t = \lambda_t\left[\rho + n - f'(k_t)\right]$

$$(3)\quad \lim_{t\to\infty} k_t \lambda_t e^{-\rho t} = 0 = k_t u'(c_t) e^{-\rho t}$$

$= u'(c_t)$ from (1)

RAMSEY–CASS–KOOPMANS MODEL

Simple case (cont.)

Euler equation

From (1), $\dot{u}'(c_t) = \dot{\lambda}_t$ and $u'(c_t) = \lambda_t$

From (2), $\dot{\lambda}_t = \lambda_t \left[\rho + n - f'(k_t)\right]$

hence $\dot{u}'(c_t) = u'(c_t)\left[\rho + n - f'(k_t)\right]$

or

Euler equation $\left.\right\}$ $$\frac{\dot{u}'(c_t)}{u'(c_t)} = \rho + n - f'(k_t)$$

Note: given a starting value of c_0 & k_0, Euler equation defines an entire path for c over time

because c_0 & k_0 define $\dot{k}$: where k goes, hence where $f'(k)$ goes,

and $\dot{u}'(c_t)$ defines $\dot{c}_t$: where c goes

(but different values for c_0 set you off on different paths: which possible path satisfies Transversality condition?)

RAMSEY–CASS–COOPMANS

Simple case

Focus on Euler equation

$$\frac{\dot{u}'(c_t)}{u'(c_t)} = \rho + n - f'(k_t)$$

Note $\dot{u}'(c_t) = \frac{\partial c_t}{\partial t} \cdot \frac{\partial u'(c_t)}{\partial c_t} = \frac{\partial c_t}{\partial t} u''(c_t)$

so Euler equation can be written

$$\frac{u''(c_t)}{u'(c_t)} \frac{\partial c_t}{\partial t} = \rho + n - f'(k_t)$$

Multiply top & bottom by c_t

$$\underbrace{\frac{u''(c_t) \cdot c_t}{u'(c_t)}}_{} \frac{\dot{c}_t}{c_t} = \rho + n - f'(k_t)$$

Negative of
coefficient of relative risk aversion.

Let's make it constant! For $u(c_t)$, choose a utility function with "constant relative risk aversion"

RAMSEY–CASS–COOPMANS MODEL

Simple case

CRRA utility function: $U(C) = \dfrac{C_t^{1-\theta}}{1-\theta}$

then $U'(C) = C_t^{-\theta}$ $\quad U''(C) = -\theta\, C_t^{-\theta-1}$

$$\Rightarrow -\frac{U''(C_t)\, C_t}{U'(C_t)} = \theta$$

Euler equation becomes:

$$\frac{\dot{C}_t}{C_t} = \frac{1}{\theta}\left[f'(k_t) - \rho - n\right]$$

Interpretation:

Consumption grows faster when MPK is big, time-discount small

Ramsey - Cass - Koopmans

Simple Case

Focus on Transversality Condition

$$\lim_{t \to \infty} k_t \, U'(c_t) e^{-\rho t} = 0$$

If this equation were not satisfied, that would mean either:

$\longrightarrow$ $k_\infty < 0$ impossible!

or

$\longrightarrow$ $\left\{ \begin{array}{l} k_\infty > 0 \\ U'(c_\infty) > 0 \end{array} \right.$ carrying a capital stock into infinity but I'd benefit from eating up that capital

Analogy: suppose world ends at time T
then TVC corresponds to

$$k_T \, U'(C_T) e^{-\rho T} = 0$$

& failure of TVC means

$k_T > 0$
$U'(C_T) > 0$

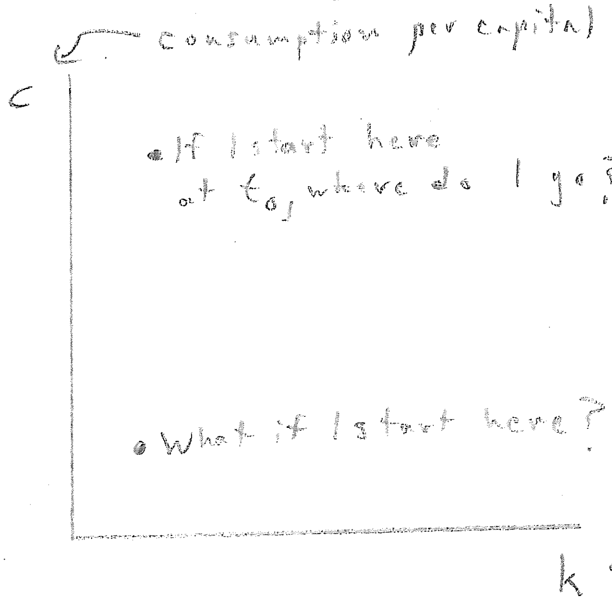
I ought to eat up that k_T, not leave it on the ground when I die.

Failure of TVC means I'm not consuming up to my "lifetime wealth."

RAMSEY - CASS - KOOPMANS

Simple Case

Phase Diagram



Summarizes story about

- Technology, production opportunities
  - ① Stuff from Solow model
- Preferences for consumption across time
  - ② Euler equation
  - ③ TVC

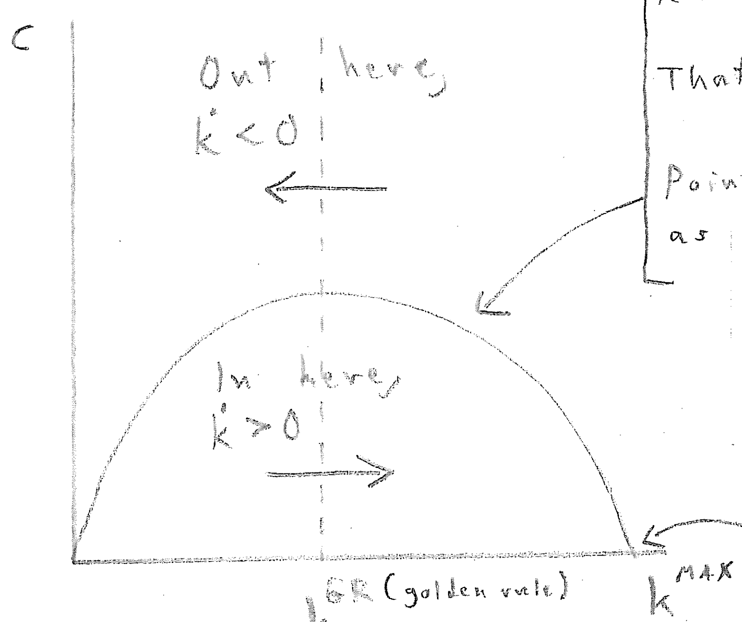
① Stuff from Solow model (applies here too)

Recall $\dot{k}_t = f(k_t) - c_t - nk$

That defines

Points where $\dot{k} = 0$

as $c = f(k) - nk$



where $f'(k) = n$

(recall here we assume no depreciation)

here all production goes to maintain capital stock, so

$k > 0$

$U'(c) = \infty$

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Simple Case

Phase Diagram (cont.)

