# RATIONAL EXPECTATIONS

$X$ A variable, e.g. $\pi$, $P$

$x^e$ An agent's expected value for that variable, may affect his behavior

e.g. $\pi^e$, $P^e$

affect expenditure through $r = i - \pi^e$

$E[x]$ True, statistical expected value for $X$

$E[x \mid z]$ conditional on variables in $z$

a vector

<u>Rational expectations</u>:

$$x^e = E[x / z]$$

where $z$ is variables observed by agent.

# RATIONAL EXPECTATIONS (cont.)

A model defines X as Fn of exogenous variables & agents' expectations

$$X = Fn(X^e, \ldots)$$

e.g. in IS/LM:

$$Y = Fn(\pi^e, M, G, T, \ldots)$$

with AS

$$P = Fn(\pi^e, M, G, T, \ldots)$$

$$\pi = Fn(\pi^e, P_{t-1}, M, G, T, \ldots)$$

If all variables determining X are certain (not random)

Model defines a value for X.

"Model-consistent" rational expectations or "rational expectations equilibrium":

$$X = Fn(X^e, \ldots)$$

Assume $X^e = X$, which gives

$$X = Fn(X, \ldots)$$

Solve for X.

# RATIONAL EXPECTATIONS

If some variables determining $x$ are uncertain, random variables

because not yet realized,
or realized but not yet known.

$$x = F(x^e, \varepsilon, \ldots)$$

random variable or vector of random variables

then given true distribution for $\varepsilon$
(e.g. mean zero, variance $\sigma_\varepsilon^2$)

model determines
distribution for $x$

$$E[x] = E[F(x^e, \varepsilon, \ldots)]$$

"Model-consistent" rat. exp. or
"rational expectations equilibrium":

$$x^e = E[x] = E[F(x^e, \varepsilon, \ldots)]$$

Solve for $x^e$.

When $\varepsilon$ is realized, $x^e$ and realized $\varepsilon$ value will determine realized $x$.

# RATIONAL EXPECTATIONS

## If some variables are... random (cont.)

Note: realized $x$ may not equal $x^e$,

because realized $x$ may not equal $E[x]$

(if model contains random variables).

## What does R.E.E. represent?

1) What would happen if expectations are rational and true model is known by agents

2) A long-term equilibrium that economy will go to if agents learn from experience.

Learning: agents' models of economy may be incorrect in short run, but converge to true model in long run.

# MONETARY POLICY

## KYDLAND & PRESCOTT (1977): DYNAMIC INCONSISTENCY OF LOW-INFLATION MONETARY POLICY

### ASSUMPTIONS

AS: $y_t = \bar{y} + b(\pi_t - {}_{t-1}\pi_t^e)$ ; where $b > 0$

natural rate of output

Note: this is another way to write down "expectations-augmented Phillips curve"

$$y_t - \bar{y} = b(\pi_t - {}_{t-1}\pi_t^e)$$

$$\pi_t - {}_{t-1}\pi_t^e = \frac{1}{b}(y_t - \bar{y})$$

$$\pi_t = \frac{1}{b}(y_t - \bar{y}) + {}_{t-1}\pi_t^e$$

Preferences of Central Bank

Key point: same as everyone else's preferences

MINIMIZE $L_t = \frac{1}{2}(y_t - y^*)^2 + \frac{1}{2}a(\pi_t - \pi^*)^2$

desired inflation rate

Key point: $y^* > \bar{y}$

desired output level

What Central Bank can do:

At time $t$, take ${}_{t-1}\pi_t^e$ as given

Choose $\pi_t$ (implies $y_t$)

or

choose $y_t$ (implies $\pi_t$, given ${}_{t-1}\pi_t^e$)

(from $y = -d(r - \bar{r}) = -d(i - \pi^e - \bar{r})$)

