Numerical investigation of indentation tests on a transversely isotropic elastic material by power-law shaped axisymmetric indenters

Congrui Jin

To cite this article: Congrui Jin (2016) Numerical investigation of indentation tests on a transversely isotropic elastic material by power-law shaped axisymmetric indenters, Journal of Adhesion Science and Technology, 30:11, 1223-1242, DOI: 10.1080/01694243.2016.1150119

To link to this article: http://dx.doi.org/10.1080/01694243.2016.1150119
Numerical investigation of indentation tests on a transversely isotropic elastic material by power-law shaped axisymmetric indenters

Congrui Jin

Department of Mechanical Engineering, State University of New York at Binghamton, Binghamton, NY, USA

ABSTRACT
Nowadays, there exist many well-known classical models of frictionless adhesive contact including the Johnson–Kendall–Roberts (JKR) model, the Derjaguin–Muller–Toporov (DMT) model, and the Maugis solution for the JKR-type-to-DMT-type transition regimes. These models are very helpful for studying molecular adhesion between two contacting linearly elastic isotropic spherical bodies. However, in experimental studies, such as nanoindentation tests, the shapes of the indenters are often more general than the spherical or flat ones. Moreover, very often, the materials to be tested are anisotropic. A special case of anisotropy is transverse isotropy, which contains a plane of isotropy, implying that the material can be rotated with respect to the loading direction about one axis without measurable effect on the material’s response. In this paper, numerical studies on JKR-type frictionless adhesive contact between power-law shaped indenters and transversely isotropic materials are presented. It shows that the formulae for numerical simulations of JKR-type frictionless adhesive contact for transversely isotropic materials have the same mathematical form as the corresponding formulae for isotropic materials, except that the effective elastic contact modulus has different expression for different materials. The DMT-type and JKR-type-to-DMT-type transition regimes have been explored by conducting the simulations using smaller values of Tabor parameters. The good agreement between numerical simulation results and existing analytical solutions shows that this numerical simulation method can be extended to simulate indentation tests using indenters of arbitrary shapes.

Introduction
Adhesion is the tendency of dissimilar particles or surfaces to cling to one another. The problem of adhesive contact has a wide range of applications, including interactions between colloids, powders, cells and biological tissues, numerical modeling of rubber friction, polymer tribology and fretting wear, and the analysis of experimental results from surface force apparatus and atomic force microscopy.[1–4] Nowadays, there exist many well-known...
classical models of frictionless adhesive contact including the Johnson–Kendall–Roberts (JKR) model,[5] the Derjaguin–Muller–Toporov (DMT) model,[6] and the Maugis solution for the JKR-type-to-DMT-type transition regimes.[7] These models are very helpful for studying molecular adhesion between two contacting linearly elastic isotropic spherical bodies.[8,9] However, in experimental investigations, the shapes of the contacting bodies are often more general than just spherical or flat ones. For example, the indenter used in the nanoindentation technique is usually a sharp pyramid,[10] such as Vickers indenter, Berkovich indenter, and Knoop indenter, or a cone,[11] such as Rockwell indenter. The nanoindentation technique, also called depth-sensing indentation, ultra-low-load indentation, continuous-recording indentation, or instrumented indentation, is currently a popular form of mechanical testing in materials science research community, arising from the need to test very small volumes of materials, such as thin films, biomaterials, and coatings. This technique allows the continuous measurement of force and displacement as a function of time while the indentation is being made. It can record small load and displacement with high accuracy and precision. The recorded indentation force-displacement diagrams, often regarded as ‘fingerprints’ of the material, can be analyzed to determine the materials’ mechanical properties even if the indentation is on the sub-micron scale and cannot be imaged.

Bolshakov et al. [12] and Pharr and Bolshakov [13] proposed a method to evaluate load-displacement curves obtained by depth-sensing indentation, and introduced the concept of the so called ‘effective indenter’ to explain the discrepancy between the indenter shape and the resulting load-displacement curves. In particular, Pharr and Bolshakov [13] demonstrated that the elastic contact of the real indenter with a residual indent is equivalent to that of the effective indenter with a flat surface. Their study demonstrated that the shape of the effective indenter can be described by axisymmetric monomial functions.

