Economics 613, Graduate Macroeconomics, Fall 2022, Hanes.
Final examination.

Read all the questions before you begin. This is a long exam! I don't expect you to answer all the questions. What you should do is pick out the questions you know how to answer and answer those first. Then do the ones that you kind-of know how to answer.

1) Consider a model with a competitive labor market with a market-clearing nominal wage W per unit of labor. A representative-agent household maximizes

$$E_t \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\theta} C_t^{1-\theta} + \frac{1}{1-\nu}(M_t / P_t)^{1-\nu} - \frac{1}{1+\lambda} L_t^{1+\lambda} \right]$$

The agent's nominal wealth evolves as $A_{t+1} = M_t + (A_t + W_t L_t - P_t C_t - M_t)(1+i_t)$

At time t, the agent takes as given A_t, the wage W_t and the price level P_t, and chooses consumption, labor and his real money balance. *Notice that the intertemporal budget constraint is written in terms of nominal wealth, not real wealth (as in the problem set you did). A is in dollars, not real terms.*

Assume "certainty equivalence" holds, so that in the agent's optimization problem you can take expected values of future variables to be equivalent to actual known values of future variables.

a) Write down the Bellman equation for the agent's problem.

2 pts

$$V_t = \max \left[\frac{1}{1-\theta} C_t^{1-\theta} + \frac{1}{1-\nu} (M/P)_t^{1-\nu} - \frac{1}{1+\lambda} L_t^{1+\lambda} + \beta E_t V_{t+1} \right]$$

$$\text{s.t. } A_{t+1} = \ldots$$

b) Derive an equation that gives the quantity of labor the household chooses to supply at time t, L_t^S, as a function of the real wage $(W / P)_t$ and consumption C_t.

$$0 = \frac{\partial V}{\partial L_t} = -L_t^{\lambda} + \beta E_t \frac{\partial V}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial L_t} = -L_t^{\lambda} + \beta E_t \frac{\partial V}{\partial A_{t+1}} W_t (1+i_t)$$

$$0 = \frac{\partial V}{\partial C_t} = C_t^{-\theta} + \beta E_t \frac{\partial V}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial C_t} = C_t^{-\theta} + \beta E_t \frac{\partial V}{\partial A_{t+1}} (-(1+i_t) P_t)$$

solve for L

$$L_t = C_t^{-\frac{\theta}{\lambda}} (w/P)_t^{\frac{1}{\lambda}}$$

5 pts

Does this make sense? Yes!

(1) $(w/P)_t \uparrow \rightarrow L_t \uparrow$

(2) $C_t \uparrow \rightarrow L_t \downarrow$

c) Derive an equation that gives the agent's demand for real money balance $(M/P)_t$ as a function of consumption C_t and the nominal interest rate i_t .

$$\frac{\partial V}{\partial (M/P)_t} = (M/P)^{-\nu} + \beta E_t \frac{\partial V}{\partial A_{t+1}} \left(1 \cdot \frac{\partial M_t}{\partial (M/P)_t} - (1+i_t)\left(\frac{\partial M_t}{\partial (M/P_t)}\right)\right)$$

5 pts

What is $\frac{\partial M_t}{\partial (M/P)_t}$? $M_t = \left(\frac{M}{P}\right)_t \cdot P_t$ so $\frac{\partial M_t}{\partial (M/P)_t} = P_t$

so

$$\frac{\partial V}{\partial (M/P)_t} = (M/P)^{-\nu} + \beta E_t \frac{\partial V}{\partial A_{t+1}} \left(1 \cdot P_t - (1+i_t) P_t\right)$$

$$= \text{-- -- -- -- -- --} \; P(-i_t)$$

Using $\partial V / \partial C_t$ from b), solve for $(M/P)_t$

$$(M/P)_t = C_t^{\theta/\nu} \left(\frac{i}{1+i}\right)_t^{-1/\nu}$$

Does this make sense? Yes:

1. $C_t \uparrow \rightarrow (M/P)_t \uparrow$
2. $i \uparrow \rightarrow (M/P)_t \downarrow$

d) Derive C_t as a function of $E_t C_{t+1}$, i_t, P_t and $E_t P_{t+1}$.

