

# Intertemporal optimization: Value Functions, Bellman equations

## Review of micro stuff (Mas-Colell et al.)

Static (one-period) problems:

Maximize  $u(x)$  (vector of quantities of individual goods & services)

subject to  $p \cdot x \leq w$  ("wealth")  
(vector of prices)

$V(p, w)$  Indirect utility function (value function)

Result of utility maximization

$p, w$  are what agent takes as given.

### "Envelope Theorem":

IF you give a little more wealth to the guy, how much does it increase his utility?

$$\frac{\partial V(p, w)}{\partial w} = \frac{\partial u(x)}{\partial x_1} \cdot \frac{1}{p_1} = \frac{\partial u(x)}{\partial x_2} \cdot \frac{1}{p_2} = \dots$$

where  $\frac{1}{p_1} = \frac{\partial x_1}{\partial w}$  if all of  $\partial w$  is spent on  $x_1$

$\frac{1}{p_2} = \frac{\partial x_2}{\partial w}$  " " " " " " "  $x_2$

etc.

Intertemporal...

Intertemporal problem without uncertainty

Example similar to Romer ch-5

$$U_t = \sum_{\tau=0}^{\infty} \beta^{\tau} u(c_{t+\tau}, 1-l_{t+\tau}) \quad 0 < \beta < 1$$

(discount factor)

Labor time (fraction of time)  
Leisure = (1-l)

$$s.t. \sum_{\tau=0}^{\infty} \left(\frac{1}{1+r_{t+\tau}}\right)^{\tau} c_{t+\tau} \leq A_t + \sum_{\tau=0}^{\infty} \left(\frac{1}{1+r_{t+\tau}}\right)^{\tau} (w l)_{t+\tau}$$

real assets at time t

real wage

Value of  $U_t$  that results from maximization is  $V_t(A_t, \dots)$  Value function

$A_t$  is one thing household takes as given as of time t (though it is result of past choices)

There may be other things household takes as given at time t. "State variables."

How can we solve this?

Turn it into a two-period problem, and use fact that  $V(\ )$  is "recursive,"

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### Value function is recursive

Because lifetime is infinite, agent's problem will be the same at time  $(t+1)$  as at  $t$ .

She's no closer to end of life.

Same number of periods (infinite) to deal with.

$V_t(A_t, \dots)$  This function has some specific form.

$V_{t+1}(A_{t+1}, \dots)$  Must have same form!  
Same function, just different inputs

### Turn it into a two-period problem

#### Utility function

Pull first period out of summation

$$U_t = u(c_t, 1-l_t) + \sum_{\tau=1}^{\infty} \beta^{\tau} u(c_{t+\tau}, 1-l_{t+\tau})$$

What will  $\sum_{\tau=1}^{\infty} \beta^{\tau} u(\dots)$  turn out to be?

Tomorrow, agent will choose  $c_{t+1}, l_{t+1}$  to maximize utility looking forward from  $t+1$ .

She'll do it correctly, of course. So:

$$U_t = u(c_t, 1-l_t) + \beta V_{t+1}(A_{t+1}, \dots)$$
$$V_t = \max_{c_t, l_t} u(c_t, 1-l_t) + \beta V_{t+1}(A_{t+1}, \dots) \leftarrow \begin{cases} \text{Bellman} \\ \text{Equation} \end{cases}$$

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Turn it into a two-period problem

Budget constraint

At time  $t+1$ , agent will take  $A_{t+1}$  as given.  
How will  $c_t, l_t$  affect  $A_{t+1}$ ?

$$A_{t+1} = (1+r_t)A_t + w_t l_t - c_t$$

What is effect of  $\partial c_t$  on  $V_{t+1}$ ?

$$\frac{\partial V_{t+1}}{\partial c_t} = \frac{\partial V_{t+1}(A_{t+1}, \dots)}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial c_t}$$
$$= \dots \cdot (-1)$$

Effect of  $\partial l_t$ ?

$$\frac{\partial V_{t+1}}{\partial l_t} = \dots \cdot \frac{\partial A_{t+1}}{\partial l_t}$$
$$= \dots \cdot w_t$$

But what is  $\frac{\partial V_{t+1}(L)}{\partial A_{t+1}}$ ?