Another important issue is that all the classical models were developed only for linearly elastic isotropic materials, but many natural and artificial materials are actually anisotropic. Besides, the presence of anisotropy in biomaterials is also very common. A special case of anisotropy is transverse isotropy, which contains a plane of isotropy, implying that the material can be rotated with respect to the loading direction about one axis without measurable effect on the material’s response. Due to its high symmetry and relative simplicity in mathematical formulae, transversely isotropic medium has become one of the most important anisotropic media. Fiber-reinforced composites with all fibers being in parallel can be regarded as transversely isotropic,[14] and many platelet and laminated systems. And, of course, many biological soft connective tissues, such as tendons and ligaments, are best described as transversely isotropic media.[15]

In this paper, numerical studies on frictionless adhesive contact between power-law shaped indenters and transversely isotropic materials are presented. We here assume that the distances between the contacting bodies can be described as axisymmetric monomial functions as follows:

\[ f(r) = B_d r^d \]  

where \( d \) is the degree of the monomial function and \( B_d \) is the constant of the shape. Note that the classical JKR and Boussinesq–Kendall [16] models can be regarded as two particular cases of the power-law indenter problem in which the degree of the indenter \( d \) is equal to two or approaching infinity, respectively.
The adhesive contact problems for bodies of monomial shapes were previously studied analytically by Galanov [17] and Borodich et al. [18] using Galin expressions. [19] Carpick et al. independently presented the analytical solutions for the particular cases in which \( d \) is an even number. [20] The adhesive contact problems for transversely isotropic materials have also been extensively studied. For example, the plane strain problem of a rigid cylinder in adhesive contact with a transversely isotropic elastic half-space without slipping has been studied by Chen and Gao [21]. Later, the adhesive behavior between a transversely isotropic piezoelectric half-space and a cylinder punch under a combined mechanical and electric loading under plane strain assumption has been investigated by Guo and Jin [22]. A spherical contact on transversely isotropic, adhesive half-space has been studied by Espinasse et al. [23] The influence of anisotropy in adhesive contact problems on the analyzed mechanical quantities has been studied analytically by Pawlik and Rogowski [24]. Barber and Ciavarella considered the effect of anisotropic elasticity by assuming the contact area to be an ellipse whose dimensions are determined by imposing the energy release rate criterion the ends of the major and minor axes. [25]

However, the analysis of a wide range of physical problems with adhesive contact on transversely isotropic materials requires a general numerical method. In all the analytical solutions, the details of the interactions, such as pressure distribution within and outside the contact region, are not provided. To our knowledge, the adhesive contact for transversely isotropic bodies involving more general shapes has not been explored numerically. In this study, we propose a numerical simulation method in which the adhesive interactions are represented by an interaction potential and the surface deformations are coupled by using half-space Green's functions discretized on the surface. The agreement between the numerical results and the analytical solution for power-law shaped indenters shows the feasibility of this numerical technique to study indentation problems with arbitrary shaped indenters.

The paper is organized as follows: In §2, mathematical formulation is presented for the power-law shaped axisymmetric adhesive contact problem of a transversely isotropic material, and essential dimensionless parameters for the problem are defined. In §3 and §4, detailed numerical simulation results are presented, as well as the comparison with existing analytical solutions. The DMT-type and JKR-type-to-DMT-type transition regimes have been explored. Finally, the conclusions drawn from the numerical studies are summarized in §5.

**Problem formulation**

In this section, the problem formulation of frictionless adhesive contact problems for power-law shaped axisymmetric indenters will be presented.

For surface interaction, the empirical potential often used is the Lennard–Jones potential. The Lennard–Jones potential is a pair potential and it describes the potential energy of interaction \( U \) between two non-bonding atoms or molecules based on their distance of separation:

\[
U(r) = 4\varepsilon [\left(\sigma/r\right)^{12} - \left(\sigma/r\right)^6]
\] (2a)
where $\varepsilon_0$ and $\sigma$ are potential parameters and $r$ is the distance between the two atoms or molecules. Integrating Equation (2a) over the surface area, we can obtain the relationship between the local pressure $p$ and the air gap $h$ as follows:

$$p(h) = \frac{8W_{ad}}{3\varepsilon} \left[ (\varepsilon/h)^9 - (\varepsilon/h)^3 \right]$$

where $W_{ad}$ is the work of adhesion, which is just the tensile force integrated over the distance necessary to pull apart the two bodies [26] and $\varepsilon$ is a length parameter equal to the range of the surface interaction. For stiff materials, its value should be in the order of interatomic spacing, however, for compliant materials, its value usually becomes much larger. This issue has been well studied in the existing literature, for example, Refs. [27,28] Detailed derivation of Equation (2b) has been shown in Appendix A.