② Stuff from Euler equation

Recall $\frac{u''(c_t)}{u'(c_t)} \dot{c}_t = \rho + n - f'(k_t)$

$$\dot{c}_t = [f'(k_t) - \rho - n]\left(-\frac{u'(c_t)}{u''(c_t)}\right)$$

$u''(c_t) < 0$

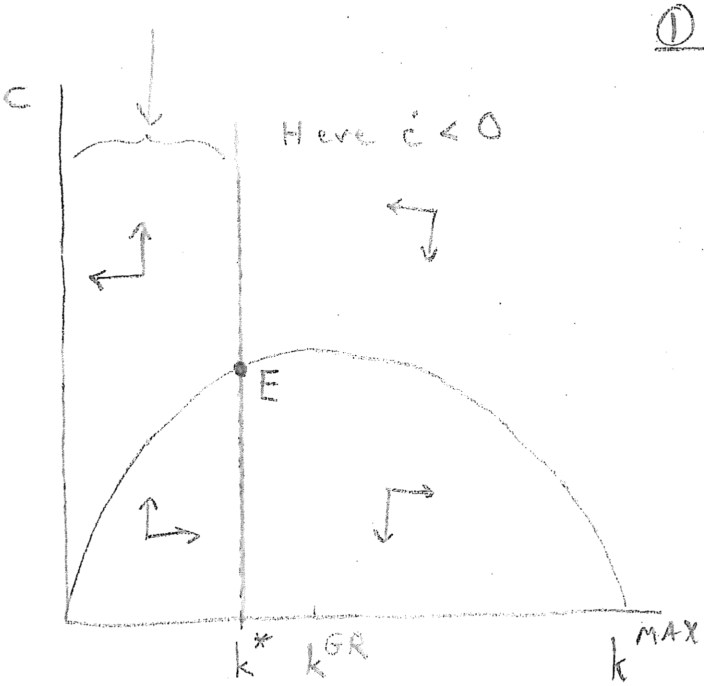
that means $\dot{c}_t = 0$ if $k_t = k^*$ such that

$$f'(k^*) = \rho + n$$

Defines k^*

Note: $k^* < k^{GR}$ (since $f'(k^{GR}) = n$)

Here $\dot{c} > 0$



①&② together define point E

At E,

$\left.\begin{array}{l}\dot{c} = 0 \\ \dot{k} = 0\end{array}\right\}$ If you get to E, you stay there.

$$\boxed{\text{But will you go to E?}}$$

That's where TVC comes in!

RAMSEY – CASS – KOOPMANS

Simple case

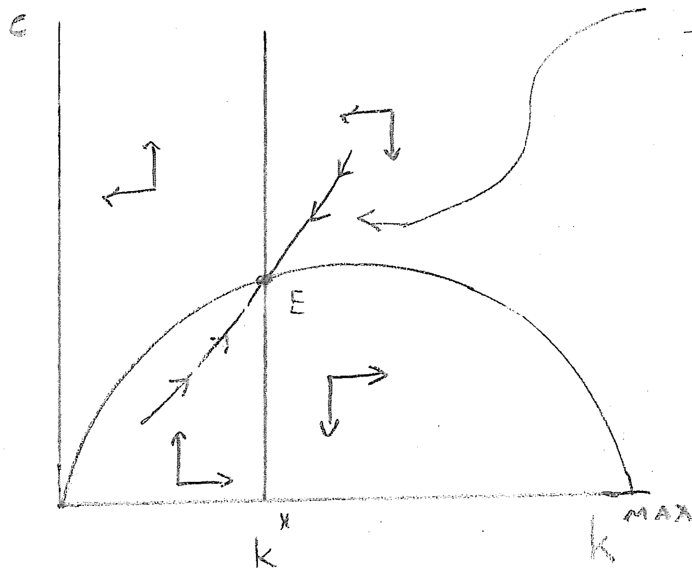
Phase diagram (cont)

③ Stuff from TVC

Suppose you start at time t_0 with $k_0 \neq k^*$. Must choose c_0.
From Euler equation & Solow stuff, choice of initial c_0 puts you on a path for c & k (defines $\dot{c}_0$ & $\dot{k}_0$ and so on into future).

There is just one choice of c_0 that will put you on the path that leads to point E. "Saddle path"

Value of c_0 that satisfies TVC is the one that puts you on the path to E.

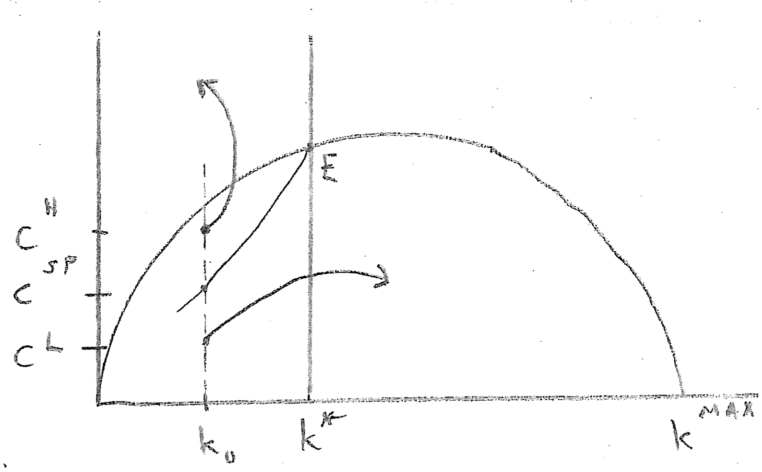


Saddle path

For any value of k_0 defines value of c_0 that will put you on path to E.

Call that value c^{SP}

See how $c \neq c^{SP}$ would violate TVC!



Consider $c^H > c^{SP}$
$c^L < c^{SP}$

Recall $\dot{k} = f(k) - nk - c$

$\Rightarrow$ at c^H, $\dot{k}$ too small, you shoot up to left.

$\Rightarrow$ at c^L, $\dot{k}$ too big, you shoot down to right.

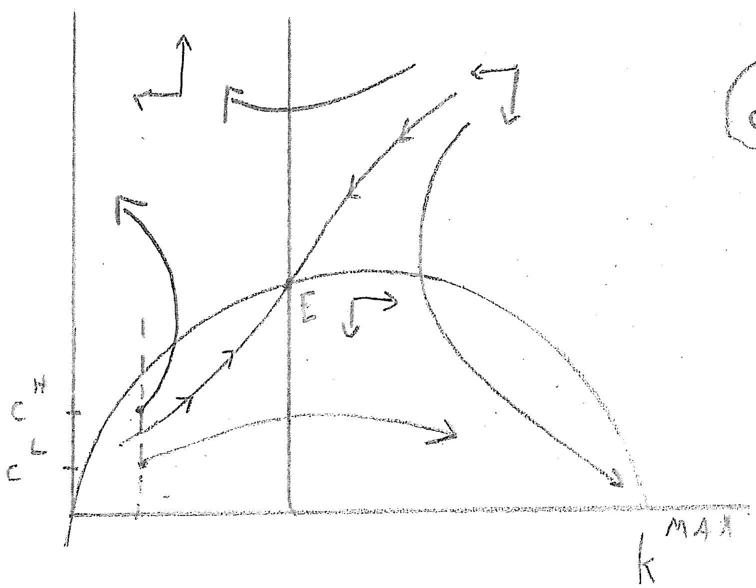
How does that violate TVC?

