Solow Model

Introduction

Assumptions

Aggregate production function

$F(K, AL)$ (1.1)

Labor-augmenting (Harrod-neutral) progress

CRS

Diminishing MPK

"Inada conditions" (we'll get back to these later)

[Cobb-Douglas meets these requirments but so do some others e.g. CES ("constant elasticity of substn) production function $Y = (K^\rho + (AL)^\rho)^{1/\rho}$]

Saving

$I = S = sY$ s is fixed, exogenous fraction (e.g. $\frac{1}{10}$)

so $\dot{K}(t) = sY(t) - \delta K(t)$ (1.16)

$C(t) = (1-s)Y(t)$

fixed fraction

Notice: $\dot{K}(t) = \underbrace{sF(K(t), A(t)L(t))}_{\text{positive but diminishing function of K: "concave"}} - \underbrace{\delta K(t)}_{\text{negative \& linear function of K}}$

Solow Model

Intro (cont.)

Behavior of L & A

Exogenous.
Fixed values or grow at steady rates.

Big ideas illustrated by model

These will also hold in most other models we look at this semester

1) Long-run equilibrium or "long-run steady state."
Economy can be out of equilibrium at a point in time, but over time it will converge to LRSS.

2) "Balanced growth path." In LRSS, some variables fixed, but most are growing at steady rates.
Many grow at same rate (hence "balanced growth").

3) Higher saving rates raises LRSS Y but does not raise growth rate of Y. LRSS growth in Y or Y/L is exogenous, determined by growth rate of A.

4) Saving rate can be too low/high (if goal is maximization of C/L).
Optimal saving rate (maxes C/L): "Golden rule"

Solow Model

Introduction

Notation

Romer uses $X(t)$, I'll use X_t

Intensive-form production function

It's useful to transform $Y = F(K, AL)$

into $y = f(k)$

$\left(\frac{Y}{AL}\right.$ $\qquad$ $\left(\frac{K}{AL}\right.$

so that $Y = ALy = ALf(k)$

AL is called "efficiency units of labor" so

K/AL is "capital per eff. unit. of labor."

This transformation possible for any CRS fn.

Won't work for others.

CRS example:

$Y = K^{\alpha}(AL)^{1-\alpha}$ $\qquad$ $\frac{Y}{L} = K^{\alpha}(AL)^{1-\alpha}\frac{1}{L} = A^{1-\alpha}K^{\alpha}L^{-\alpha} = A^{1-\alpha}k^{\alpha}$

ok!

Non-CRS example:

$Y = K^{1/2} + AL$ $\qquad$ $\frac{Y}{L} = K^{1/2}L^{-1} + A = k^{1/2}L^{-1/2} + A$

Not ok!

Solow Model (cont.)

Notation in "per efficiency-unit of labor"

It's useful to transform $Y = F(K, AL)$ ← agg. prodn. fn.

into "intensive-form production function"

$$y = f(k)$$

k ← $\frac{K}{AL}$ capital per e-unit of labor

y ← $\frac{Y}{AL}$ output per e-unit

so that

$$Y = ALy = ALf(k)$$

How do we know this is mathematically possible?

It's not possible for

$$Y = K^2 (AL)^3$$

$$y = \frac{Y}{AL} = K^2 (AL)^2 = (AL)^2 (AL)^{-2} K^2 (AL)^2 = (AL)^4 \left(\frac{K}{AL}\right)^2$$

↖ k

$$Y = ALy = (AL)^5 f(k)$$ ← k^2

Also not possible for

$$Y = K^2 + (AL)$$

$$y = \frac{Y}{AL} = \frac{K^2}{AL} + 1 = AL \frac{1}{AL^2} K^2 + 1 = AL\left(\frac{K}{AL}\right)^2 + 1$$

↖ k

etc.

It's an implication of CRS!

Solow Model (cont.)

Notation in... (cont.)

CRS says $F(cK, cAL) = cF(K, AL)$

hence c

$F\left(\frac{1}{AL}K, \frac{1}{AL}AL\right) = \frac{1}{AL}F(K, AL)$

$\underbrace{F(k, 1)}_{f(k)} \qquad = \frac{1}{AL}Y = y$

Take your agg. prodn. fn., stick k in spot where K goes and one in spot where L goes

and that's $f(k)$!

Example:

$Y = aK + bAL$ ← (has CRS

same thing

$y = ak + b \cdot 1 = f(k)$

Does $ALf(k) = Y$?

Yes! $AL(ak + b) = aAL\frac{K}{AL} + bAL = aK + bAL$

Solow Model

Notation in.... (cont.)

