ANSWERS

Problem set on maximization of expected utility and Jensen's inequality
(adapted from Romer problem 5.6)

Consider a model similar to the Diamond OLG mode. A person lives two periods, period 1 and period 2. He acts to maximize the expected value of his lifetime utility.
Lifetime utility is $U = \ln C_1 + \ln C_2$.

Hint: to get your answers to the questions below, do not set up and solve Lagrangians. Just use the budget constraint to get C_2 as a function of C_1, substitute that into the expected utility function and take one first order condition.

(1) Suppose a person receives labor income equal to W_1 in the first period and no labor income in the second period. Second-period consumption is thus $C_2 = (1+r)(W_1 - C_1)$ where r is the real return to holding a unit of capital in period 2.

a) Suppose that in period 1 people know with certainty that r will be equal to a value $\bar{r}$. What is C_1?

b) Now suppose that in period 1 r is uncertain. $r = \bar{r} + \epsilon$ where ϵ is mean-zero "white noise." Note that as of period 1 the expected value of r is equal to $\bar{r}$ from part a), and $E[\epsilon] = 0$. Will C_1 be greater than, less than or equal to the value of C_1 you found in part a)?

(2) Now suppose a person receives no labor income in the first period. Instead he receives labor income W_2 in the second period. To consume in the first period, he borrows at interest rate r. That is, in period 2 he must pay $(1+r)$ for each unit of consumption he received in period 1. Thus second-period consumption is $C_2 = W_2 - (1+r)C_1$.

a) Suppose that in period 1 people know with certainty that r will be equal to a value $\bar{r}$ and also know that W_2 will be equal to a value $\bar{W}$. What is C_1?

b) Now suppose that in period 1 r is certain, but W_2 is not. $W_2 = \bar{W} + \epsilon$ where ϵ is mean-zero "white noise." Note that as of period 1 the expected value of W_2 is equal to $\bar{W}$ from part a), and $E[\epsilon] = 0$. Will C_1 be greater than, less than or equal to the value of C_1 you found in part a)?
Hint: apply Jensen's inequality.

Answers to problem set on expected utility & Jensen's inequality

(1)

(a) $U = \ln c_1 + \ln((1+\bar{r})(w_1 - c_1))$

$$\frac{\partial U}{\partial c_1} = \frac{1}{c_1} + \frac{1}{(1+\bar{r})(w_1 - c_1)} \cdot -(1+\bar{r})$$

$$0 = \frac{1}{c_1} - \frac{1}{w_1 - c_1}$$

$$\frac{1}{c_1} = \frac{1}{w_1 - c_1}$$

$$c_1 = w_1 - c_1$$

$$2c_1 = w_1$$

$$c_1 = \tfrac{1}{2} w_1$$

(b) $E[U] = \ln c_1 + E\left[\ln((1+\bar{r}+\varepsilon)(w_1 - c_1))\right]$

$$\frac{\partial E[U]}{\partial c_1} = \frac{1}{c_1} + E\left[\frac{1}{(1+\bar{r}+\varepsilon)(w_1 - c_1)} \cdot -(1+\bar{r}+\varepsilon)\right]$$

$$0 = \frac{1}{c_1} + E\left[\frac{1}{w_1 - c_1}\right]$$ all variables here are known in first period

$$0 = \frac{1}{c_1} + \frac{1}{w_1 - c_1}$$

c_1 same as in part a).

Answers to...

(2)

(a) $U = \ln C_1 + \ln(\overline{W} - (1+\overline{r})C_1)$

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} + \frac{1}{\overline{W} - (1+\overline{r})C_1} \cdot -(1+\overline{r})$$

$$\frac{1}{C_1} = \frac{1}{\frac{\overline{W}}{1+\overline{r}} - C_1}$$

$$C_1 = \frac{\overline{W}}{1+\overline{r}} - C_1$$

$$C_1 = \frac{1}{2}\frac{\overline{W}}{1+\overline{r}}$$

(b) $U = \ln C_1 + E\left[\ln(\overline{W} + \varepsilon - (1+\overline{r})C_1)\right]$

$$\frac{\partial U}{\partial C_1} = \frac{1}{C_1} + E\left[\frac{1}{\overline{W} + \varepsilon - (1+\overline{r})C_1} \cdot -(1+\overline{r})\right]$$

$$0 = \frac{1}{C_1} - E\left[\frac{(1+\overline{r})}{\overline{W} + \varepsilon - (1+\overline{r})C_1}\right]$$

$$0 = \frac{1}{C_1} - E\left[\frac{1}{\left(\frac{1}{1+\overline{r}}\right)(\overline{W} + \varepsilon) - C_1}\right]$$

Answers to...

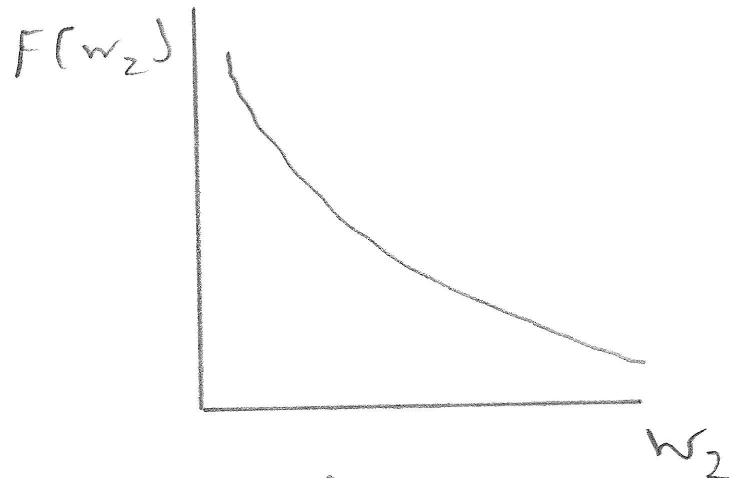
$$\frac{1}{c_1} = E\left[\frac{1}{\left(\frac{1}{1+r}\right)(\bar{w}+\varepsilon) - c_1}\right]$$

Now we can use Jensen's inequality.

$$\text{Say } F(w_2) = \frac{1}{\left(\frac{1}{1+r}\right) w_2 - c_1}$$

Is $F(w_2)$ convex or concave?

It looks like



or you could take first & second derivations, see that $F'(w_2) < 0$, $F''(w_2) \geq 0$.

Thus Jensen's inequality says

$$E\left[\frac{1}{\left(\frac{1}{1+r}\right)(\bar{w}+\varepsilon) - c_1}\right] > \frac{1}{\left(\frac{1}{1+r}\right)\bar{w} - c_1}$$

Therefore...

Answers to...

$$\frac{1}{C_1} > \frac{1}{\left(\frac{1}{1+r}\right)\overline{W} - C_1}$$

Question is whether $C_1 \lessgtr \frac{1}{2}\frac{\overline{W}}{1+\overline{r}}$ answer to a)

$$\frac{\left(\frac{1}{1+r}\right)\overline{W} - C_1}{C_1} > 1$$

$$\left(\frac{1}{1+r}\right)\overline{W}\frac{1}{C_1} - 1 > 1$$

$$\left(\frac{1}{1+r}\right)\overline{W}\frac{1}{C_1} > 2$$

$$\left(\frac{1}{1+r}\right)\overline{W} > 2C_1$$

$$C_1 < \frac{1}{2}\frac{\overline{W}}{1+\overline{r}}$$ answer to a)

Uncertainty about W_2 tends to make people consume less in first period.