RAMSEY – CASS – KOOPMANS

Simple Case

Phase diagram

Stuff from transversality condition



c^H Choosing c^H & shooting to upper left means

— more & more c

— less & less k

which would mean

$k_\infty < 0$

$\lim_{t\to\infty} k_t U'(c_t) e^{-\rho t} < 0$

$\boxed{\text{violates TVC}}$

c^L Choosing c^L & going to lower right means I go to k^{MAX} because I can't get to right of k^{MAX} (k falls back because I can't maintain a per-capita capital stock bigger than k^{MAX})

At k^{MAX},

$k_\infty > 0 \quad U'(c) = \infty$

$\lim_{t\to\infty} k_t U'(c_t) e^{-\rho t} > 0$ $\boxed{\text{violates TVC}}$

hence c^{SP} is only c_0 that satisfies TVC

Ramsey–Cass–Koopmans

Simple Case

Phase Diagram

Conclusion:

— Starting with <u>any</u> initial capital stock, we must get on path going to E, & when we get to E we'll stay there.

That is, given k_0 we choose c_0 that puts us on "saddle path."

— In LRE, $k^* < k^{GR}$

k^* is "modified golden rule" capital stock.

k^* is "dynamically efficient"

If we were at $k = k^*$, why not save a bit more, get to k^{GR}, & get more consumption in future?

Because we'd have to give up a bit of c now to get that future c, & given time-discount....

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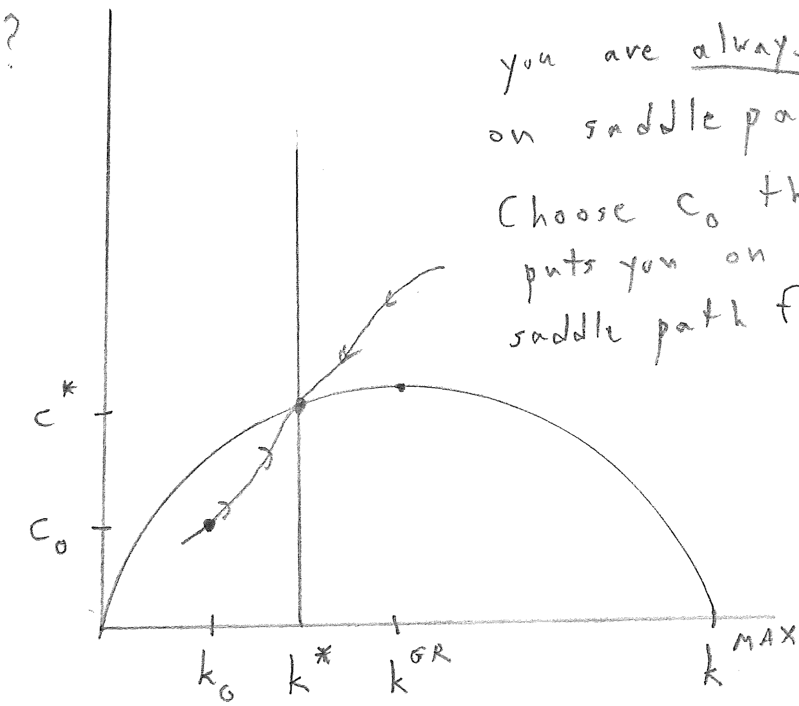
Simple case (A fixed)

Review: getting to LRE

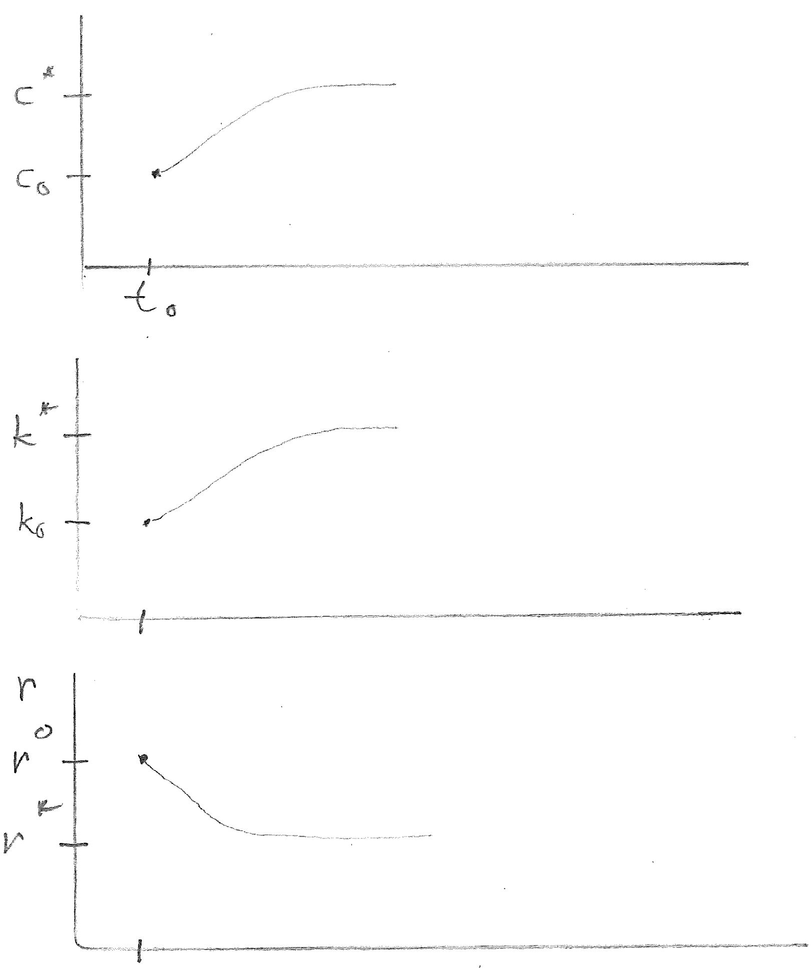
What happens if $k_0 < k^*$?

— capital destroyed, or popn. jumped up

— k^* rose, so that old k^* is below new k^*



you are always on saddle path. Choose c_0 that puts you on saddle path for k_0



transition to LRE, as you crawl along saddle path.

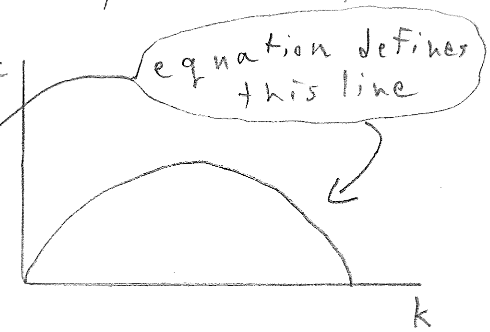
RAMSEY–CASS–KOOPMANS

PRODUCTIVITY GROWTH

As for Solow model, define c, k, y as "per efficiency unit of labor,"
g is growth rate of A.

$$\dot{k}_t = f(k_t) - c_t - (n+g)k_t$$

For $\dot{k} = 0$, $c = f(k) - (n+g)k$



Tricky: C in utility function

$$U = \int_{t=0}^{\infty} e^{-\rho t} u(C_t)\,dt \quad \text{where } u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}$$

Big θ means u diminishes fast

Big C: <u>still</u> consumption per <u>person</u> (not per efficiency-unit of labor)

A necessary condition on parameters:

$$\rho - (1-\theta)g > 0$$

means ρ (subjective time-discount) big enough

θ big enough ($u(C_t)$ diminishes fast enough)

g small enough (A doesn't grow too fast)

If this condition isn't satisfied (if $\rho - (1-\theta)g \le 0$)

math breaks down: you can't solve the optimization problem.

Why? See below.