Another thing about $f(k)$:

$$f'(k) = F_K(K, AL) = \frac{\partial F(\ )}{\partial K} \quad \text{MPK}$$

How do I know? If $Y = AL\,f(k)$,

$$\frac{\partial Y}{\partial K} = \frac{\partial (AL\,f(k))}{\partial K}$$

$$= AL\,\frac{\partial f(k)}{\partial K}$$

$$= AL\,f'(k)\,\frac{\partial k}{\partial K}$$

Since $k = \frac{1}{AL}K$, this is $\frac{1}{AL}$

$$\frac{\partial Y}{\partial K} = AL\,f'(k)\,\frac{1}{AL} = f'(k)$$

Also, $F_{KK}(K, AL) = \dfrac{\partial(\partial Y/\partial K)}{\partial K} = \dfrac{\partial(f'(k))}{\partial K}$

$$= f''(k)\,\frac{\partial k}{\partial K} = f''(k)\,\frac{1}{AL}$$

Since we assumed $F_K(K, AL) > 0$

$F_{KK}(K, AL) < 0$

then we know $f'(k) > 0$

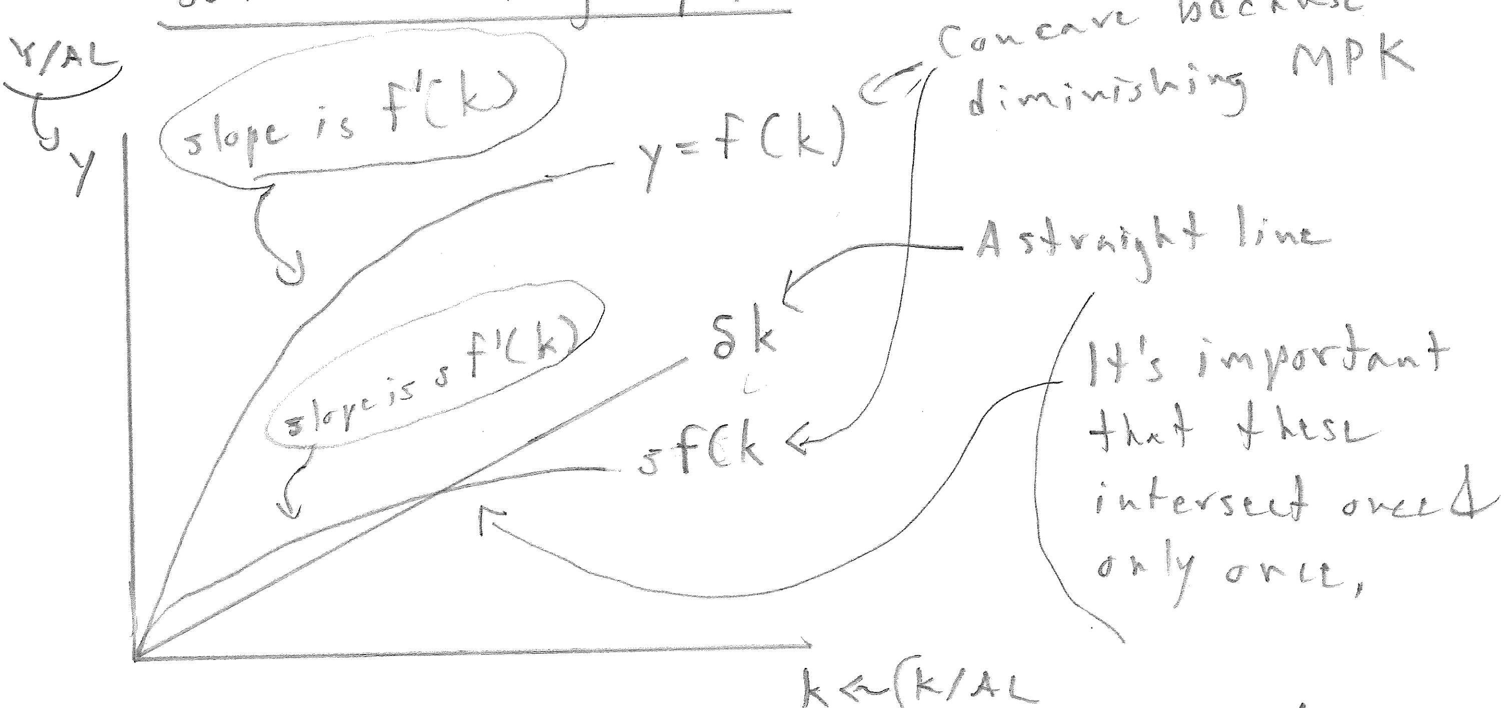
$f''(k) < 0$

$f(k)$ is <u>concave</u> fn of k.

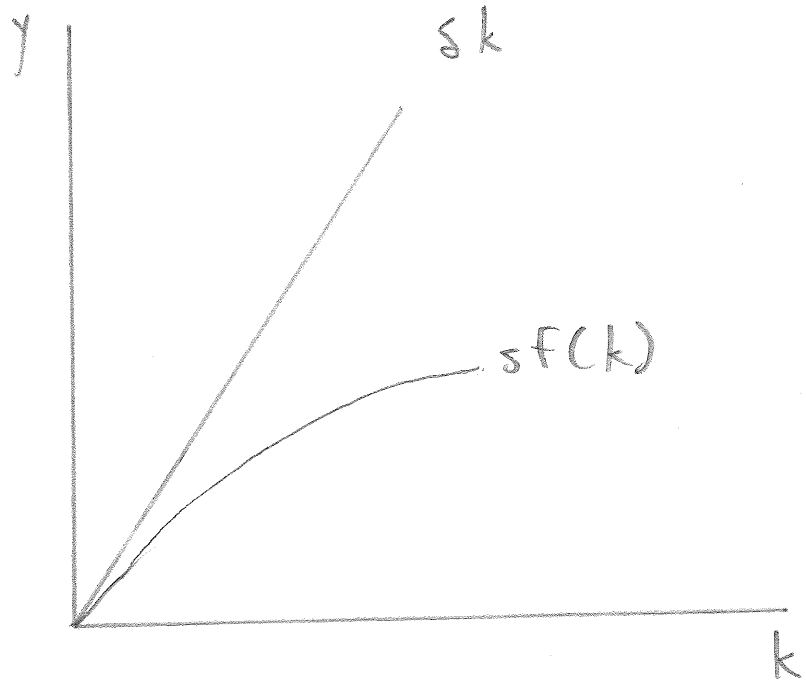
Solow model

Intro...