The Derjaguin’s approximation [29] is then applied to Equation (2). This approximation relates the force law between two curved surfaces to the interaction energy per unit area between two planar surfaces, which makes this approximation a very useful tool, since forces between two planar bodies are often much easier to calculate. This approximation is widely used to estimate forces between colloidal particles. Note that Greenwood also adopted this approximation in his simulation of the adhesive contact between two inclined surfaces. [30] The separation between the two surfaces due to the surface interaction as well as the external loads, denoted by $h$, as shown in Figure 1, will be expressed by the following equation:

$$h(x, y) = -\alpha + \varepsilon + h_0 + \frac{1}{\pi E\Omega} \iint_{\Omega} \frac{p(x', y')dx'dy'}{\sqrt{(x-x')^2 + (y-y')^2}}$$

Figure 1. The parameter $h_0$ is the initial air gap, i.e. the separation of the surfaces in the absence of applied and adhesive forces, and then due to surface interaction as well as the external loads, the surfaces will deform and the separation between the two points will change from $h_0$ to $h$. $\alpha$ is the displacement (in the z direction) at infinity of body 1 with respect to body 2. In the case of indentation test, $\alpha$ represents the indentation depth.
where the parameter $\alpha$ is the displacement between the two surfaces with respect to the zero force position $h = \epsilon$, i.e. the so called indentation depth and the parameter $E_{TI}$ represents the effective elastic contact modulus. Note that $E_{TI}$ has different expression for different materials. The parameter $h_0$ is the initial air gap, i.e. the separation of the surfaces.

Figure 2. The power-law shaped axisymmetric indenters for five different values of $d$: (a) 1.00, (b) 1.25, (c) 1.50, (d) 1.75, and (e) 2.00.
in the absence of applied and adhesive forces, and it can be written in rectangular coordinates as follows:

\[ h_0 = B_d (x^2 + y^2)^{d/2} \]  

(4)

For the adhesive contact between two linearly elastic isotropic materials with Young’s modulus \( E_i \) and Poisson’s ratio \( \nu_i \), where \( i = 1, 2 \), \( E^{TI} \) reduces to \( E^* \), which is defined as the effective Young’s modulus, i.e. if both the materials feature significant compliances, the compliances add up as the following:

\[ \frac{1}{E^*} = \frac{1 - \nu_1^2}{E_1} + \frac{1 - \nu_2^2}{E_2} \]  

(5a)

For the adhesive contact between a rigid indenter and a transversely isotropic material, \( E^{TI} \) is more complicated and can be written as the following Equation [31]:

\[ E^{TI} = -\frac{2D^{1/2}(AC - D)}{(S_1 + S_2)[(D - 2BD + AC)a_{11} - (2D - BD - AC)a_{12}]} \]  

(5b)
with

\[
\begin{align*}
A &= \frac{a_{12}(a_{11}-a_{12})}{a_{11}a_{33}-a_{12}^2}, & B &= \frac{a_{11}(a_{11}+a_{13})-a_{12}a_{33}}{a_{11}a_{33}-a_{12}^2}, & D &= \frac{a_{11}^2-a_{12}^2}{a_{11}a_{33}-a_{12}^2}, \\
C &= \frac{a_{11}(a_{12}-a_{13})+a_{11}a_{44}}{a_{11}a_{33}-a_{12}^2}, & S_{1,2} &= \sqrt{\frac{A+C}{2D}} \pm \left(\frac{A+C}{2D} - \frac{1}{D}\right)^{1/2}
\end{align*}
\]  

(5c)

where \(a_{11}, \ a_{12}, \ a_{13}, \ a_{33},\) and \(a_{44}\) are the five elastic constants for the transversely isotropic material. Assume that the z-axis is normal to the plane of isotropy, and then Hooke's law can be written as:

\[
\begin{pmatrix}
\varepsilon_{11} \\
\varepsilon_{22} \\
\varepsilon_{33} \\
\varepsilon_{12} \\
\varepsilon_{13} \\
\varepsilon_{23}
\end{pmatrix} = \begin{pmatrix}
 a_{11} & a_{12} & a_{13} & 0 & 0 & 0 \\
 a_{12} & a_{11} & a_{13} & 0 & 0 & 0 \\
 a_{13} & a_{13} & a_{11} & 0 & 0 & 0 \\
 0 & 0 & 0 & a_{11} - a_{12} & 0 & 0 \\
 0 & 0 & 0 & 0 & a_{44}/2 & 0 \\
 0 & 0 & 0 & 0 & 0 & a_{44}/2 \\
\end{pmatrix}
\begin{pmatrix}
\sigma_{11} \\
\sigma_{22} \\
\sigma_{33} \\
\sigma_{12} \\
\sigma_{13} \\
\sigma_{23}
\end{pmatrix}
\]  