RAMSEY-CASS-KOOPMANS

PRODUCTIVITY GROWTH (cont)

LRSS

would mean $\dot{c}=0, \dot{k}=0, \dot{y}=0$ as in Solow

per AL

$\dot{c}=0$ means $\frac{\dot{C}}{C}=g$ from Solow

per person

Euler equation says

per person $\longrightarrow$ $\frac{\dot{C}}{C}=\frac{1}{\theta}\left[f'(k_t)-\rho-n\right]$

In LRSS,

$$g=\frac{1}{\theta}\left[f'(k^E)-\rho-n\right]$$

$$\theta g=f'(k^E)-\rho-n$$

$$\text{or } f'(k^E)=\rho+n+\theta g$$

Intuition: real interest rate/return to saving higher, LRE reward for postponing consumption must be higher, if

- ρ higher (more impatient)
- n higher (capital just leaks to new babies anyway)
- θ higher: less willing to substitute future C for C TODAY because MU diminishes rapidly
- g higher: even if I don't save, c tomorrow can be higher than c today.

RAMSEY-CASS-KOOPMANS

PRODUCTIVITY GROWTH (cont.)

But why does social planner choose LRSS?

Anything else causes $k_\infty \to \infty$ impossible

$k_\infty \to 0$, $u'(c_\infty) \to \infty$

violates TVC $\lim_{t\to\infty} k_t u'(c_t) e^{-\rho t} = 0$

because $u''(c)<0$, $u'(c)$ rises faster than k falls

Why the condition on parameter values: $\rho - (1-\theta)g > 0$?

Recall $U = \int_{t=0}^{\infty} e^{-\rho t} u(c_t)\,dt$ where $u(c_t) = \frac{c_t^{1-\theta}}{1-\theta}$

Big θ, u diminishes fast

Math won't work if U can be <u>infinite</u>

What prevents possibility of $U = \infty$, if c growing at rate g ?

tends to make future $u(c)$ big!

To prevent $U = \infty$, need things that pull down present discounted utility value of future consumption:

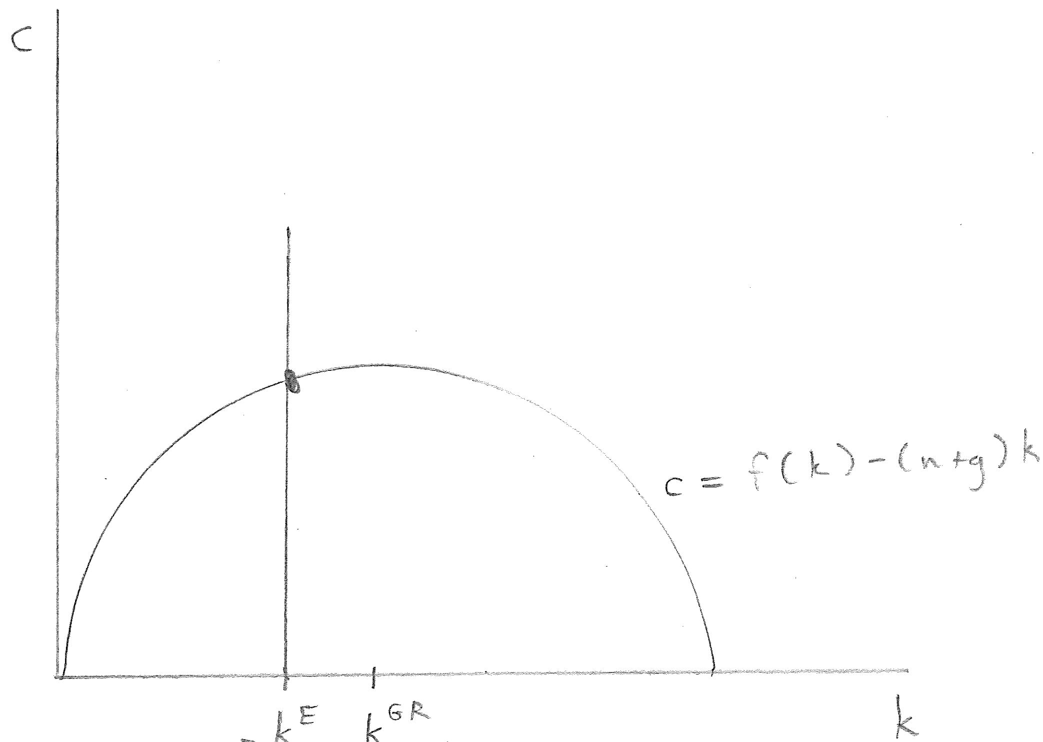
- high ρ
- low g
- high θ

rapidly diminishing MU, so high future consumption (caused by g) has smaller contribution to felicity & utility

RAMSEY-CASS-KOOPMANS

PRODUCTIVITY GROWTH (cont.)

Graph



$f'(k^{GR}) = n+g$

$f'(k^E) = \rho + n + \theta g$

How do I know $k^{GR} > k^E$

which is to say $f'(k^{GR}) < f'(k^E)$?

$$\Rightarrow n+g < \rho + n + \theta g$$

$$\Rightarrow g < \rho + \theta g$$

$$0 < \rho - g + \theta g$$

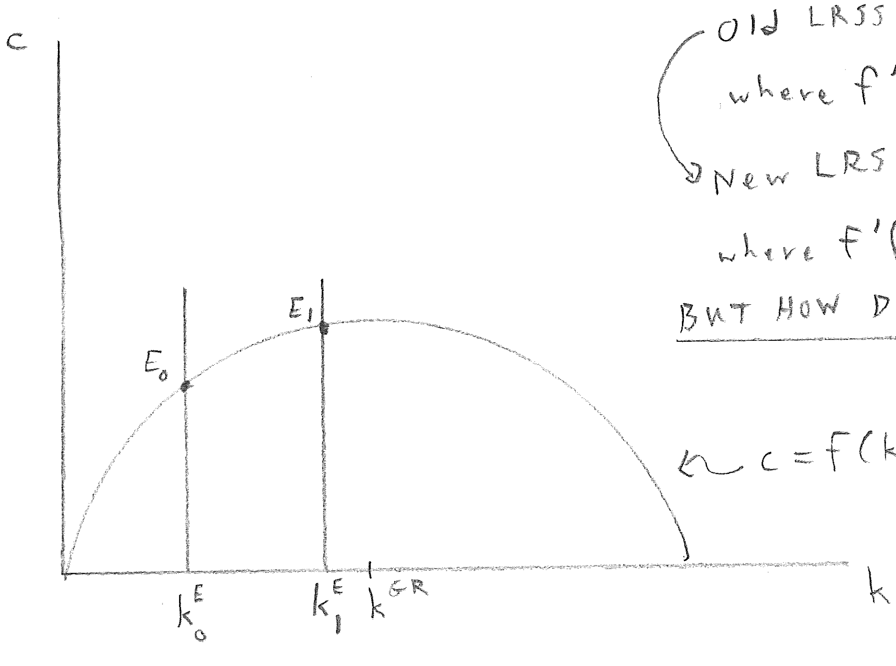
$$0 < \rho - (1-\theta) g$$

the necessary condition on parameters

RAMSEY–CASS–KOOPMANS

USING THE MODEL: A FALL IN DISCOUNT RATE ρ

What happens if ρ falls from ρ_0 to ρ_1?