Solow model graph



To make δk & sf(k) intersect, we need sf'(k) to be __big__ for small k.



BAD!

How do we rule out this bad thing?

With assumptions...

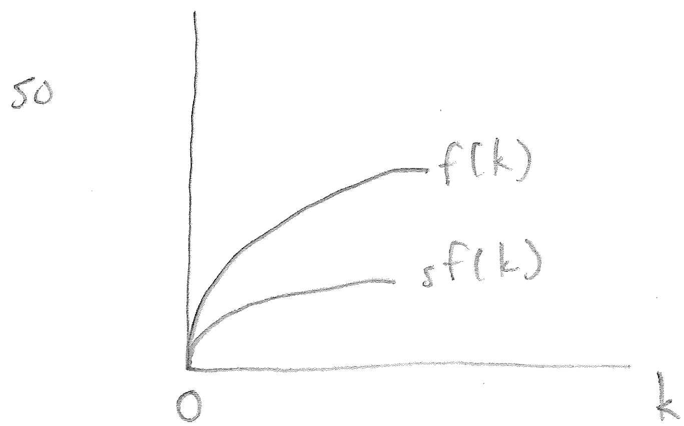
Solow Model

Intro...

Inada conditions

$$\lim_{k \to 0} f'(k) = \infty$$

$$\lim_{k \to \infty} f'(k) = 0$$



Holds for Cobb-Douglas and some others (e.g. CES)

Solow Model

Simple case: no growth in A or L

Say $A = 1$

then $y = \frac{Y}{L}$ $\quad k = \frac{K}{L}$ $\quad c = \frac{C}{L}$

"per capita"

Recall $\dot{K} = sY - \delta K$

$$\frac{1}{L}\dot{K} = s\frac{1}{L}Y - \delta\frac{1}{L}K = sy - \delta k$$

If AL fixed,

$$\dot{k} = \frac{\partial(K/AL)}{\partial t} = \frac{1}{AL}\frac{\partial K}{\partial t} = \frac{1}{AL}\dot{K}$$

with $A = 1$,

$$\dot{k} = \frac{1}{L}\dot{K} = sy - \delta k$$

Recall $C = (1-s)Y$

$$\frac{C}{AL} = (1-s)\frac{Y}{AL}$$

$$c = (1-s)y$$

$$\dot{c} = (1-s)\dot{y}$$

If AL fixed,

$$\dot{c} = \frac{\partial(C/AL)}{\partial t} = \frac{1}{AL}\frac{\partial C}{\partial t} = \frac{1}{AL}\dot{C}$$

with $A = 1$,

$$\dot{c} = \frac{1}{L}\dot{C}$$

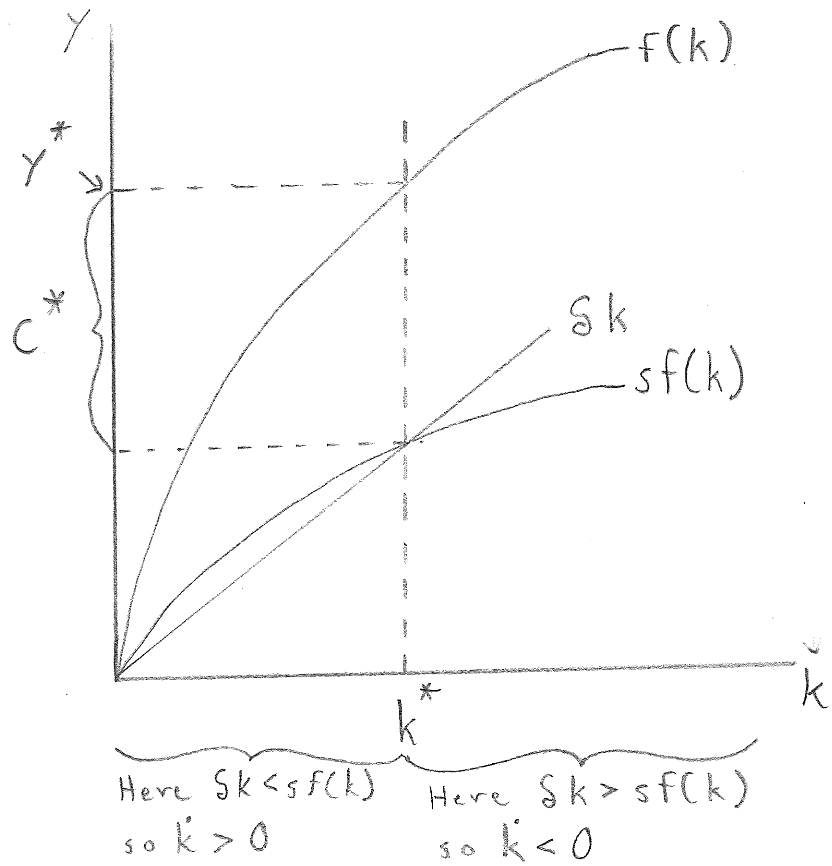
growth in total consumption

Solow model

Equilibrium with fixed A & L

Growth in capital per worker:

$$\dot{k}_t = sf(k_t) - \delta k_t$$



LOOK! There's an equilibrium k^*, y^*, c^*

For which $sf(k^*) = \delta k^*$

or $k^* = s \dfrac{f(k^*)}{\delta}$

and $c^* = f(k^*) - sf(k^*)$

NOTE! Values of k^*, y^*, c^* depend on s.