# MONETARY POLICY

## Dynamic inconsistency of optimal policy (cont.)

## Central bank behavior:

$$\min_{\pi_t} L_t = \frac{1}{2}\left[\bar{y} + b(\pi_t - {}_{t-1}\pi_t^e) - y^*\right]^2 + \frac{1}{2}a(\pi_t - \pi^*)^2$$

F.O.C.: $\frac{\partial L_t}{\partial \pi_t} = 0 = \left[\bar{y} + b(\pi_t - {}_{t-1}\pi_t^e) - y^*\right]b + a(\pi_t - \pi^*) = 0$

$$\Rightarrow \pi_t = \pi^* + \frac{b}{a+b^2}(y^* - \bar{y}) + \frac{b^2}{a+b^2}({}_{t-1}\pi_t^e - \pi^*)$$

NOTE:

— If ${}_{t-1}\pi_t^e = \pi^*$, $\pi_t > \pi^*$

so $\pi = \pi^*$ can't be long-run equilibrium!

— Looking back at AS,

$y_t > \bar{y}$ only if $\pi_t > {}_{t-1}\pi_t^e$,

so $y_t > \bar{y}$ can't be long-run equilibrium!

Equilibrium in long run, or if public knows Central Bank's goals:

$${}_{t-1}\pi_t^e = \pi_t \quad \text{for all } t$$

$$\Rightarrow \pi = \pi^* + \frac{b}{a}(y^* - \bar{y}) > \pi^*$$ Inflation is "too high"

$y = \bar{y}$ output is at natural rate

$$L = \frac{1}{2}(\bar{y} - y^*)^2 + \frac{1}{2}\frac{b^2}{a}(\bar{y} - y^*)^2 = \frac{1}{2}\left(1 + \frac{b^2}{a}\right)(\bar{y} - y^*)^2$$

even though AS allows us to have $\pi = \pi^*$, $Y = \bar{Y}$

# MONETARY POLICY

## Dynamic inconsistency of optimal policy

## Derivation of long-run equilibrium

Apply rational expectations:

$$ {}_{t-1}\pi_t^e = \underset{t-1}{E}[\pi_t] = E\left[\pi^* + \frac{b}{a+b^2}(y^* - \bar{y}) + \frac{b^2}{a+b^2}\left({}_{t-1}\pi_t^e - \pi^*\right)\right] $$

Knowing structure of economy & central bank's goals means

$$ E[\pi^*] = \pi^* $$

$$ E[y^* - \bar{y}] = y^* - \bar{y} $$

and $\underset{t-1}{E}\left[{}_{t-1}\pi_t^e\right] = {}_{t-1}\pi_t^e$

so

$$ {}_{t-1}\pi_t^e = \pi^* + \frac{b}{a+b^2}(y^* - \bar{y}) + \frac{b^2}{a+b^2}\,{}_{t-1}\pi_t^e - \frac{b^2}{a+b^2}\pi^* $$

Solve for ${}_{t-1}\pi_t^e$

$$ \left(1 - \frac{b^2}{a+b^2}\right){}_{t-1}\pi_t^e = \left(1 - \frac{b^2}{a+b^2}\right)\pi^* + \frac{b}{a+b^2}(y^* - \bar{y}) $$

$$ {}_{t-1}\pi_t^e = \pi^* + \frac{b/(a+b^2)}{a/(a+b^2)}(y^* - \bar{y}) = \pi^* + \frac{b}{a}(y^* - \bar{y}) \quad \text{For all } t $$

Recall $\pi_t = \pi^* + \frac{b}{a+b^2}(y^* - \bar{y}) + \frac{b^2}{a+b^2}\left({}_{t-1}\pi_t^e - \pi^*\right)$

$$ \pi_t = \pi^* + \frac{b}{a+b^2}(y^* - \bar{y}) + \frac{b^2}{a+b^2}\left(\pi^* + \frac{b}{a}(y^* - \bar{y}) - \pi^*\right) $$

$$ = \pi^* + \left(\frac{b + b^2\frac{b}{a}}{a+b^2}\right)(y^* - \bar{y}) = \pi^* + \left(\frac{b}{a}\,\frac{1+\frac{b^2}{a}}{1+\frac{b^2}{a}}\right)(y^* - \bar{y}) $$

$$ = \pi^* + \frac{b}{a}(y^* - \bar{y}) = {}_{t-1}\pi_t^e \quad \text{For all } t $$

Dynamic inconsistency

Derivation of long-run equilibrium (cont.)