From b), $0 = \frac{\partial V}{\partial C_t} = C_t^{-\theta} + \beta E_t \frac{\partial V}{\partial A_{t+1}} \left(-(1+i_t) P_t \right)$

5 pts

What's $\frac{\partial V}{\partial A_{t+1}}$? Benveniste-Scheinkman (envelope theorem)

says $\frac{\partial V}{\partial A_{t+1}} = \frac{\partial u_{t+1}}{\partial C_{t+1}} \cdot \frac{\partial C_{t+1}}{\partial A_{t+1}}$ holding fixed A_{t+2},

which means spending all of ∂A_{t+1} on C_{t+1},

so $\partial C_{t+1} = \partial A_{t+1} \cdot \frac{1}{P_{t+1}}$

with certainty equivalence,

$$0 = C_t^{-\theta} + \beta E_t C_{t+1}^{-\theta} \frac{1}{E_t P_{t+1}} \left(-(1+i_t) P_t \right)$$

solve for C_t:

$$C_t = E_t C_{t+1} \left[\beta (1+i_t) \frac{P_t}{E_t P_{t+1}} \right]^{-\frac{1}{\theta}}$$

e) Using $P_{t+1} = (1+\pi_{t+1})P_t$ and an approximation, derive an equation that gives C_t as a function of $E_t C_{t+1}$ and the "real interest rate" $r_t = i_t - E_t \pi_{t+1}$.

Replace $E_t P_{t+1}$ in c) with $(1+\pi_{t+1})P_t$:

5 pts

$$C_t = E_t C_{t+1} \left[\beta (1+i_t) \frac{1}{1+E_t \pi_{t+1}} \right]^{-1/\theta}$$

The approximation is $\frac{1+i_t}{1+E_t \pi_{t+1}} \approx 1 + (i_t - E_t \pi_{t+1}) = 1 + r_t$

so

$$C_t = E_t C_{t+1} \left[\beta (1+r_t) \right]^{-1/\theta}$$

Does this make sense?

2) Consider an economy that can be described by the Solow model with a fixed savings rate s, a rate of population growth n, a rate of "technological improvement" g, and a rate of depreciation δ. The aggregate production function is $Y = K^{\alpha}(AL)^{1-\alpha}$ *where* $0 < \alpha < 1$.

a) Taking the savings rate s as given, derive expressions showing the long-run steady-state values of:
- k (capital per efficiency-unit of labor).
- w (real wage per efficiency-unit of labor)
- the real wage *per worker*.

as a function of s and exogenous parameters.

See answers to midterm, 3).

6 pts

$$k^* = \left(\frac{s}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$w = (1-\alpha)\left(\frac{s}{n+g+\delta}\right)^{\frac{\alpha}{1-\alpha}}$$

w is wage per efficiency-unit, so multiply by A to get wage per worker

$Aw = \ldots$

b) Now consider the "Golden rule" long-run steady-state. Derive the Golden rule values of:
- k (capital per efficiency-unit of labor).
- w (real wage per efficiency-unit of labor)
- the real wage per worker.

as a function of exogenous parameters.

Again see midterm 3).

$$k^*_{GR} = \left(\frac{\alpha}{n+g+\delta}\right)^{\frac{1}{1-\alpha}}$$

$$w^*_{GR} = (1-\alpha)\,k^{*\alpha}_{GR} = \ldots$$

$$Aw^* = \ldots$$

3) Consider an IS/MP model:

$Y = E(Y, r, G, T)$ where $r = i - \pi^e$ and $E_Y > 0$, $E_r < 0$, $E_G > 0$, $E_T < 0$

$r = r(Y, \pi)$ where $r_Y > 0$, $r_Y > 0$

$\pi = \pi(Y)$ where $\pi_Y > 0$

a) Derive expressions for $\partial r / \partial G$ and $\partial Y / \partial G$. 5 pts

$$Y = E(Y, r, G, T)$$

$$\frac{\partial Y}{\partial G} = E_Y \frac{\partial Y}{\partial G} + E_r \frac{\partial r}{\partial G} + E_G$$

$$r = r(Y, \pi)$$

$$\frac{\partial r}{\partial G} = r_Y \frac{\partial Y}{\partial G} + r_\pi \frac{\partial \pi}{\partial G}$$

what is $\frac{\partial \pi}{\partial G}$?

$$\pi = \pi(Y) \text{ so } \frac{\partial \pi}{\partial G} = \pi_Y \frac{\partial Y}{\partial G}$$

$$\text{so } \frac{\partial r}{\partial G} = r_Y \frac{\partial Y}{\partial G} + r_\pi \pi_Y \frac{\partial Y}{\partial G}$$

One way to solve is to substitute this into equation for $\frac{\partial Y}{\partial G}$

$$\frac{\partial Y}{\partial G} = E_Y \frac{\partial Y}{\partial G} + E_r \left(r_Y \frac{\partial Y}{\partial G} + r_\pi \pi_Y \frac{\partial Y}{\partial G} \right) + E_G$$

$$= \frac{\partial Y}{\partial G} \left(E_Y + E_r r_Y + E_r r_\pi \pi_Y \right)$$

$$\left(1 - E_Y - E_r r_Y + E_r r_\pi \pi_Y\right) \frac{\partial Y}{\partial G} = E_G$$

$$\frac{\partial Y}{\partial G} = E_G \frac{1}{1 - E_Y - E_r r_Y - E_r r_\pi \pi_Y}$$

b) In an IS/MP model, the central bank is adjusting the supply of real money balances to keep r at the value given by the interest-rate rule. Assume $M^S / P = L(i,Y)$ where $L_i < 0$, $L_Y > 0$ and $\pi^e = 0$.