"Golden Rule" Capital Stock

Value of k^* that maximizes c^*. Call it k^{GR}

$$\max_{k^*} c^* = f(k^*) - sf(k^*)$$

In long-run equilibrium, $sf(k) = \delta k$ so problem becomes

$$\max_{k^*} c^* = f(k^*) - \delta k^*$$ Take derivative (first order condition)

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - \delta$$

Hence $f'(k^{GR}) = \delta$ MPK (real interest rate) equals depreciation rate

Solow Model

Equilibrium with fixed A, growing L

n Population growth rate $= \dfrac{\dot{L}_t}{L_t}$ ← Change in L per unit time; L_t ← Level of L at that instant

$\dot{L}_t = n L_t$

$L_t = L_0 e^{nt}$ ← t: amount of time from beginning to now; L_0: initial value of L, at beginning of time

Application of a math rule.

Proof it's true: Differentiation rule $\dfrac{\partial e^{g(x)}}{\partial x} = e^{g(x)} f'(x)$

If $L_t = L_0 e^{nt}$ then should be true that

$$n L_t = \dot{L}_t = \frac{\partial (L_0 e^{nt})}{\partial t} = \underbrace{L_0 e^{nt}}_{L_t} n \quad \text{True!}$$

What's $\dot{k}_t$ if L growing?

Recall $\dot{K}_t = s Y_t - \delta K_t$

Differentiation rule $g(h(x), j(x))$

$$\frac{\partial g}{\partial x} = g_h \cdot h'(x) + g_j \cdot j'(x)$$

Apply to $k = K/AL$ ← L growing, A fixed

$$\dot{k}_t = \frac{\partial k}{\partial K} \dot{K}_t + \frac{\partial k}{\partial L} \dot{L}_t = \frac{1}{AL}\overbrace{(sY_t - \delta K_t)}^{\dot{K}_t} - \frac{K}{AL^2} \overbrace{n L_t}^{\dot{L}_t}$$

$$= s \frac{Y}{AL} - \delta \frac{K}{AL} - n \frac{K}{AL}$$

$$= sy - \delta k - nk$$

$$= s f(k) - (\delta + n) k$$

SOLOW MODEL

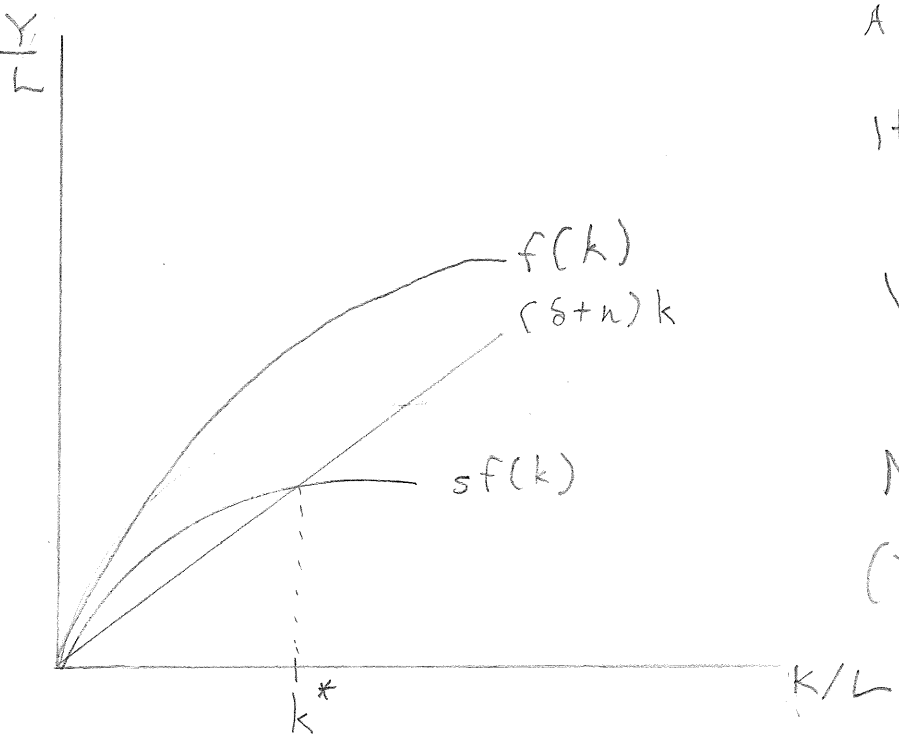
Equilibrium with fixed A, growing L (cont.)