Inter temporal

Turn it into

Benveniste-Scheinkman equation

Recall that envelope theorem says effect on utility of giving agent a little more "wealth," assuming utility is being maximized, is equal to addition to utility from spending it all on  $x_1$ , or  $x_2$ , or whatever:

$$\frac{\partial V(p, w)}{\partial w} = \frac{\partial u(x)}{\partial x_i} \cdot \frac{1}{P_i} \text{ etc.}$$

So here

$\frac{\partial V_{t+1}(A_{t+1}, \dots)}{\partial A_{t+1}}$  must be equal to effect on

lifetime utility of spending all  $\partial A_{t+1}$

on  $c_{t+1}$ , not carrying any on into  $A_{t+2}$ .

So if I spend it all on  $c_{t+1}$  what's  $\frac{\partial c_{t+1}}{\partial A_{t+1}}$ ?

Recall  $A_{t+2} = (1+r_{t+1})A_{t+1} + w_{t+1}l_{t+1} - c_{t+1}$

$$\partial A_{t+2} = (1+r_{t+1})\partial A_{t+1} + w_{t+1}\partial l_{t+1} - \partial c_{t+1}$$

setting  $\partial A_{t+2} = \partial l_{t+1} = 0$ ,

$$\partial c_{t+1} = (1+r_{t+1})\partial A_{t+1} \text{ So...}$$

Intertemporal...

Turn it into...

Benveniste - Scheinkman

Applying envelope theorem to  $c_{t+1}$  gives

B-S condition with respect to  $c_{t+1}$ :

$$\frac{\partial V_{t+1}(\cdot)}{\partial A_{t+1}} = \frac{\partial u(c_{t+1}, 1-l_{t+1})}{\partial c_{t+1}} \cdot (1+r_{t+1})$$

You can also apply envelope theorem to  $(1-l_{t+1})$ , using effect on utility of "spending" it all on leisure in  $t+1$ . Setting  $\partial A_{t+2} = \partial c_{t+1} = 0$ ,

$$\partial l_{t+1} = (1+r_{t+1}) \partial A_{t+1} \frac{1}{w_{t+1}} (-1)$$

$$\frac{\partial V_{t+1}(\cdot)}{\partial A_{t+1}} = \frac{\partial u(\cdot)}{\partial (1-l_{t+1})} \cdot (1+r_{t+1}) \frac{1}{w_{t+1}}$$

# Intertemporal optimization

## First-order conditions from two-period problem without uncertainty

Choose  $c_t, l_t$  to maximize

$$u_t = u(c_t, 1-l_t) + \beta V_{t+1}(A_{t+1}, \dots)$$

$$\text{s.t. } A_{t+1} = (1+r_t)A_t + w_t l_t - c_t$$

At optimal choices for  $c_t, l_t$ ,

$$\frac{\partial u_t}{\partial c_t} = 0 = \frac{\partial u(c_t, 1-l_t)}{\partial c_t} + \beta \frac{\partial V_{t+1}}{\partial A_{t+1}} \cdot (-1)$$

$$\frac{\partial u_t}{\partial (1-l_t)} = 0 = \frac{\partial u(\quad)}{\partial (1-l_t)} + \beta \frac{\partial V_{t+1}}{\partial A_{t+1}} \cdot (-w_t)$$

$$\text{where } \frac{\partial V_{t+1}}{\partial A_{t+1}} = \frac{\partial u(c_{t+1}, 1-l_{t+1})}{\partial c_{t+1}} \cdot (1+r_{t+1})$$

or

$$= \frac{\partial u(\quad)}{\partial (1-l_{t+1})} \cdot (1+r_{t+1}) \frac{1}{w_{t+1}}$$

This can tell us a lot!