Recall

$$y_t - \bar{y} = b\left(\pi_t - {}_{t-1}\pi_t^e\right)$$

Substitute in $\pi_t = {}_{t-1}\pi_t^e$

$$y_t - \bar{y} = 0$$

$$y = \bar{y} \text{ for all } t$$

while $\pi = \pi^* + \frac{b}{a}(y^* - \bar{y})$

$$L = \tfrac{1}{2}(\bar{y} - y^*)^2 + \tfrac{1}{2}a\left(\pi^* + \frac{b}{a}(y^* - \bar{y}) - \pi^*\right)^2$$

$$= \tfrac{1}{2}(\bar{y} - y^*)^2 + \tfrac{1}{2}a\left(\frac{b}{a}(y^* - \bar{y})\right)^2$$

$$= \tfrac{1}{2}(\bar{y} - y^*)^2 + \tfrac{1}{2}\frac{b^2}{a}(\bar{y} - y^*)^2$$

Pre-commitment

What if Central Bank can bind its future behavior: at time 0, choose a value of $\pi$ for all future periods, denoted $\hat{\pi}$, and (with rational expectations)

${}_{t-1}\pi_t^e = \hat{\pi}$ for all $t$?

## Dynamic inconsistency

## Pre-commitment

Result of ${}_{t-1}\pi^e_t = \pi_t = \hat{\pi}$ for all $t$:

$$y_t - \bar{y} = b(\pi_t - {}_{t-1}\pi^e_t) = b(\hat{\pi} - \hat{\pi}) = 0$$

$$\Rightarrow y = \bar{y} \quad \text{for all } t$$

$$L = \tfrac{1}{2}(\bar{y} - y^*)^2 + \tfrac{1}{2}a(\hat{\pi} - \pi^*)^2 \quad \text{For all } t$$

To minimize $L$, choose $\hat{\pi} = \pi^*$

hence

$$L = \tfrac{1}{2}(\bar{y} - y^*)^2$$

compare with $L$ if central bank is free to choose $\pi_t$ at every time $t$:

$$L = \tfrac{1}{2}(\bar{y} - y^*)^2 + \tfrac{1}{2}\frac{b^2}{a}(\bar{y} - y^*)^2$$

Loss is smaller with pre-commitment, that is if Central Bank restricts its future freedom, binds its future behavior!

Recall $L$ is same for society (public), so this is better for everyone!

## Analogies

Prisoner's dilemma

Taxation of Fixed capital

# AD & AS with Rational Expectations: Example

## Derivation of AD

$m - p_t = a y_t - b i_t \quad (M^S = M^D)$

$y_t = -d i_t$ Spending equation excluding ${}_t\pi^e_{t+1}$ for simplicity

Note: $y$ denotes log of "output gap": $\ln\left(\frac{Y_t - \bar{Y}}{\bar{Y}}\right)$

$i$ denotes interest rate less rate at which $y = 0$,

so when $i = 0$, $y = 0$

Solve for $y$:

$$y_t = \frac{d}{b + da}(m_t - p_t) = \frac{1}{a + \frac{b}{d}}(m_t - p_t)$$

Reverse to get equation of AD curve:

$$p_t = m_t - \left(a + \frac{b}{d}\right) y_t$$

slope of AD curve

shifts AD curve

## AS

$\pi_t = {}_{t-1}\pi^e_t + f y_t$ Friedman-Phelps Phillips curve

$p_t - p_{t-1} = {}_{t-1}\pi^e_t + f y_t$

$$p_t = \underbrace{{}_{t-1}\pi^e_t + p_{t-1}} + f y_t \quad \text{AS curve}$$

slope of AS curve

shifts AS curve, note ${}_{t-1}\pi^e_t + p_{t-1} = {}_{t-1}p^e_t$

reverse to get

$$y_t = \frac{1}{f} p_t - \frac{1}{f} p_{t-1} - \frac{1}{f}\,{}_{t-1}\pi^e_t$$

## AD & AS, .... example

### Solve for $\pi_t$ in terms of ${}_{t-1}\pi_t^e$

Set $y_t$ from AS equal to $y_t$ from AD

$$\frac{1}{f}p_t - \frac{1}{f}p_{t-1} - \frac{1}{f}\,{}_{t-1}\pi_t^e = \frac{d}{b+da}(m_t - p_t)$$

Solve for $p_t$

$$p_t = \frac{1}{\frac{1}{f} + \frac{d}{b+da}}\left(\frac{d}{b+da}m_t + \frac{1}{f}p_{t-1} + \frac{1}{f}\pi^e\right)$$

Subtract $p_{t-1}$ from both sides

$$p_t - p_{t-1} = \frac{1}{\frac{1}{f} + \frac{d}{b+da}}\left(\frac{d}{b+da}m_t + \frac{1}{f}\pi^e\right) + \left(\frac{\frac{1}{f}}{\frac{1}{f} + \frac{d}{b+da}} - \frac{\frac{1}{f} + \frac{d}{b+da}}{\frac{1}{f} + \frac{d}{b+da}}\right)p_{t-1}$$

$$\pi_t = \frac{d/b+da}{\frac{1}{f} + \frac{d}{b+da}}(m_t - p_{t-1}) + \frac{1/f}{\frac{1}{f} + \frac{d}{b+da}}\,{}_{t-1}\pi_t^e$$

? But what's ${}_{t-1}\pi_t^e$ ?

### Apply assumption of model-consistent R. E.

Recall this means, for variable $x$,

$${}_t x_{t+j}^e = E_t[x_{t+j}]$$

(statistical expected value of variable conditional on info available at time $t$

Here, it means

$${}_{t-1}\pi_t^e = E_{t-1}[\pi_t]$$

## AD & AS . . .

## Apply assumption of . . . (cont.)

$${}_{t-1}\pi_t^e = E_{t-1}[\pi_t] = E_t\left[\frac{d/b+da}{\frac{1}{F}+\frac{d}{b+da}}\left(m_t - p_{t-1}\right) + \frac{1/F}{\frac{1}{F}+\frac{d}{b+da}}\,{}_{t-1}\pi_t^e\right]$$

$$E_{t-1}[p_{t-1}] = p_{t-1}$$ ← (known at time $t-1$ (observable)

$$E_{t-1}[m_t] = {}_{t-1}m_t^e$$ ← (depends on process generating $m$

$$E_{t-1}[{}_{t-1}\pi_t^e] = {}_{t-1}\pi_t^e$$ ← (known at time $t-1$) (in agents' heads)

$${}_{t-1}\pi_t^e = \frac{d/b+da}{\frac{1}{F}+\frac{d}{b+da}}\left({}_{t-1}m_t^e - p_{t-1}\right) + \frac{1/F}{\frac{1}{F}+\frac{d}{b+da}}\,{}_{t-1}\pi_t^e$$

Solve for ${}_{t-1}\pi_t^e$

(multiply right-hand side by $\frac{\frac{1}{F}+\frac{d}{b+da}}{\frac{1}{F}+\frac{d}{b+da}}$, subtract left-hand side term with ${}_{t-1}\pi_t^e$ from both sides)

$${}_{t-1}\pi_t^e = {}_{t-1}m_t^e - p_{t-1}$$

Now what? Substitute this into AS/AD system, calculate $p_t$ & $y_t$ in terms of ${}_{t-1}m_t^e$ etc.