Intuition for $\dot{k}_t = sf(k) - (\delta + n)k$

As L grows, any quantity of K must be shared among more workers, reducing quantity of equipment any worker has.

As K shrinks, same thing happens.

Equilibrium & Golden-rule equilibrium



At k^*, $sf(k^*) = (\delta + n)k^*$

If $n \uparrow$ (higher popu. growth)

$k^* \downarrow$, $y^* \downarrow$, $c^* \downarrow$

Values of k^*, y^* depend on s vs. n

Note: $\frac{Y}{L}$ $\frac{K}{L}$ $\frac{C}{L}$ fixed in LRE

(Y, K, C, L all growing at same rate n)

Golden Rule:

$$\max_{k^*} c^* = f(k^*) - sf(k^*) \quad \text{where } sf(k^*) = (\delta + n)k^*$$

$$\frac{\partial c^*}{\partial k^*} = f'(k^*) - (\delta + n)$$

At $\frac{\partial c^*}{\partial k^*} = 0$, $f'(k^{GR}) = \delta + n$

Solow Model

With growth in A & L

g Growth rate for $A = \frac{\dot{A}_t}{A}$

$$\dot{A}_t = g A_t \qquad A_t = A_0 e^{gt}$$

What's $\dot{k}_t$ if A & L growing?

$$\dot{K}_t = sY_t - \delta K_t$$

$$k = K/AL$$

$$\dot{k}_t = \frac{\partial k}{\partial K}\dot{K}_t + \frac{\partial k}{\partial L}\dot{L}_t + \frac{\partial k}{\partial A}\dot{A}_t$$

$$= \frac{1}{AL}(sY_t - \delta K_t) - \frac{K}{AL^2} n L_t - \frac{K}{A^2 L} g A_t$$

$$= s\frac{Y}{AL} - \delta\frac{K}{AL} - \frac{K}{AL} n - \frac{K}{AL} g$$

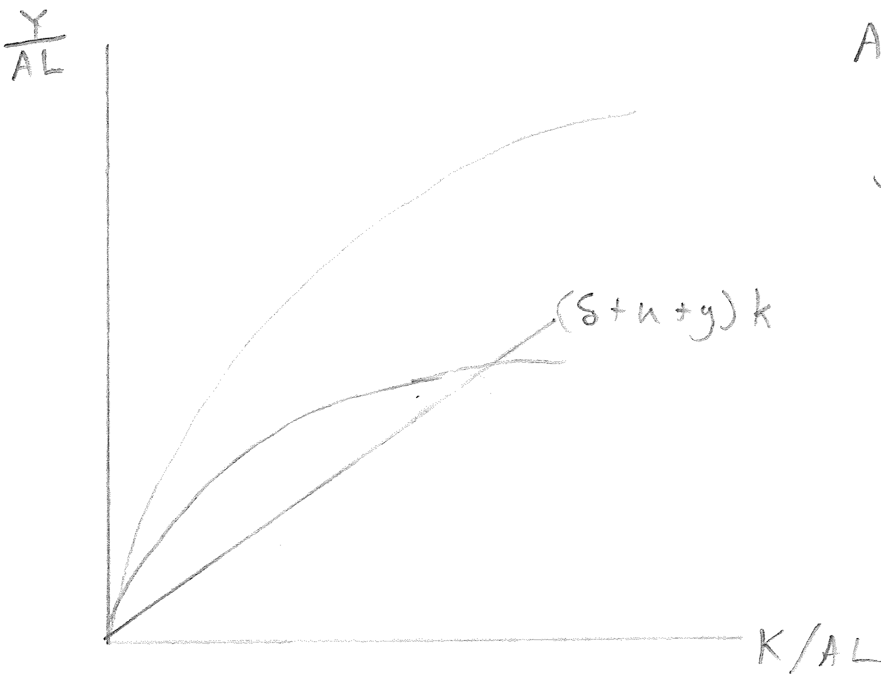
$$= sy - \delta k - nk - gk$$

$$= sf(k_t) - (\delta + n + g)k_t \qquad (1.19)$$

What matters in model is $(A \cdot L)$ ← efficiency-units of labor

so growth in A looks like growth in L.

Solow Model
With growth in A & L
Equilibrium & Golden Rule



At k^*, $sf(k^*) = (\delta+n+g)k^*$

$g\uparrow$ is like $n\uparrow$

Golden rule:

$f'(k^{GR}) = \delta + n + g$

In every LRE, $\frac{Y}{AL}$ $\frac{K}{AL}$ $\frac{C}{AL}$ fixed

but that does not mean

$\frac{Y}{L}$ $\frac{K}{L}$ $\frac{C}{L}$ fixed!

In LRE, what's happening to

K, Y, C, and $\frac{K}{L}$, $\frac{Y}{L}$, $\frac{C}{L}$?

Describe in terms of growth rates

$\frac{\dot{K}_t}{K_t}$ $\frac{\dot{Y}_t}{Y_t}$ etc.

Solow Model with Growth in A & L

LRE evolution of observable variables

What is $\frac{\dot{K}}{K}$ in LRE?