(5d)

It can be seen that the formulae for numerical simulation for the adhesive contact for transversely isotropic materials have the same mathematical form as the corresponding formulae for isotropic materials, except that the effective elastic contact modulus \(E^T\) has different expression for different materials.

Theoretically, the contact domain \(\Omega\) should be infinite, however, due to the fact that the pressure decreases very rapidly as the air gap between the two surfaces becomes larger, the computational domain can be taken as a finite rectangle \(\Omega = [-a, a] \times [-b, b]\). The total normal load \(f\) can be written as follows:

\[
f = \int_{\Omega} p(x,y) dx dy
\]  

(6)

To implement the formulae into numerical simulation, we then introduce the following dimensionless variables:

\[
H = \frac{h}{\varepsilon} - 1, \ D = \frac{a}{\varepsilon}, \ \mu = \left(\frac{W_{ad}}{W_{ad}^{1/3}}\right)^{1/3}, \\
U_0 = \frac{h}{\varepsilon}, \ X = x\left(\frac{b}{\varepsilon}\right)^{1/3}, \ Y = y\left(\frac{b}{\varepsilon}\right)^{1/3}, \\
P = \frac{p\varepsilon}{W_{ad}}, \ F = \frac{P^{1/2}f^{1/2}}{3\varepsilon W_{ad}}
\]  

(7)

where \(D\) is the normalized indentation depth. The parameter \(\mu\) is the so called Tabor parameter.[32] This parameter, proposed by Tabor in 1976, is often used to decide whether the JKR or DMT model would best describe a contact system, as Greenwood [30] concluded that the limits of the Maugis–Dugdale model correspond to Tabor’s limits, i.e. the small values of Tabor parameter describe material behavior in DMT-type regime, and the large values of the Tabor parameter describe the contact behavior for JKR-type regime. It has been recently shown that, for low values of the Tabor parameter, a modification of the length scale can lead to a better agreement with the contact model.[33]
Then the normalized Equations (2b), (3), and (6) can be written in the following forms:

\[ P = \frac{8}{3} [(H + 1)^{-9} - (H + 1)^{-3}] \quad (8) \]

\[ H = -D + U_0 + \frac{8\mu^{3/2}}{3\pi} \int_\Omega \int_\Omega \frac{[H(X', Y') + 1]^{-9} - [H(X', Y') + 1]^{-3}}{\sqrt{(X - X')^2 + (Y - Y')^2}} \, dX' \, dY' \quad (9) \]

\[ F = \frac{1}{3\pi} \int_\Omega \int_\Omega P(X, Y) \, dX \, dY \quad (10) \]

where \( U_0 = (X^2 + Y^2)^{d/2} \). The power-law shaped axisymmetric indenters for five different values of \( d \): 1.00, 1.25, 1.50, 1.75, and 2.00 are plotted in Figure 2.

Equation (9) is then solved by a virtual state relaxation (VSR) method: the indentation depth \( D \) is gradually increased, and the \( H \) vector obtained from the previous step is used as an initial state for computing \( H \) vector in the next step. In each step, we let time evolve until the final state of equilibrium is reached. This method accurately plots all the stable equilibria for each value of \( D \). In all the simulation cases, we first increase the value of \( D \) from minimum to the maximum indentation depth to simulate the approach process, and then we decrease the value of \( D \) back to minimum to simulate the detachment process.