Old LRSS: $k = k_0^E$

where $f'(k_0^E) = \rho_0 + n + \theta g$

New LRSS: $k = k_1^E$

where $f'(k_1^E) = \rho_1 + n + \theta g$

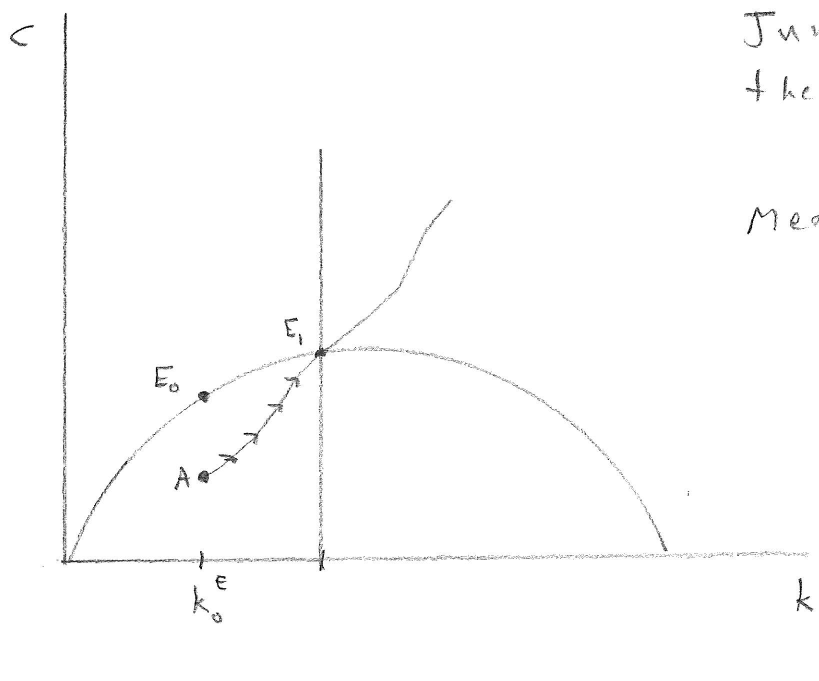
BUT HOW DO WE GET THERE?

$c = f(k) - (n+g)k$

when ρ changes, k remains fixed at k_0^E, but c can jump!

c jumps to put you on new equilibrium's saddle path,

& follow new saddle path to E_1.



when ρ falls,
Jump from E_0 to A
then crawl to E_1.

Meanwhile, $r = f'(k)$
falls gradually from
its original value to
its new LRSS value.

RAMSEY-CASS-KOOPMANS

USING THE MODEL: A FALL IN POPN. GROWTH RATE n

n falls from n_0 to n_1 $\quad n_1 < n_0$

Old LRSS: $f'(k_0^E) = \rho + n_0 + \theta g$

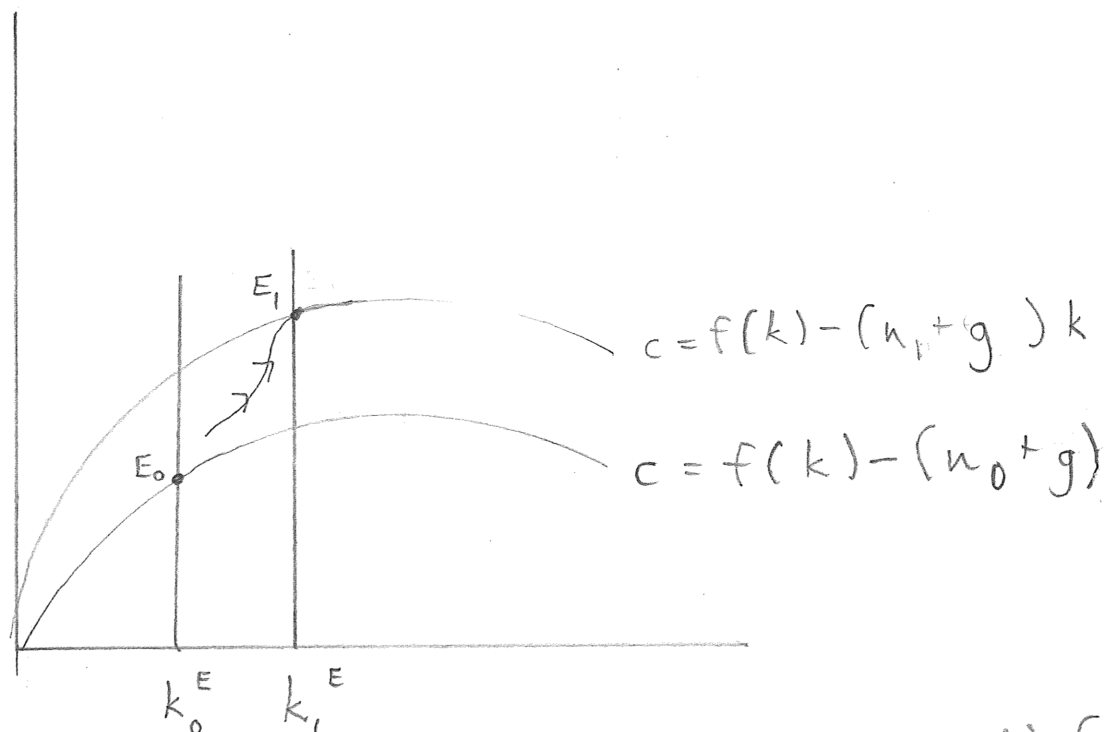
New LRSS: $f'(k_1^E) = \rho + n_1 + \theta g$

hence $f'(k_0^E) > f'(k_1^E)$

$k_0^E < k_1^E$

but also

$c = f(k) - (n+g)k$ ← this changes, too



Is E_0 above, below, or on new saddle path (to E_1)?

Don't know (it depends on....)

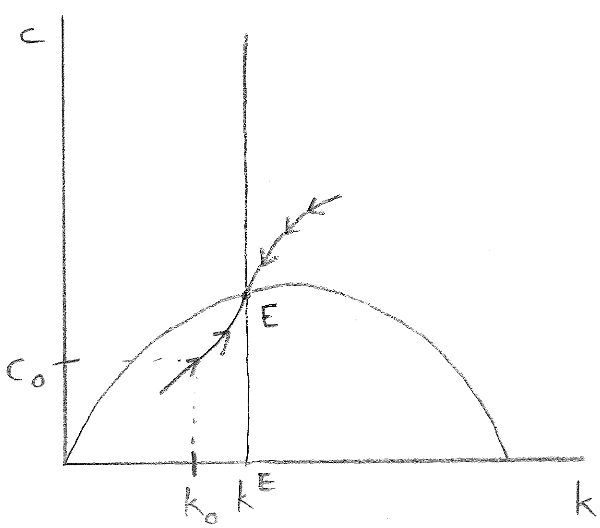
so c may rise, fall or stay same at moment n falls

RAMSEY–CASS–KOOPMANS MODEL

Rate of adjustment & slope of saddle path

Introduction

We know:



If a parameter changes (like ρ, A, n) then you have a new steady state (E) & a saddle path taking you there.

Given existing k, you jump to path...

But:

— How long does it take to get to E?

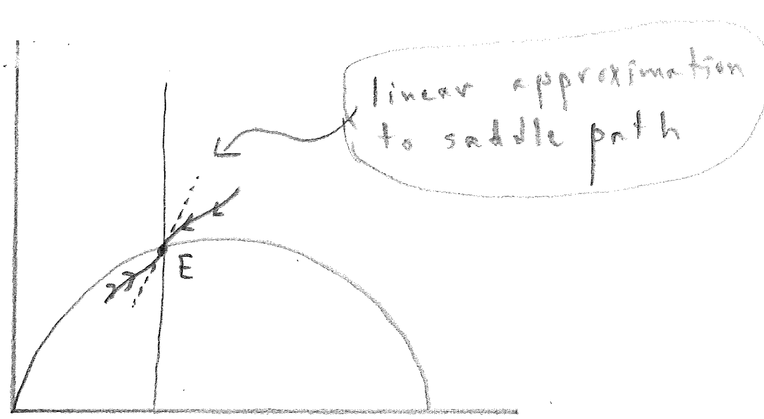
— What's slope of saddle path?