Recall $$\dot{k}_t = \frac{1}{AL}\dot{K}_t - \frac{K}{AL^2}\dot{L}_t - \frac{K}{A^2L}\dot{A}_t$$

$$= \frac{1}{AL}\dot{K}_t - \frac{K}{AL}\frac{\dot{L}}{L} - \frac{K}{AL}\frac{\dot{A}}{A}$$

In LRE, $\dot{k}_t = 0$. So in LRE

$0 =$ etc. Solve equation for $\frac{\dot{K}}{K}$

$$\frac{1}{AL}\dot{K}_t = \frac{K}{AL}\frac{\dot{L}}{L} + \frac{K}{AL}\frac{\dot{A}}{A}$$

Multiply by AL, divide by K

$$\dot{K}/K = \frac{\dot{L}}{L} + \frac{\dot{A}}{A} = n + g$$

What's $\dot{Y}/Y$?

$$\frac{Y}{AL} \rightarrow \dot{y}_t = \frac{\partial y}{\partial Y}\dot{Y}_t + \frac{\partial y}{\partial L}\dot{L}_t + \frac{\partial y}{\partial A}\dot{A}_t$$

$$= \frac{1}{AL}\dot{Y}_t - \frac{Y}{AL^2}\dot{L}_t - \frac{Y}{A^2L}\dot{A}_t$$

Set $\dot{y}_t = 0$ & solve for $\dot{Y}/Y$

$$\dot{Y}/Y = g + n$$

What's $\dot{C}/C$?

Recall $C = (1-s)Y$

$$\dot{C} = (1-s)\dot{Y}$$

$$\frac{\dot{C}}{C} = \frac{(1-s)\dot{Y}}{C} = \frac{(1-s)\dot{Y}}{(1-s)Y} = \frac{\dot{Y}}{Y} = n + g$$

Solow Model with Growth in A&L

LRE evolution of observable variables (cont.)

What about C/L, Y/L, $\frac{K}{L}$?

For any x_1, x_2,

$$\dot{(x_1/x_2)} = \frac{1}{x_2}\dot{x}_1 - \frac{x_1}{x_2^2}\dot{x}_2$$

$$\frac{\dot{(x_1/x_2)}}{x_1/x_2} = \frac{\dot{x}_1}{x_1} - \frac{\dot{x}_2}{x_2}$$

so

$$\frac{\dot{(C/L)}}{C/L} = \frac{\dot{C}}{C} - \frac{\dot{L}}{L} = g+n-n = g$$

$$\frac{\dot{(Y/L)}}{Y/L} = \frac{\dot{Y}}{Y} - \frac{\dot{L}}{L} = g$$

$$\frac{\dot{(K/L)}}{K/L} = g$$

$\Rightarrow$ In LRE, rate of growth in output, consumption, capital per worker completely determined by rate of growth of "technology"

but s & n affect levels of Y/L, etc.

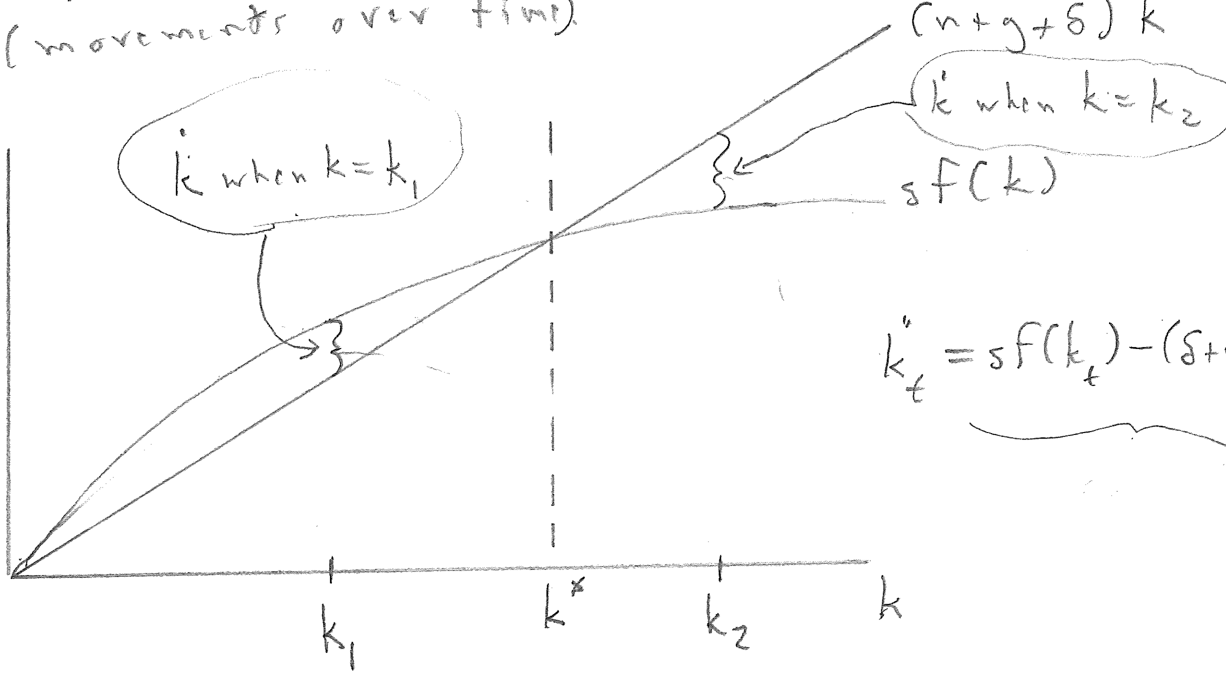
$s\uparrow \rightarrow \frac{Y}{L}\uparrow, \frac{K}{L}\uparrow, \frac{C}{L}$ ← depends on whether you're above/below Golden Rule.