In the next section, the force-displacement curve and the contact area will be discussed based on the numerical results. As discussed by Greenwood [30], any criterion to define contact area can be disputable because the air gap in this problem formulation is assumed to be always nonzero. In the current study, we will adopt Greenwood’s definition [30] for contact area, which means the edge of contact area will be regarded as the location of the

**Figure 3.** The normalized force vs. the normalized displacement curves for the monomial indenters with different values of \( d \). It shows that our numerical results agree very well with the analytical solutions, and the magnitude of the normalized pull-off force is monotonically increasing with increasing \( d \).
Figure 4. Normalized pressure distributions $P$ for the elastic samples indented by the monomial indenters with five different values of $d$ when the normalized indentation depth: $D = 0.1$ during approach: (a) $d = 1.00$; (b) $d = 1.25$; (c) $d = 1.50$; (d) $d = 1.75$; and (e) $d = 2.00$. The edge of contact area is colored by the deepest shade of blue.
tensile peak stress. In the following discussion, the normalized contact radius obtained directly from the numerical simulation is written as \( R_c = \sqrt{X_c + Y_c} \), where \((X_c, Y_c)\) is the coordinate for the tensile peak stress.

**Numerical simulation results**

The numerical simulation results will be compared with existing analytical solutions. Therefore, we first provide a brief review of some basic analytical results presented by Borodich et al. [18]

The normal force \( f \), the separation distance \( \alpha \), and the contact radius \( c \) have the following relationship as shown below [14]:

\[
f = C(d)B_d E'^{d+1}c^d - \sqrt{8\pi W_{ad}E'^{d+1}} \frac{c^3}{E'^{d+1}d}
\]

\[
\alpha = B_d C(d) \frac{d+1}{2d}c^d - \left( \frac{2\pi W_{ad}c}{E'^{d+1}} \right)^{1/2}
\]
where $C(d) = \frac{d^2}{d+1} 2^{d-1} \frac{[\Gamma(d/2)]^2}{\Gamma(d)}$ with $\Gamma$ being the Euler gamma function. We then define the following dimensionless variables:

$$\bar{f} = \frac{f}{f^*}, \quad \bar{\alpha} = \frac{\alpha}{\alpha^*}, \quad \bar{c} = \frac{c}{c^*}$$

(13)

$$f^* = \left( \frac{8\pi W_{ad} d + \left( \frac{8\pi W_{ad}}{E^{1/2}} \right)^{1/(2d-1)} \Gamma(d/2)^2}{C(d) d B^2} \right)^{1/(2d-1)},$$

$$\alpha^* = \left( \frac{8\pi W_{ad}}{E^{1/2}} \right)^{1/(2d-1)},$$

$$c^* = \left( \frac{8\pi W_{ad}}{E^{1/2} C(d) d B^2} \right)^{1/(2d-1)}$$

(14)

Figure 5. The normalized contact radius vs. the normalized force for the monomial indenters with five different values of $d$.

Figure 6. The normalized contact radius vs. the normalized indentation depth for the monomial indenters with five different values of $d$. 
Equations (11), (12) can be written as the following dimensionless relations:

\[
\begin{align*}
\tilde{f} &= \tilde{c}^{d+1} - \tilde{c}^{3/2}, \\
\tilde{\alpha} &= \frac{d+1}{d} \tilde{c}^{d} - \tilde{c}^{1/2}
\end{align*}
\]  

(15)

Note that Equation (15) does not depend on any elastic material property, and therefore, it has the same dimensionless form for both isotropic and transversely isotropic materials. When we use Equation (14) to obtain the dimensional results, for transversely isotropic materials, we can directly use \( E_{TI} \); for isotropic materials, however, we need to replace \( E_{TI} \) by \( E \).

Particularly, when \( d = 2 \), by replacing \( E_{TI} \) by \( E \), it can be seen that Equation (15) reduces to the classical JKR model for spherical indenters. In the JKR model, the Hertz contact pressure is superposed to the singular adhesive tractions, akin to the elastic crack-tip stress fields. The Griffith fracture criterion then yields the decohesion criterion. A contact close to a true JKR case is for the polymeric surfaces where low modulus and high effective adhesion energies are expected. This model can be written as [34]:

\[
\begin{align*}
\tilde{f} &= \tilde{c}^{3} - \tilde{c}^{3/2}, \\
\tilde{\alpha} &= \frac{3}{2} \tilde{c}^{2} - \tilde{c}^{1/2}
\end{align*}
\]  

(16)

In the Hertz contact theory, any surface interactions such as near contact van der Waals interactions, or contact adhesive interactions are neglected, and it gives the following relationships [34]:

\[
\begin{align*}
\tilde{f} &= \tilde{c}^{3}, \\
\tilde{\alpha} &= \frac{3}{2} \tilde{c}^{2}
\end{align*}
\]  