Intertemporal optimization. - - -

--- without uncertainty

Euler equation

Recall the Euler equation from RCK model

$$\frac{\text{Change in MU} \rightarrow \dot{u}'(c_t)}{\text{MU} \rightarrow u'(c_t)} = \rho + n - f'(k)$$

$\leftarrow r$

$\leftarrow \dot{c}$

where  $\dot{u}'(c_t) = \frac{\partial c_t}{\partial t} \cdot \frac{\partial u'(c_t)}{\partial c_t} = \frac{\partial c_t}{\partial t} u''(c_t)$

Implicitly defines path for  $c$  over time.

with CRRA  $u'(c) = c^{-\theta}$   $u''(c) = -\theta c^{-\theta-1}$

so Euler equation becomes

$$\frac{\dot{c}_t (-\theta c_t^{-\theta-1})}{c_t^{-\theta}} = \rho + n - f'(k)$$

$$\frac{\dot{c}_t}{c_t} = \frac{1}{\theta} [f'(k_t) - \rho - n]$$

$\leftarrow r$

- defines path of  $c$  over time given either
- starting point
  - ending point (work backwards)



Intertemporal ...

... without uncertainty

Euler equation (cont.)

Using B-S equation, we can derive an Euler eqn. from Bellman equation

$$U_t = u(c_t, 1-l_t) + \beta V_{t+1} [A_{t+1} \dots]$$

where  $A_{t+1} = (1+r_t)A_t + w_t l_t - c_t$

$$0 = \frac{\partial U_t}{\partial c_t} = u'(c_t, \dots) + \beta \frac{\partial V_{t+1}(A_{t+1})}{\partial A_{t+1}} \cdot \frac{\partial A_{t+1}}{\partial c_t}$$

$$\text{B-S: } \frac{\partial V_{t+1}(A_{t+1})}{\partial A_{t+1}} = u'(c_{t+1}, \dots) \cdot (1+r_{t+1})$$

$$0 = u'(c_t, \dots) + \beta u'(c_{t+1}, \dots) (1+r_{t+1}) (-1)$$

$$\frac{u'(c_{t+1}, \dots)}{u'(c_t, \dots)} = (\beta(1+r_{t+1}))^{-1}$$

Specific felicity functions:  $\frac{1}{1-\theta} c_t^{1-\theta} + v(1-l_t)$

$$\frac{c_{t+1}^{-\theta}}{c_t^{-\theta}} = (\beta(1+r_{t+1}))^{-1}$$

$$\frac{c_{t+1}}{c_t} = \left( \beta(1+r_{t+1}) \right)^{\frac{1}{\theta}}$$

(Recall high discounting means small  $\beta$ )

Intertemporal optimization

With uncertainty

I don't know what  $V_{t+1}$  will be

$$V_t = \max_{c_t, l_t} u(c_t, 1-l_t) + \beta E_t V_{t+1}(A_{t+1}, \dots)$$

F.O.C.'s become:

$$0 = \frac{\partial u(c_t, 1-l_t)}{\partial c_t} + \beta \frac{\partial E_t V_{t+1}(A_{t+1})}{\partial c_t}$$

etc.

plus a handy thing:

$$\frac{\partial E_t V_{t+1}(\cdot)}{\partial c_t} = E_t \left[ \frac{\partial V_{t+1}(\cdot)}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial c_t} \right]$$

How do you know?

probability of state  $j$

$$E_t V_{t+1}(\cdot) = \sum_{j=1}^n \psi_j V_{t+1,j}$$

where there are  $n$  possible futures, each with probability  $\psi_j$ , and a value of  $V_{t+1}$  in that future.

$$\frac{\partial E_t V_{t+1}(\cdot)}{\partial X_t} = \sum_{j=1}^n \psi_j \frac{\partial V_{t+1,j}}{\partial X_t} = E_t \left[ \frac{\partial V_{t+1}(\cdot)}{\partial X_t} \right]$$

↑  
anything

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## With uncertainty (cont.)

### B-S condition with uncertainty

Because of envelope theorem, we know that in each possible future state  $j$

$$\frac{\partial V_{t+1,j}}{\partial C_t} = \frac{\partial u(\cdot)}{\partial C_{t+1,j}} \frac{\partial C_{t+1,j}}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial C_t}$$

so

$$\sum_{j=1}^n \nu_j \frac{\partial V_{t+1,j}}{\partial A_{t+1}} = \sum_{j=1}^n \nu_j \frac{\partial u(\cdot)}{\partial C_{t+1,j}} \frac{\partial C_{t+1,j}}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial C_t}$$

so

$$E_t \left[ \frac{\partial V_{t+1}(\cdot)}{\partial A_{t+1}} \right] = E_t \left[ \frac{\partial u(\cdot)}{\partial C_{t+1,j}} \right]$$

Now we can define Euler equation.