$n\uparrow \rightarrow \frac{Y}{L}\downarrow, \frac{K}{L}\downarrow, \frac{C}{L}\downarrow$ ← try it on a graph

Solow Model

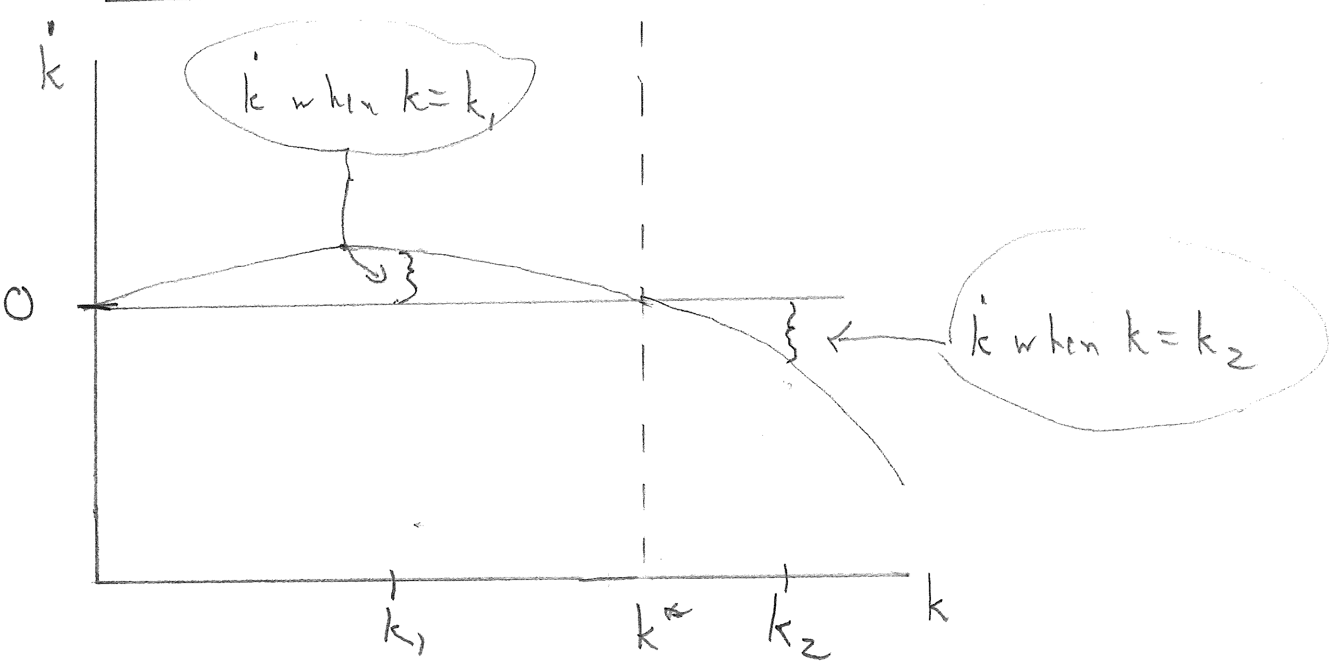
Phase diagram

A phase diagram is a graph that shows dynamics (movements over time)



$$\underbrace{\dot{k}_t = sf(k_t) - (\delta+n+g)k_t}$$

phase diagram



Solow Model with Growth in A & L

What is effect on y^* of Δs?

An approximation that may be OK for small Δs:

calculate $\partial y^*/\partial s$

$$\frac{\partial y^*}{\partial s} = f'(k)\frac{\partial k^*}{\partial s}$$

So what is $\frac{\partial k^*}{\partial s}$?

At k^*, $s f(k^*) = (n+g+\delta) k^*$

Means

$$s = \frac{(n+g+\delta) k^*}{f(k^*)}$$

Take derivative of both sides w.r.t. s

$$s f'(k^*)\frac{\partial k^*}{\partial s} + f(k^*) = (n+g+\delta)\frac{\partial k^*}{\partial s} \quad (1.24)$$

$$\Longrightarrow \frac{\partial k^*}{\partial s} = \frac{f(k^*)}{(n+g+\delta) - s f'(k^*)} \quad (1.25)$$

$$\text{Thus } \frac{\partial y^*}{\partial s} = f'(k)\frac{f(k^*)}{(n+g+\delta) - s f'(k^*)} \quad (1.26)$$

Multiply both sides by (s/y) to get:

$$\underbrace{\frac{\partial y^*/y}{\partial s/s}}_{\text{Elasticity}} = \frac{s}{\underset{\uparrow}{y}}\,\frac{f'(k^*)\,f(k^*)}{(n+g+\delta) - s f'(k)}$$

$f(k)$

Solow Model with Growth in A & L

What is effect on y^* of Δs? (cont.)