(17)

A more involved model, the DMT model, also considers van der Waals interactions outside the elastic contact regime, which gives rise to an additional load, and it can be written as [34]:

\[
\begin{align*}
\tilde{f} &= \tilde{c}^{3} - \frac{1}{3}, \\
\tilde{\alpha} &= \frac{3}{2} \tilde{c}^{2}
\end{align*}
\]  

(18)

The relationships between \( F \) and \( \tilde{f} \), \( R_c \) and \( \tilde{c} \), and \( D \) and \( \tilde{\alpha} \) are as follows:

\[
\begin{align*}
\frac{3\pi C(d)^{3/(2d-1)}}{(8\pi)^{(d+1)/(2d-1)}} F &= \tilde{f} \mu^{\frac{d+3}{2d-1}} \\
\frac{C(d)^{2/(2d-1)}}{(8\pi)^{1/(2d-1)}} R_c &= \tilde{c} \mu^{\frac{1}{3d-1}} \\
\frac{C(d)^{1/(2d-1)}}{2^{(d+1)/(2d-1)} \pi^{d/(2d-1)}} D &= \tilde{\alpha} \mu^{\frac{d}{4d-2}}
\end{align*}
\]  

(19-21)

To further understand the adhesive contact for power-law shaped axisymmetric indenters, we here numerically simulate the indentation tests with power-law shaped axisymmetric
indenters for five different values of $d$: 1.00, 1.25, 1.50, 1.75, and 2.00. In the numerical simulations, we assume that the value of the Tabor parameter is always equal to 1.5. The normalized force $\bar{f}$ vs. the normalized displacement $\bar{\alpha}$ curves for monomial indenters with different values of $d$ are shown in Figure 3. The analytical solutions presented by Borodich et al. are also plotted for the purpose of comparison.\cite{18} It shows that our numerical results agree very well with the analytical solutions, and the magnitude of the normalized pull-off force $\bar{f}_c$ is monotonically increasing with increasing $d$. The pull-out force is the maximum tensile force which needs to be applied to break the adhesive contact and rip the indenter.
off the surface. The discrepancies between numerical solution and analytical solution when the contact radius is small is caused by the fact that when the value of Tabor parameter is not very small, the contact and separation between two surfaces do not occur smoothly, and there are sudden jumping in or jumping out of contact behaviors. The adhesive contact between the two surfaces is unstable during both initial contact and separation. This elastic instability is regarded as a common mechanism for energy dissipation during adhesive contact. When two bodies move closer from a large separation, a turning point exists indicating the jumping-on of contacting surfaces when they move infinitesimally closer. On the other hand, when two bodies are pulled off from a contact state, the turning point corresponds to the jumping-off of contacting surfaces. The middle part of the solution branch between the two turning points represents unstable equilibrium states. This issue has been discussed in detail by Greenwood [30]. The relaxation technique we used can only plot the stable solutions. The unstable solutions which cannot be observed in real-world experiments, can be

![Normalized pressure distributions](image)

**Figure 9.** Normalized pressure distributions $P$ for the elastic samples indented by the monomial indenter with $d = 1.50$ when the normalized indentation depth: $D = 2.0$ during approach with four different values of Tabor parameter: (a) $\mu = 0.3$; (b) $\mu = 0.5$; (c) $\mu = 0.7$; and (d) $\mu = 1.5$.  

Downloaded by [Syracuse University Library] at 11:34 11 April 2016
obtained using the Riks arc length method of integration.\cite{35} It is indeed a powerful method to find the unstable equilibria, but the computational cost will be significantly increased.

One of the advantages of conducting numerical simulations is that the pressure distribution within the contact area can be obtained. There exists no analytical solution for the pressure distribution within the contact region. Figure 4 plots a series of normalized pressure distributions in the simulation domain $[0, 4.0] \times [0, 4.0]$ for the elastic samples indented by the monomial indenters with five different values of $d$ when the normalized indentation depth $D = 0.1$ during approach. Positive values of $P$ represent compressive forces between surfaces. The edge of contact area can be regarded as the location of the tensile peak stress, which is colored by the deepest shade of blue in our color scheme. It is an interesting phenomenon that by decreasing the value of $d$ from 2.0 to 1.0, the maximum compressive contact pressure at the center of the contact area gradually forms a sharp peak. This is because as the value of $d$ is decreased from 2.0 to 1.0, the shape of the indenter changes from a sphere to a sharp cone, as illustrated in Figure 2. Neither the classical JKR nor the DMT model shows the sharp peak of compressive stress because those classical models only deal
with sphere indenters which do not exhibit the stress concentration inherent in the sharp indenters. Figure 5 shows the normalized contact radius $\overline{c}$ vs. the normalized force $\overline{f}$ for the monomial indenters with five different values of $d$, and Figure 6 shows the curves for the normalized contact radius $\overline{c}$ vs. the normalized indentation depth $\overline{\alpha}$.