I.E., given $k_0 - k^E_{NEW}$, what's $c_0 - c^E_{NEW}$?

Approximate answers, ∂k for "small" $k_0 - k^E_{NEW}$:

— Get slope of saddle path at E

— Calculate rates of change ... at $E + \partial k$



R – C – K MODEL

Rate... (cont.)

Approximation of slope

Start with two expressions describing E:

$$f(k_t) = c_t + \dot{k}_t + nk \quad (\text{set } \dot{k}_t = 0 \text{ for } E)$$

$$\frac{\dot{c}}{c} = -\frac{u'(c_t)}{u''(c_t)}\left[f'(k) - n - \rho\right] \quad (\text{set } \dot{c} = 0 \text{ for } E)$$

Romer assumes $U(c_t) = \frac{c_t^{1-\theta}}{1-\theta} \Rightarrow \frac{-u'(c_t)}{u''(c_t)} = \frac{1}{\theta}$

$$\text{so } \frac{\dot{c}}{c} = \frac{1}{\theta}\left[f'(k_t) - n - \rho\right]$$

$$\Rightarrow \dot{c} = \frac{1}{\theta} c \left[f'(k_t) - n - \rho\right]$$

Near E, taking Taylor approximation,

$$\dot{c} \simeq \dot{c}^E + \frac{\partial \dot{c}}{\partial k}\left[k - k^E\right] + \frac{\partial \dot{c}}{\partial c}\left[c - c^E\right] \qquad (2.29)$$

$$\simeq 0 + \frac{f''(k)c}{\theta}\left[k - k^E\right] + \underbrace{\frac{\left[f'(k_t^E) - n - \rho\right]}{\theta}}_{= 0 \text{ because } f'(k_t^E) - n - \rho = 0}\left[c - c^E\right]$$

$$\simeq \frac{f''(k)c}{\theta}\left[k - k^E\right]$$

$$\Rightarrow \underbrace{\dot{(c - c^E)}}_{\text{rate at which } c \text{ converges to } c^E} = \underbrace{\dot{c}}_{\text{because } \dot{c}^E = 0:\ c^E \text{ is a fixed target}} = \frac{f''(k)c}{\theta}\left[k - k^E\right] \qquad (2.31)$$

R-C-K Model

Rate...

Approximation of slope

Near E, taking Taylor approximation,

$$\dot{k} \simeq \dot{k}^E + \frac{\partial \dot{k}}{\partial k}\left[k - k^E\right] + \frac{\partial \dot{k}}{\partial c}\left[c - c^E\right] \qquad (2.30)$$

Recall $\dot{k}_t = f(k_t) - c_t - nk$ so

$$\Rightarrow \dot{k} \simeq \underbrace{\dot{k}^E}_{\text{zero}} + \underbrace{\left[f'(k_t) - n\right]}_{\substack{\text{At } k^E,\ f'(k) = n + \rho \\ \text{so } f'(k_t^E) - n = \rho}}\left[k - k^E\right] - 1 \cdot \left[c - c^E\right]$$

$$\Rightarrow \dot{k} \simeq \rho\left[k - k^E\right] - \left[c - c^E\right]$$

$(k \dot{-} k^E) \simeq$ same $\qquad (2.32)$

Get growth rate of $(c \dot{-} c^E)$ (divide 2.31 by $c - c^E$)

$$\frac{(c \dot{-} c^E)}{c - c^E} \simeq \frac{f''(k^E)c^E}{\Theta}\frac{k - k^E}{c - c^E} \qquad (2.33)$$

Likewise, divide (2.32) by $k - k^E$:

$$\frac{k \dot{-} k^E}{k - k^E} \simeq \rho - \frac{c - c^E}{k - k^E} \qquad (2.34)$$

Slope of saddle path, or its inverse, which is $\simeq$ fixed

Note $\frac{c \dot{-} c^E}{c - c^E}$ & $\frac{k \dot{-} k^E}{k - k^E}$ must be < 0

(if $c > c^*$, $\dot{c} < 0$....)

R – C – K MODEL

~~Rate...~~

Rates of change in c & k

From (2.33), rearranging gives:

$$\frac{c-c^E}{k-k^E} \simeq \frac{f''(k^E)\,c^E}{\Theta}\,\frac{1}{\frac{\dot{(c-c^E)}}{c-c^E}} \qquad (2.35)$$

It must be true that

$$\frac{\dot{(c-c^E)}}{c-c^E} = \frac{\dot{(k-k^E)}}{k-k^E}$$

Speed at which c approaches c^E (or k approaches k^E) is proportional to distance $c-c^E$ (or $k-k^E$)

otherwise we couldn't reach c^E at same time we reach k^E.

Recall (2.34): $\dfrac{\dot{k-k^E}}{k-k^E} \simeq \rho - \dfrac{c-c^E}{k-k^E}$

From (2.33) $\dfrac{f''(k^*)c^*}{\Theta}\dfrac{1}{\frac{\dot{c-c^E}}{c-c^E}}$

hence if $\dfrac{\dot{k-k^E}}{k-k} = \dfrac{\dot{c-c^E}}{c-c} \Rightarrow \dfrac{\dot{c-c^E}}{c-c} \simeq \rho - \overbrace{\dfrac{c-c^E}{k-k^E}}$

so $$\frac{\dot{c-c^E}}{c-c} \simeq \rho - \frac{f''(k^E)\,c^E}{\Theta}\,\frac{1}{\frac{\dot{c-c^E}}{c-c^E}}$$

$$\left(\frac{\dot{c-c^E}}{c-c}\right)^2 - \rho\left(\frac{\dot{c-c^E}}{c-c^E}\right) + \frac{f''(k^E)\,c^E}{\Theta} = 0$$

This is a quadratic equation in $\left(\frac{\dot{c-c^E}}{c-c}\right)$. Apply quadratic formula:

R – c – K MODEL

Rate

Rates of change (cont.)

Recall $f''(k) < 0$

Solutions to quadratic formula:

$$\frac{\dot{\overline{c - c^E}}}{c - c^E} = \frac{\rho + \left[\rho^2 - \frac{4 f''(k^E) c^E}{\Theta}\right]^{1/2}}{2} > 0$$

Can't be this one, because $\frac{\dot{\overline{c - c^E}}}{c - c^E} < 0$

so it must be

$$\frac{\dot{\overline{c - c^E}}}{c - c^E} = \frac{\rho - \left[\rho^2 - \frac{4 f''(k^E) c^E}{\Theta}\right]^{1/2}}{2} \qquad (2.39)$$

$$= \frac{\dot{\overline{k - k^E}}}{k - k^E}$$

Also recall (2.35) $$\underbrace{\frac{c - c^E}{k - k^E}}_{\text{slope of saddle path}} = \frac{f''(k^E) c^E}{\Theta} \, \frac{1}{\underbrace{\frac{\dot{\overline{c - c^E}}}{c - c^E}}_{(2.39)}}$$

Thus we have expressions for

- slope of saddle path
- rate of change in c & k

RAMSEY – CASS – KOOPMANS

Rate

Calibration

Quantify it assuming $f(k) = k^{\alpha}$ (Cobb-Douglas).