$$\frac{\partial y^*/y}{\partial s/s} = \frac{(n+g+\delta)k^*/f(k^*)}{f(k^*)} \frac{f'(k^*)f(k^*)}{(n+g+\delta) - \frac{(n+g+\delta)k^*}{f(k^*)} f'(k^*)}$$

$$= \frac{(n+g+\delta)k^* f'(k^*)f(k^*)/f(k^*)}{f(k^*)(n+g+\delta) - (n+g+\delta)k^* f'(k^*)} \qquad (1.27)$$

Divide top & bottom by $(n+g+\delta)f(k^*)$

$$= \frac{k^* f'(k^*)/f(k^*)}{1 - k^* f'(k^*)/f(k^*)} \qquad (1.27)$$

What is $\dfrac{k^* f'(k^*)}{f(k)} = \dfrac{\frac{K}{AL} \cdot MPK}{\frac{Y}{AL}} = \dfrac{K \cdot MPK}{Y}$

If return to $K = MPK$ (perfect markets)

then $K \cdot MPK$ = total income received by capital

= GDP — labor income

hence $\dfrac{K \cdot MPK}{Y} = \dfrac{\text{GDP} - \text{labor income}}{\text{GDP}} \approx \dfrac{1}{3}$

hence $\dfrac{\partial y^*/y}{\partial s/s} = \dfrac{1/3}{1 - 1/3} = 1/2$

That is, $\% \Delta$ in LRSS output is about half as big as $\% \Delta$ in savings rate.

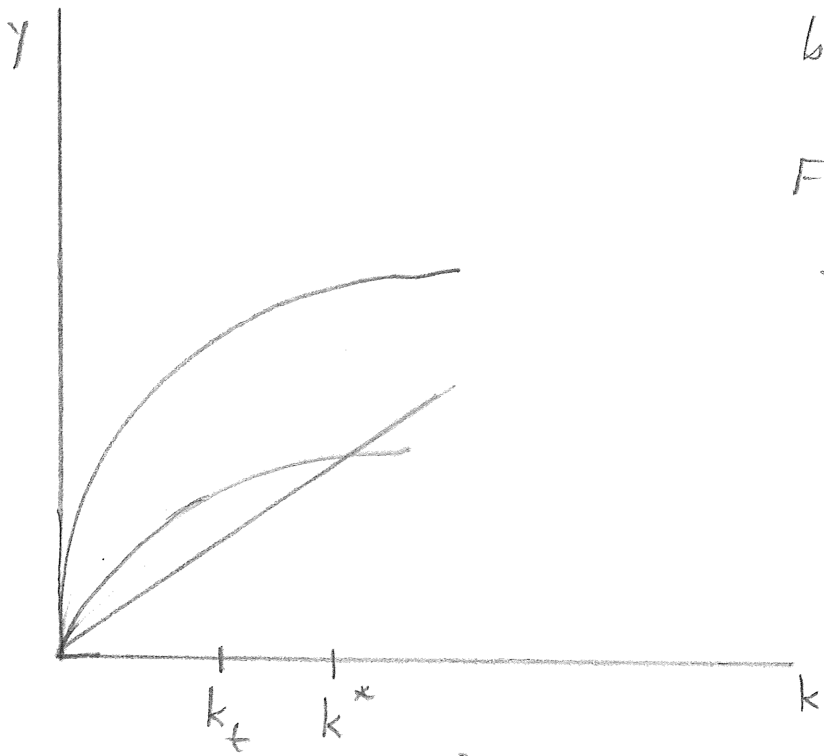
Double s ($\% \Delta = 100$) $\longrightarrow$ output about 50% higher.

Solow Model with Growth in A & L

A change in LRSS

At time t we're below LRSS k^*, at k_t

For example, k_t was old LRSS, then
$s \uparrow$ or
$n \downarrow$ or
$g \downarrow$ or
capital destroyed
immigrants arrive



Note: $(k_t - k^*) < 0$

Recall: for small ΔX,

$$F(X + \Delta X) \approx F(X) + F'(X)\Delta X$$

Solow Model with Growth in A & L

If $k^* \uparrow$, how fast does $k \to k^*$?

An approximation that may be OK for small Δ's:

$$\text{At } k^*,\ \dot{k}_t = \dot{k}\Big|_{k=k^*} + \left(\frac{\partial \dot{k}}{\partial k}\Big|_{k=k^*}\right)\partial k$$

$$\text{Around } k^*,\ \dot{k}_t \approx 0 + \left(\frac{\partial \dot{k}}{\partial k}\Big|_{k=k^*}\right)(k-k^*)$$

Recall $\dot{k}_t = sf(k_t) - (n+g+\delta)k_t$

hence $\dfrac{\partial \dot{k}_t}{\partial k} = sf'(k_t) - (n+g+\delta)$

$$s = \frac{(n+g+\delta)k^*}{f(k^*)}$$

$$\Rightarrow \frac{\partial \dot{k}}{\partial k}\Big|_{k=k^*} = \underbrace{(n+g+\delta)}_{\approx 6\%}\left[\frac{f'(k^*)k^*}{f(k^*)} - 1\right] \approx -4\%$$

$$\frac{MPK \cdot K^*}{Y^*} \approx 1/3$$

hence $\dot{k}_t \approx -4\% \underbrace{(k_t - k^*)}_{\text{This is negative if } k<k^*}$

Time to get halfway to k^*: about 18 years