## AD & AS

## Find $p_t$ & $y_t$

Recall

$$p_t = \frac{1}{\frac{1}{f} + \frac{d}{b+da}} \left( \frac{d}{b+da} m_t + \frac{1}{f} p_{t-1} + \frac{1}{f} \, {}_{t-1}\pi_t^e \right)$$

$${}_{t-1}m_t^e - p_{t-1}$$

$$p_t = \frac{1}{\frac{1}{f} + \frac{d}{b+da}} \left( \frac{d}{b+da} m_t + \frac{1}{f} p_{t-1} + \frac{1}{f} \, {}_{t-1}m_t^e - \frac{1}{f} p_{t-1} \right)$$

$$p_t = \frac{d/b+da}{\frac{1}{f} + \frac{d}{b+da}} m_t + \frac{1/f}{\frac{1}{f} + \frac{d}{b+da}} \, {}_{t-1}m_t^e$$

Rearrange to a more useful expression: add to RHS

$${}_{t-1}m_t^e - \frac{\frac{1}{f} + \frac{d}{b+da}}{\frac{1}{f} + \frac{d}{b+da}} \, {}_{t-1}m_t^e$$

and rearrange RHS to get

$$p_t = {}_{t-1}m_t^e + \frac{d/ba}{\frac{1}{f} + \frac{d}{b+da}} \left( m_t - {}_{t-1}m_t^e \right)$$

Recall $y_t = \frac{d}{b+da} (m_t - p_t)$ (substitute in here above equation for $p_t$)

gives

$$y_t = \frac{a + b/d}{1 + \frac{f}{a + b/d}} \left( m_t - {}_{t-1}m_t^e \right)$$

# AD & AS .....

## Conclusion

$$p_t = {}_{t-1}m_t^e + \frac{d/(ba}{\frac{1}{f} + \frac{d}{b+da}} (m_t - {}_{t-1}m_t^e)$$

$$y_t = \frac{a + b/d}{1 + \frac{f}{a+b/d}} (m_t - {}_{t-1}m_t^e)$$

what is $(m_t - {}_{t-1}m_t^e)$?

Error in expectation of $m_t$

Variation in $m_t$ you can't forecast.

Example: if you know

$m_t = \bar{m} + \varepsilon_t$ where $\varepsilon$ is mean-zero i.i.d., but you can't observe $\varepsilon_t$,

then ${}_{t-1}m_t^e = \bar{m}$

and $(m_t - {}_{t-1}m_t^e) = \varepsilon_t$

Another example: if you know

$m_t = m_{t-1} + \varepsilon_t$ ($m$ is a "random walk")

then ${}_{t-1}m_t^e = m_{t-1}$

and $(m_t - {}_{t-1}m_t^e) = \varepsilon_t$

Another example: if you know

$m_t = k + nt + \varepsilon_t$ (trend growth with disturbance)

then ${}_{t-1}m_t^e = k + nt$

and ———— $= \varepsilon_t$

# AD & AS with Rational Expectations: Example

## Conclusion (cont.)

Note: rational expectations does not mean perfect foresight!

$$p_t = {}_{t-1}m_t^e + \frac{d/ba}{\frac{1}{f} + \frac{d}{b+da}} \varepsilon_t$$

where $\varepsilon_t = m_t - {}_{t-1}m_t^e$

$$y_t = \frac{a + b/d}{1 + \frac{f}{a+b/d}} \varepsilon_t$$

R.E. means $E_{t-1}[\varepsilon_t] = 0$ ← (Expected value of error is zero

hence

$E[p_t] = {}_t m_{t+1}^e$, $E[y_t] = 0$

but if $\varepsilon_t \neq 0$, p & y will deviate from expected values.

And you (agent with rational expectations) know that you may <u>turn out</u> to be wrong.

"Expected value of error is zero" does <u>not</u> mean "I don't believe I can be wrong."