Derive an expression for $\partial(M^S / P) / \partial G$. Hint: use your answer to a)!

$$\frac{\partial(M^S/P)}{\partial G} = L_i \frac{\partial i}{\partial G} + L_Y \frac{\partial Y}{\partial G}$$

with $\pi^e = 0$, $i = r$ so

$$\frac{\partial(M^S/P)}{\partial G} = L_i \frac{\partial r}{\partial G} + L_Y \frac{\partial Y}{\partial G}$$

Substitute in $\frac{\partial r}{\partial G}$, $\frac{\partial Y}{\partial G}$ from a).

5 pts

4) Consider a model like Romer's "baseline" RBC model. To simplify it,
- there are no taxes or government spending
- there is no population growth
- there is no trend growth in total factor productivity.

A representative household has one unit of time which it divides between leisure and work. It maximizes

$$U_t = E_t \sum_{\tau=0}^{\infty} \beta^{\tau} \left(\ln C_{t+\tau} + b \ln\,(1 - l_{t+\tau}) \right) \qquad where\ \ 0 < \beta < 1\,,\ 0 < b < 1$$

where C is household consumption per household and l is the fraction of household time devoted to labor.

The household saves by holding capital (it can turn one unit of real income into one unit of capital). The capital it will hold in period $t+1$ is $K_{t+1} = Y_t - C_t + (1-\delta)K_t$ where δ is the depreciation rate and Y is (real) household income.

One part of household income is (real) labor income. The household earns a real wage w for every unit of labor supplied, so real labor income in period t is $w_t l_t$.

The other part of household income is income from renting out capital. The real rental rate or "real interest rate" is r, which is equal to the marginal product of capital minus the depreciation rate. Thus (real) income from renting out capital in period t is $r_t K_t$.

a) Write down the value function and the intertemporal budget constraint.

$$V_t = \max_{K_t} \left[\ln C_t + b \ln(1 - l_t) + E_t\, V(K_{t+1}) \right]$$

$$\text{s.t. } K_{t+1} = w_t l_t + \overbrace{(r_t + \delta)}^{MPK} K_t - C_t + (1-\delta) K_t$$

5pts

$$= w_t l_t + (1 + r_t) K_t - C_t$$

But question was confusing, so I also took $w_t l_t + (1 + r_t - \delta) K_t - C_t$

b) Using the value function, derive current consumption C_t as a function of the representative agent's beliefs about future consumption C_{t+1} and the future real interest rate r_{t+1}. Notice I did not say "assume certainty equivalance."

5pts

$$\frac{\partial V_t}{\partial C_t} = \frac{1}{C_t} + \beta E_t \left[\frac{\partial V(\,)}{\partial k_{t+1}} \cdot \frac{\partial K_{t+1}}{\partial C_t} \right] = 0$$

$$\frac{\partial K_{t+1}}{\partial C_t} = -1$$

(holding K_{t+2} Fixed.

$$\frac{\partial V(\,)}{\partial k_{t+1}} = \frac{1}{C_{t+1}} \cdot \frac{\partial C_{t+1}}{\partial K_{t+1}}$$

$$= \frac{1}{C_{t+1}} (1 + r_{t+1})$$

So ---

4) conts

$$0 = \frac{1}{c_t} + \beta E_t \left[(-1) \frac{1}{c_{t+1}} (1+r_{t+1}) \right]$$

$$\frac{1}{c_t} = \beta E_t \left[\frac{1+r_{t+1}}{c_{t+1}} \right]$$

$$c_t = \frac{1}{\beta} \frac{1}{E_t \left[\frac{1+r_{t+1}}{c_{t+1}} \right]}$$

Note: it makes sense that $\beta \uparrow \to c_t \uparrow$.

When β is closer to one, I (subjectively) discount future c less, so at given r I save more, consume less now.

c) Now consider the "nonstochastic long-run steady state." Assume that there is no trend growth in total factor productivity, what is the steady state real interest rate r^*?