(Note: you can do same for  $Z_t$ .)

Intertemporal  
With uncertainty (cont.)

Euler equation

$$0 = u'(c_t, \dots) + \beta E_t \left[ u'(c_{t+1}, \dots) \cdot (1 + r_{t+1}) \right]$$

But here it gets tricky. What is  $E_t [ \dots ]$ ?

You might think

$$\dots = u'(E_t [c_{t+1}, \dots]) \cdot (1 + E_t [r_{t+1}])$$

But no! Two reasons. Math things.

1)  $c_{t+1}$  &  $r_{t+1}$  might both be uncertain (random vars.)  
at time  $t$ . For two random variables  $X$  &  $Z$ ,

$$E_t [X_{t+1} \cdot Z_{t+1}] \neq E_t [X_{t+1}] \cdot E_t [Z_{t+1}]!$$

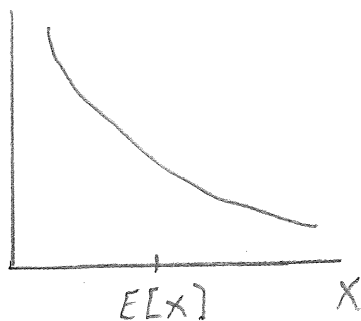
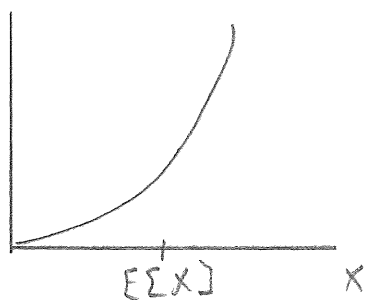
$$E_t [X_{t+1} \cdot Z_{t+1}] = E_t [X_{t+1}] \cdot E_t [Z_{t+1}] + \text{Cov}(X_{t+1}, Z_{t+1})$$

2) Jensen's inequality

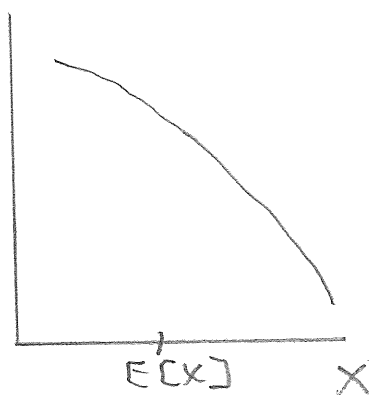
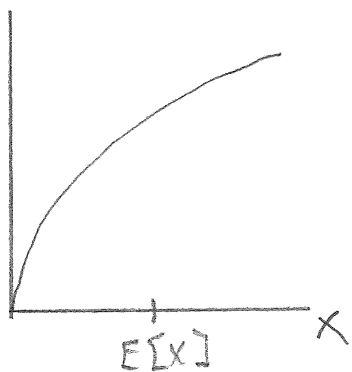
Inter temporal...With uncertainty (cont.)Jensen's inequalityRandom variable  $X$ ,  $E[X]$ , function  $F(X)$ 

$$E[F(X)] = F(E[X]) \text{ ; if } F(X) \text{ linear in } X, F''(X) = 0$$

$$E[F(X)] > F(E[X]) \text{ ; if } F(X) \text{ convex } F''(X) > 0$$



$$E[F(X)] < F(E[X]) \text{ ; if } F(X) \text{ concave } F''(X) < 0$$



So here, even if only  $c_{t+1}$  is uncertain,  
 we need to know if  $u'(c_{t+1}, \dots)$  is convex  
 or concave. Diminishing MU means concave.

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With uncertainty (cont.)

Another approximation: "certainty equivalence"

It's often hard to get a usable expression

for  $E_t[u'(c_{t+1}, \dots)]$

so we often approximate with  
"certainty equivalence":

$$E_t[F(X)] \approx F(E[X])$$

We can use Jensen's inequality to figure out the direction of the resulting error if we can at least figure out  $F''(X)$ .