Economics 614, Advanced Macro. Problem set on capacity utilization

Consider a model in which a representative-agent household maximizes $E_t \sum_{t=0}^{\infty} \beta^t \left[\frac{1}{1-\theta} C_t^{1-\theta} - \frac{1}{1+\lambda} L_t^{1+\lambda} \right]$

The agent can hold capital K and choose the capital utilization rate u_t. The agent earns a nominal rental rate R for each unit of "effective capital" $(uK)_t$. The agent's nominal wealth is A. At time t, the agent takes as given A_t, the wage W_t, the nominal interest rate i_t and the price level P_t. She chooses consumption, labor, investment I_t (purchases of more capital, or sales in which case I is negative) and u_t, among other things. The price of a unit of capital is always equal to the price of a unit of consumption. The effective-capital rental rate is determined in a competitive market. For a firm, capital is a "variable factor" like labor.

1) Suppose that the agent's nominal wealth evolves as:

$$A_{t+1} = (1+i_t)[A_t + W_t L_t + R_t u_t K_t - P_t(C_t + I_t + u_t^2 K_t)] + P_{t+1}[I_t + (1-\delta)K_t]$$

The term u^2 represents a current cost of running a unit of capital harder. δ is the depreciation rate (a fixed parameter). Starting from the Bellman equation, derive the value of u_t that the agent will set for period t.

$$V_t(A_t) = \text{Max}\left\{ \frac{1}{1-\theta} C_t^{1-\theta} - \frac{1}{1+\lambda} L_t^{1+\lambda} + E_t V(A_{t+1}) \right\}$$

F.O.C.:

$$0 = \frac{\partial E_t V(A_{t+1})}{\partial A_{t+1}} \frac{\partial A_{t+1}}{\partial u_t} = \frac{\partial E_t V}{\partial A_{t+1}} \left[(1+i_t)(R_t K_t - P_t 2u K_t) \right]$$

$$\Rightarrow 0 = R_t K_t - P_t 2u K_t$$

$$u_t = \frac{1}{2} \left(\frac{R}{P} \right)_t$$

2) Now suppose that the agent's nominal wealth evolves as:

$$A_{t+1} = (1+i_t)[A_t + W_t L_t + R_t u_t K_t - P_t(C_t + I_t)] + P_{t+1}[I_t + (1-\theta u_t^2)K_t]$$

Here there is no current cost of running capital harder, but the depreciation rate increases with u_t.

Starting from the Bellman equation, derive the value of u_t that the agent will set for period t in terms of the real interest rate r, using the usual approximation that $(1+r)_t \approx (1+i_t)/(1+E_t \pi_{t+1})$ where π is the inflation rate.

F.O.C.:

$$0 = \frac{\partial E_t V(A_{t+1})}{\partial A_{t+1}} \left[(1+i_t)(R_t K_t + P_{t+1}(-\theta 2 u_t) K_t \right]$$

$$\Rightarrow u = \frac{(1+i_t)}{2\theta} \frac{R_t}{P_{t+1}} = \frac{(1+i_t)}{2\theta} \frac{R_t}{(1+\pi_{t+1}) P_t} = \frac{1}{2\theta} \frac{1+i_t}{1+\pi_{t+1}} \frac{R_t}{P_t}$$

$$\Rightarrow u = \frac{1}{2\theta} \left(\frac{R}{P} \right)_t (1+r_t)$$