From b),
$$c^* = \frac{1}{\beta} \frac{1}{\frac{1+r^*}{c^*}} = \frac{1}{\beta} \frac{c^*}{1+r^*}$$

5

$$1 = \frac{1}{\beta} \frac{1}{1+r^*}$$

$$1+r^* = \frac{1}{\beta}$$

$$r^* = \frac{1}{\beta} - 1$$

5) Consider an economy that can be described by the Diamond OLG model. There is no depreciation and no population growth (n=0). "Technology" is not improving (g=0). A person has one unit of labor that he provides to firms in exchange for a real wage w. The real rental rate on a unit of capital is r.

A person's lifetime utility function $U = \left(C^{\beta}_1 + C^{\beta}_2\right)^{1/\beta}$ *where* $0 < \beta < 1$

His budget constraint is $C_{2t+1} = (1+r_{t+1})(w_t - C_{1t})$ See midterm 5)

a) Write down the *Lagrangian* that describes the utility-maximization problem of a young person in period t. (A young person is a person in the first period of life.)

3 pts
$$\mathcal{L} = (c_1^{\beta} + c_2^{\beta})^{\frac{1}{\beta}} + \lambda\left[w - c_1 - \frac{1}{1+r_{t+1}} c_2\right]$$

b) Derive first-period consumption C_1 as a function of the things a person takes as given.

6 pts See midterm. Take two F.O.C.'s;

$$\frac{\partial \mathcal{L}}{\partial c_1} = 0 = \ldots$$

$$\frac{\partial \mathcal{L}}{\partial C_2} = 0 = \ldots$$

get $c_2/c_1 = (1+r)^{\frac{1}{1-\beta}}$ or $c_1 = c_2(1+r)^{-\frac{1}{1-\beta}}$

From budget constraint, $c_2 = (1+r)(w - c_1)$

so

$$c_1 = \frac{1}{1+(1+r)^{\frac{\beta}{1-\beta}}} W$$

c) Using your answer to b), derive s as defined for the OLG model,that is the fraction of a young person's income (labor income) devoted to saving as a function of the things a person takes as given.

$$s = \frac{w - c_1}{w} = \frac{w - \frac{1}{1+(1+r)^{\beta/1-\beta}} w}{w}$$

6

$$= \frac{1}{1+(1+r)^{-\frac{\beta}{1-\beta}}}$$

6) Consider a Keynesian model in which the expectations-augmented Phillips curve is of the Friedman-Phelps type:

$$y_t = -r_t$$

$$\pi_t = \pi_t^e + y_t + \epsilon_t$$

where y is the output gap (the difference between log output and the log of the natural rate of output), r is the difference between the real interest rate and the natural rate of interest, π_t^e is *last* period's expected value for *this* period's inflation rate, and ϵ is a "supply shock" or "cost-push shock" to the Phillips curve. ϵ is "white noise" (no serial correlation, expected value equal to zero) with variance σ_ϵ^2.

The central bank sets r to minimize a loss function: $L_t = \frac{1}{2}E[y_t^2] + \frac{1}{2}E[\pi_t^2]$

At the time the central bank chooses r_t, it knows π_t^e and it *also* knows ϵ_t.

(4) **a)** Write down an equation that gives loss L_t as a function of r_t.

(4) **b)** Use your answer to a) to derive the value of r_t that the central bank will set, as a function of π_t^e and ϵ_t.

(4) **c)** Given your answer to **b)**, what will π_t be, as a function of π_t^e and ϵ_t?

(4) **d)** Now assume that the public has rational expectations. At the time that the public forms its expectation π_t^e, it does not know what ϵ_t will turn out to be, but it does know the distribution of ϵ. So what is the value of π_t^e?

a) $L_t = \frac{1}{2}(-r_t)^2 + \frac{1}{2}(\pi^e - r_t + \varepsilon_t)^2$

because central bank knows ε_t

b) $\frac{\partial L}{\partial r_t} = 0 = (-r_t)(-1) + (\pi^e - r_t + \varepsilon_t)(-1)$

$$r_t = \frac{\pi^e + \varepsilon_t}{2}$$

c) $\pi_t = \pi^e - \frac{\pi^e + \varepsilon_t}{2} + \varepsilon_t = \frac{1}{2}(\pi^e + \varepsilon_t)$

d) $E_{t-1}\pi_t = E_{t-1}\left[\frac{1}{2}(E_{t-1}[\pi_t] + 0\right]$

$\Rightarrow \pi^e = E_{t-1}[\pi_t] = \frac{E_{t-1}\pi}{2}$

so $\pi^e = E_{t-1}[\pi_t] = 0$