**Discussion**

By assuming that the value of the Tabor parameter is equal to 1.5, the numerical simulations have been so far carried out only in the JKR adhesion regime. In the JKR-type contact problems, an important assumption is that the surface interaction is absent outside the contact area. The DMT theory [6] deploys cohesive surface forces outside the contact zone, while retaining the Hertzian force-deformation characteristics in the core. This results in the adhesive stress being zero inside the contact area and finite outside it. The contact problems for stiff materials, small spheres, and weak adhesion are usually considered as DMT-type contact problems.

All the analytical solutions are dealing with limit cases, i.e. when the value of Tabor parameter is very large or very small. There exist no analytical solutions for the contact problems with the monomial indenters in the JKR-type-to-DMT-type transition regimes. To explore the DMT-type regimes and the transition regimes, the contact problem with the monomial indenters is simulated for the case of $d = 1.50$ for five different values of Tabor parameter: 0.1, 0.3, 0.5, 0.7, and 1.5. Figure 7 plots the normalized force $\overline{f}$ vs. normalized displacement $\overline{\alpha}$. Figure 8 plots the normalized contact radius $\overline{c}$ vs. normalized force $\overline{f}$. The classical JKR model, Hertz contact model, and the DMT model for spherical indenters are also plotted for the purpose of comparison. It can be seen that as the value of the Tabor parameter is increased, the load-displacement curves develop a cusp and resulting unstable jumps. With decreasing value of Tabor parameter, the force vs. displacement curve becomes closer to the curve predicted by the DMT model. However, the contact radius vs. force curve does not converge to the DMT prediction.

Figure 9 plots a series of normalized pressure distributions in the simulation domain $[0, 4.2] \times [0, 4.2]$ for the elastic samples indented by the monomial indenter with $d = 1.50$ at the normalized indentation depth $D = 2.0$ during approach. This simulation has been performed using four different values of the Tabor parameter. When the values of Tabor parameter is small, very large values of pressure can be observed within the contact region. Numerical simulation packages for elastic-plastic adhesive contact problems can be developed based on the current numerical scheme, but this endeavor is left for future work.

**Conclusion**

In contact mechanics, the well-known JKR and Boussinesq-Kendall models describe frictionless adhesive contact between two linearly elastic isotropic spheres, and between a flat indenter and a linearly elastic isotropic half-space, respectively. However, in experimental studies, such as nanoindentation tests, the shapes of the indenters are often more general than spherical or flat ones. Moreover, very often, the materials to be tested are anisotropic. The analysis of a wide range of physical problems with adhesive contact on transversely isotropic materials requires a general numerical method.
In this paper, numerical studies on JKR-type frictionless adhesive contact between power-law shaped indenters and transversely isotropic materials are presented. It shows that the formulae for numerical simulations for JKR-type frictionless adhesive contact for transversely isotropic materials have the same mathematical form as the corresponding formulae for isotropic materials, except that the effective elastic contact modulus has different expression for different materials. The numerical results show good agreement with existing analytical solutions. The DMT-type and JKR-type-to-DMT-type transition regimes have been explored by conducting the simulations using smaller values of Tabor parameters.

In the numerical simulation, the adhesive interactions are represented by an interaction potential and the surface deformations are coupled using half-space Green's functions discretized on the surface. The resulting set of equations is highly nonlinear and is solved by VSR technique. The good agreement between the numerical results and the analytical solution for power-law shaped indenters shows that this numerical technique can be applied to study indentation problems with arbitrary shaped indenters.[36,37] In the next step, this numerical simulation package will be improved by adding the effect of surface tension, as recently observed deformations of soft substrates, such as plasticized polystyrene,[38] hydrogels [39], and silicone gels [40] caused by adhesion of micro-/nano-scale bodies show considerable deviation from the JKR model.