What's $f''(k^E)$?

$$f'(k) = \alpha k^{\alpha-1} \qquad f''(k) = (\alpha-1)\alpha k^{\alpha-2}$$

$$\frac{[f'(k)]^2}{f(k)} = \frac{\alpha^2 k^{2\alpha-2}}{k^{\alpha}} = \alpha^2 k^{\alpha-2} = \frac{1}{\alpha-1}\alpha f''(k)$$

hence $f''(k) = \dfrac{\alpha-1}{\alpha}\dfrac{[f'(k^E)]^2}{f(k^E)}$ Recall $f'(k^E) = \rho + n$

multiply by -1

hence (2.39) becomes $\dfrac{\dot{c} - c^E}{c - c^E} = \dfrac{\rho - \left[\rho^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(\rho+n)^2 \frac{c^E}{f(k^E)}\right]^{1/2}}{2}$

Note $\dfrac{c^E}{f(k^E)} = 1 - s^E \leftarrow$ saving rate at k^E

$$s^E = \frac{n k^E}{k^{E\alpha}} = n k^{E^{1-\alpha}} = \frac{n}{k^{E^{\alpha-1}}} = \frac{\alpha n}{\rho + n}$$

$\frac{1}{\alpha} f'(k)$

$$\frac{\dot{c} - c^E}{c - c^E} = \frac{\dot{k} - k^E}{k - k^E} = \frac{\rho - \left[\rho^2 + \frac{4}{\theta}\left(\frac{1-\alpha}{\alpha}\right)(\rho+n)(\rho+n-\alpha n)\right]^{1/2}}{2}$$

To quantify, need numbers for ρ, θ, α and n.

R – C – K MODEL

Simple case

Calibration

Quantify it (cont.)

Propose "reasonable" values for parameters:

$\alpha = 1/3$

$\rho + n = r^E$ = real interest rate (?)

$\rho = r^E - n$ = real interest rate - population growth rate

Θ = Coefficient of relative risk aversion

$$\left(\text{recall } u(c) = \frac{c^{1-\Theta}}{1-\Theta} \right.$$

and

if Θ big, consumption is not substitutable between periods
and c (& s) respond little to $f'(k)$)

Whatever values you choose, adjustment will be much more rapid than Solow model, because in Solow s is fixed, while here s is bigger the further k_0 is below k^E.

RAMSEY-CASS-KOOPMANS

Government purchases

G_t Government consumption of output, per unit of effective labor

Government takes part of total output:

$$f(k_t) = c_t + \dot{k}_t - (n+g)k_t + G_t \qquad (2.41)$$

Household lifetime budget constraint becomes:

$$\int_{t=0}^{\infty} e^{-R(t)} c_t e^{(n+g)t} dt = k_0 + \int_{t=0}^{\infty} e^{-R(t)} [w_t - T_t] e^{(n+g)t} dt$$

lump-sum tax paid by household in period t, per unit of AL

No-Ponzi-game constraint for government:

$$\underbrace{\int_{t=0}^{\infty} e^{-R(t)} G_t e^{(n+g)t} dt}_{\text{p.d.v. spending}} = \underbrace{\int_{t=0}^{\infty} e^{-R(t)} T_t e^{(n+g)t} dt}_{\text{p.d.v. taxes}}$$

R – C – K

Government purchases constant G

How government changes the phase diagram

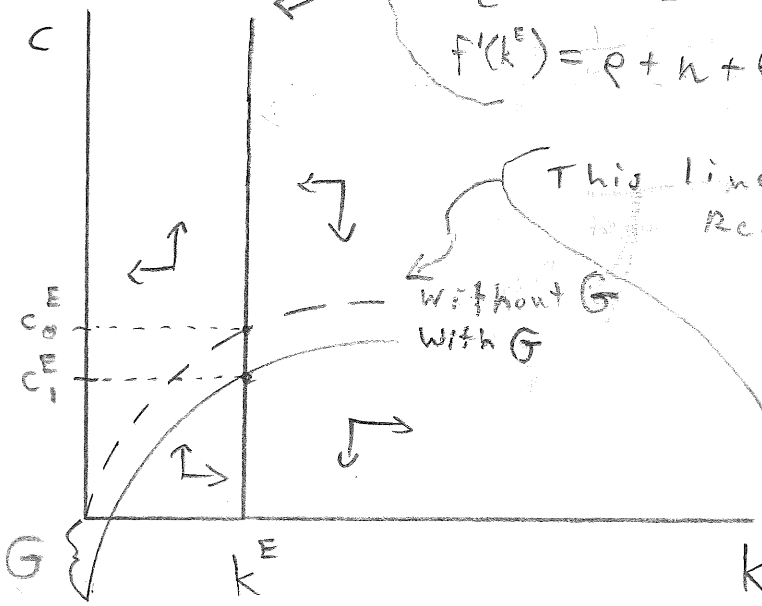
This line is same as before (without G)
Recall it expresses Euler equation:

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left[f'(k_t) - \rho - n\right] \quad \text{in steady state } \frac{\dot{c}}{c} = g$$

c.A

$$f'(k^E) = \rho + n + \theta g$$

position of line



This line is different
Recall it expresses division of output

$$f(k) = \dot{k} + (n+g)k + c + G \quad \leftarrow \text{new}$$

in steady state $\dot{k} = 0$

$$0 = f(k) - c - (n+g)k - G$$

$$c = f(k) - (n+g)k - G$$

Result of adding (subtracting) G:

- c^E falls by G
- k^E unchanged
- Lower saddle path

Things to note:

- Euler equation didn't change because we assumed G has no effect on household's utility. Realistic?
- "Solow line" shifted down by G because we assumed G has no effect on production. Realistic?

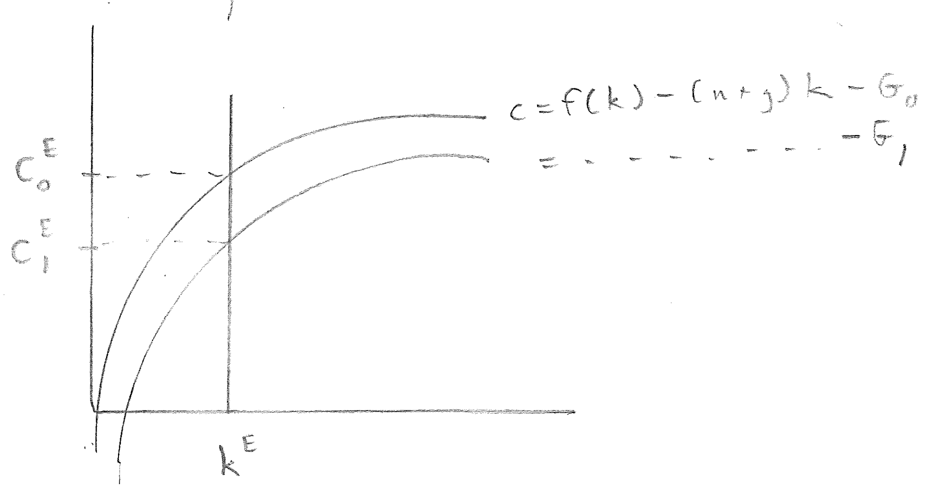
R-C-K Model

Unexpected vs. expected changes in parameters

So far, we've been talking about changes in parameters that were unexpected, came as a surprise to people in model.

Why do I say this?

