Economics 614 Macroeconomic Theory II
Problem on money demand and consumption

Suppose an economy's representative household is infinitely-lived and acts to maximize a utility function subject to *no* uncertainty - future values of variables are known with certainty. Felicity is increasing in consumption C_t , decreasing in the quantity of labor supplied L_t , and increasing in holdings of real money balance M_t/P_t , as follows:

$$U = \sum_{t=0}^{\infty} \beta^t \left(\ln(C_t) - \tfrac{1}{2}\theta L_t^2 + \frac{1}{1-\sigma} (M_t/P_t)^{1-\sigma} \right) \qquad where \quad 0<\beta<1$$

$$\text{subject to} \quad Z_{t+1} = \frac{P_t}{P_{t+1}} \left[\frac{M_t}{P_t} + (1+i_t)\left(Z_t - \frac{M_t}{P_t}\right) + W_t L_t - C_t \right]$$

where Z_{t+1} is real wealth entering period *(t+1)*, M_t is the nominal money balance held across period *t*, i_t is the nominal interest rate paid on nonmoney assets held across period *t*, and W_t is the real wage. At time *t*, the household takes Z_t as given and chooses consumption, labor supply, and real money balances to maximize this lifetime utility function.

a) Write down the value function for the household's problem. Hint: Z is a state variable.

$$V(Z_t, \ldots) = \max_{C_t, L_t, (M/P)_t} \left(\ln C_t - \tfrac{1}{2}\theta L_t^2 + \frac{1}{1-\sigma}(M/P)_t^{1-\sigma} + \beta V(Z_{t+1}, \ldots) \right)$$

b) Derive the quantity of real money balance M_t/P_t that a household will choose to hold, as a function of consumption C_t and the nominal interest rate i_t .

$$\frac{\partial V}{\partial C_t} = \frac{1}{C_t} + \beta V_Z(\;)\frac{P_t}{P_{t+1}}(-1)$$

$$\Rightarrow 1/C_t = \beta V_Z(\;)\, P_t/P_{t+1}$$

$$\frac{\partial V}{\partial (M/P)_t} = (M/P)_t^{-\sigma} + \beta V_Z(\;)\frac{P_t}{P_{t+1}}(1-(1+i_t))$$

$$\Rightarrow \left(\frac{1}{i_t}\right)\left(\frac{M}{P}\right)_t^{-\sigma} = \beta V_Z(\;)\, P_t/P_{t+1}$$

same thing

so
$$\frac{1}{C_t} = \frac{1}{i_t}\left(\frac{M}{P}\right)_t^{-\sigma}$$

$$\Rightarrow \left(\frac{M}{P}\right)_t = \left(\frac{C_t}{i_t}\right)^{1/\sigma}$$

c) Derive C_t as a function of C_{t+1} , i_{t+1} and $\frac{P_t}{P_{t+1}}$

$$\frac{1}{C_t} = \beta \, V_Z(\ ) \, P_t / P_{t+1}$$

BS condition (envelope theorem) says

$$V_Z(\ ) = \frac{\partial U}{\partial C_t} \cdot \frac{\partial C_t}{\partial Z} \quad \text{holding fixed } (M/P)_t, L_t, Z_{t+1}$$

Look at budget constraint: holding fixed Z_{t+1}, etc.,

$$\frac{\partial C_t}{\partial Z_t} = (1+i_t)$$

hence $V_Z(\ ) = \frac{1}{C_t}(1+i_t)$ $\quad \frac{\partial u}{\partial C_t}$

In period $t+1$, $V_Z(\ ) = \frac{1}{C_{t+1}}(1+i_{t+1})$

so

$$\frac{1}{C_t} = \beta \frac{1}{C_{t+1}} (1+i_t) \frac{P_t}{P_{t+1}}$$

$$\Rightarrow C_t = \frac{1}{\beta} C_{t+1} \frac{1}{1+i_{t+1}} \frac{P_{t+1}}{P_t}$$

d) If the inflation rate is denoted π, then $P_{t+1} = (1+\pi)P_t$.
As long as i and π are realistically small (e.g. no more than 5 percent or 0.05), $1+r \approx \frac{1+i}{1+\pi}$

Using this notation rewrite your answer to c) in terms of C'_{t+1} and the *real* interest rate *r*.

From above, $P_{t+1}/P_t = 1+\pi$

$$\Rightarrow C_t = \frac{1}{\beta} C_{t+1} \frac{1+\pi_{t+1}}{1+i_{t+1}} = \frac{1}{\beta} C_{t+1} \frac{1}{1+r_{t+1}}$$