Acknowledgment

The author gratefully thanks the anonymous reviewers for their critical comments.

Funding

This work is supported by start-up funds provided by the Department of Mechanical Engineering at State University of New York at Binghamton.

References

Appendix A. Detailed derivation of integration of Lennard–Jones potential over surface area

Following the derivation provided by Jagota and Argento [41] and Yu and Polycarpou,[42] the atomic interactions of macroscopic bodies can be represented by surface tractions. The volume integrals that describe the interaction force between atoms in one body and atoms in the other body could be replaced by surface integrals, by considering surface traction $p = \mathbf{h} \times \mathbf{n}$, where $\mathbf{h}$ is a second order tensor, defined by $h = \rho_1 \rho_2 \int S \mathbf{n}_2 G dS_2$ with the integration over the surface of the second body. The parameters $\rho_1$ and $\rho_2$ are the number density of the atoms of the two bodies; $\mathbf{n}_1$ and $\mathbf{n}_2$ are the outer surface normals to the two bodies, and $G$ is defined as $G = (X_2 - X_1) \nu(s)$ with $X_2 - X_1$ being a vector from the surface of the second body to the surface of the first body, and $\nu(s)$ is defined as $\nu(s) = \frac{1}{r} \int_0^s U(x) x^2 dx$ where $U$ is the interatomic potential. When both surfaces are planes, the integrations could be carried out analytically. Substituting the equation for $U$ as provided by the Equation (2a) into the equations for $p$, the surface traction due to the interatomic Lennard–Jones potential is in the $z$ direction and can be obtained as $p(h) = \frac{2\pi \rho_1 \rho_2 \sigma^6 (15h^2 - 2h^3)}{4h^6}$. Making use of the definition of Hamaker constant [43] $A = 4\pi \rho_1 \rho_2 \sigma^6$, we can obtain the following results: $\frac{p(h)}{h} = \frac{\Lambda}{6\pi h^3} - \frac{\Lambda\sigma^6}{45\pi h^{10}}$. When the two surfaces are at equilibrium distance, i.e. $p$ is equal to zero, we can obtain $h = \varepsilon = (2/15)^{1/6} \sigma$. The adhesion energy $W_{ad}$ is the work done to move two surfaces from equilibrium separation $\varepsilon$ to infinity, i.e. $W_{ad} = \int_0^\infty p(h) dh = A/(16\pi \varepsilon^2)$. Therefore, $p$ could be rewritten as $p(h) = \frac{8W_{ad}}{3\varepsilon} \{(\varepsilon/h)^9 - (\varepsilon/h)^7\}$, which is just Equation (2b).

Appendix B. Validity region of the DMT model in predicting pull-off forces.

The DMT model assumes that the surface forces do not change the deformed profile from the solution corresponding to the Hertz equations. The attractive forces only lie outside the contact area with the contact region under compression with a Hertzian stress distribution. The DMT model calculates the attractive forces outside the contact zone pulling the bodies together. When the deformation is such that the elastic reaction force balances the combined effect of the surface forces plus any externally applied load, equilibrium is reached. According to the DMT model, for a sphere of radius $R$ in contact with a flat surface, at point contact prior to any deformation, the attractive force is equal to $2\pi R \sigma d$, but as the sphere deforms, this force rapidly decreases to $\pi R \sigma d$. If an external load $P$ is then applied, the deformation corresponds to the Hertz solution for an applied load of $P + \pi R \sigma d$. This model shows that, as the surfaces are separated, the maximum force at separation occurs at point
of contact, the force of adhesion being equal to $2\pi R W_a d$. The difference between the JKR model and the DMT model regarding the pull-off behavior lies in the fact that the JKR model predicts separation at a finite contact area, whereas the DMT model predicts separation at point contact. For small, fairly rigid spheres with low surface energy, the contact behavior is the same as the one predicted by the DMT model, but for larger, more elastic spheres with high surface energy, separation of the sphere and flat occurs at a finite contact area even when the deformation is still assumed to be Hertzian. The behavior is similar to that predicted by the JKR model where a different deformation mode is assumed. However, the instability at a finite contact area is not due to infinite tensile stresses around the contact periphery, but arises from the sharp fall in surface forces as the contact area is decreased. Therefore, the DMT model does predict many important features of a more rigorous analysis, but the quantitative agreement is not good, as further explained by Pashley.[44]