Example: increase in G



C jumps down from C_0 to C_1

But along any planned consumption path

$$\frac{\dot{c}}{c} = \frac{1}{\theta}\left(f'(k) - \rho - n\right) \quad \text{Euler equation}$$

A jump in C would mean $\frac{\dot{c}}{c} = \infty$ or $-\infty$

Ruled out by Euler equation!

So this ΔC must have be unplanned: change in G came as a surprise.

? What happens if people expect a future change in parameter (for example a future change in G)?

$\Rightarrow$ C can jump at moment we change forecast of future parameter, & form new plan, but not after that!

R - C - K MODEL

Effect of a temporary hike in government spending

G has been G_P. At t_0, we get news:

G has risen to $G_W > G_P$. At future time t_1, G will return to G_P

Things to keep in mind:

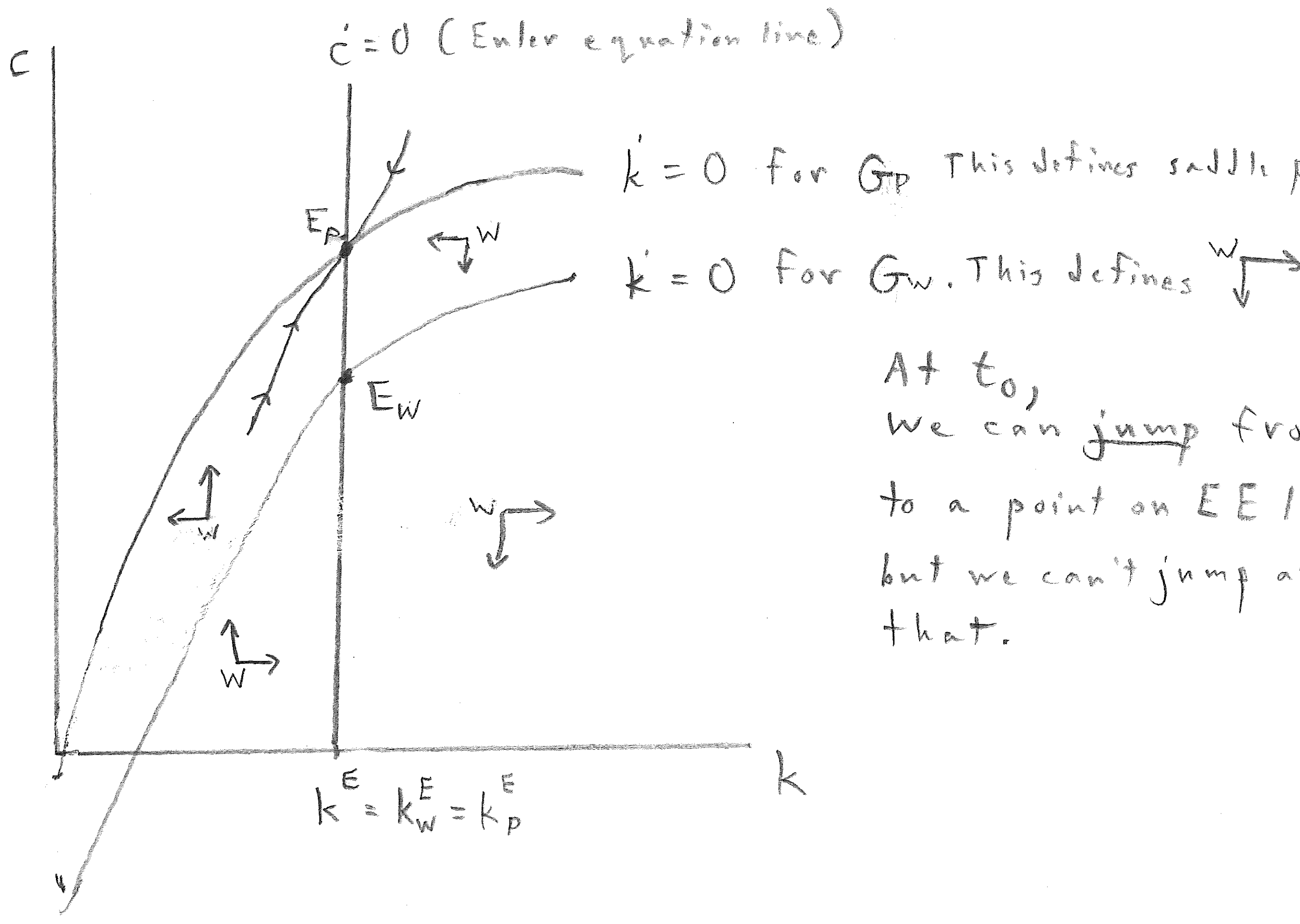
1) $r = f'(k)$ cannot jump because k cannot jump, k must evolve

2) c can jump only on receipt of news;

you cannot plan a jump from t to $t + dt$

because on any planned path $\frac{\dot{c}}{c} = \frac{1}{\theta}[f'(k) - \rho - n]$

& $f'(k)$ evolves slowly.

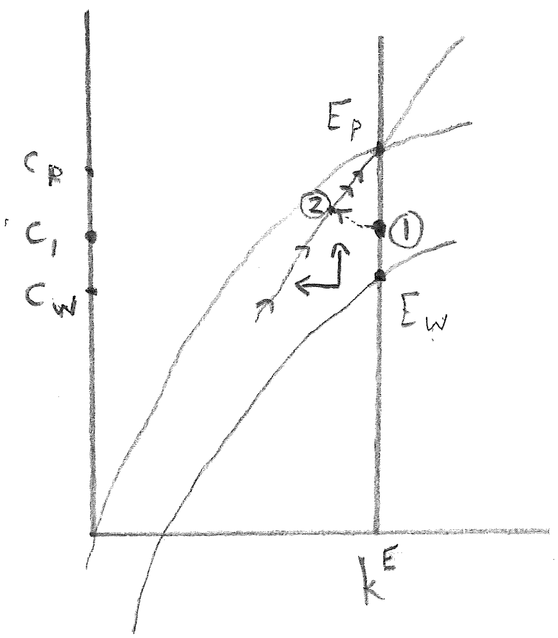


At t_0,
we can jump from E_P
to a point on EE line,
but we can't jump after
that.

R – C – K Model

Effect of a temporary hike in government spending (cont.)

Path must be:



At t_0,
Jump from E_P to ①; then
crawl according to $\leftarrow\!\uparrow_W$;
hit ② at t_1; then
crawl on saddle path to E_P.

That's the only way to get back to E_P (eventually) without jumps other than the jump at t_0

Why?

- You must be at saddle path as of t_1, or you'll never get back to E_P

- At t_0 you must jump to a point on k^E line that will cause you to be on saddle path as of t_1, given that you move according to $\leftarrow\!\uparrow_W$ in period from t_0 to t_1

- $C_1 < C_P$ ∵ news lowers lifetime income (net of taxes) so lifetime consumption must fall, while level of consumption must eventually return to C_P

R-C-K MODEL

Effect of a temporary hike in government spending (cont.)

Intuition: why does k fall during period of high G?

Saving less, letting k fall now transfers consumption from the future to the present.

Benefit: reduction in current consumption caused by temporarily high G is not as bad as it would be if you tried to maintain capital stock.

Cost: you lose future consumption, & also reduce p.d.v. of lifetime consumption, relative to maintaining capital stock (because dip in k means less lifetime y)

Why is the benefit worth the cost?

Because $U''(C) < 0$ (a dip in consumption is very painful)

What happens to interest rate during period of high G?

$r = f'(k)$

k falls, then rises to original level,

so r rises, then falls to original level.