The Structure of the Election-Generating Universe

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Abstract:

We use two sets of ranking data, one from actual elections and the other from surveys of voters, to examine whether the outcomes of three-candidate vote-casting processes follow a discernible pattern. We evaluate twelve statistical models that make different assumptions about such a pattern, and find that a spatial model describes the observations in both data sets much more accurately than any of the other eleven models. Our finding implies that any conclusions about the probability of voting phenomena in actual elections are suspect if they are reached on the basis of models other than the spatial model.

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1. INTRODUCTION

How often do events of interest to voting theorists occur in actual elections? For example, what is the probability of observing a voting cycle—an outcome in which no candidate beats all other candidates in pairwise comparison by majority rule? When there is a candidate who beats all others in such pairwise comparisons—a Condorcet winner—what is the probability that a voting method chooses this candidate? What is the probability that voters have an incentive to vote strategically—that is, cast their votes in ways that do not reflect their true preferences? Voting theorists have analyzed these questions in great detail, using a variety of statistical models that describe different distributions of candidate rankings. But there has been no systematic effort to determine which statistical model comes closest to describing the distribution of rankings of candidates in actual elections. Thus we know how often various voting events occur under different statistical models, but not how often voting events occur in actual elections. This paper provides a first step toward finding an answer to this question.

We consider elections in which each voter is asked to submit a strict ranking of M candidates. Such elections can be viewed as outcomes of a statistical process in which each trial has $M!$ possible outcomes (the possible strict rankings of the M candidates). The set of all possible vectors of length $M!$ whose components sum to 1 forms the space of feasible generating probabilities for elections, which we call the "election generating universe." The vector of probabilities that apply to the voters in one election represents one point in this space. We refer to the probability distribution over feasible vectors of probabilities of rankings that generate actual—as opposed to theoretically imaginable—elections as the "structure of the election-generating universe."[1] In this paper we identify a model that comes surprisingly close to describing this structure.

[1] We borrow our terminology from cosmology, using the distribution of stars within the universe as an analogy for the distribution of candidate rankings within the space of feasible generating probabilities for elections. The "structure of the election generating universe" is analogous to the term "large-scale structure of the universe" that cosmologists use to describe the observable—as opposed to theoretically imaginable—distributions of matter and light in the universe.

We consider ten models of the structure of the election-generating universe that have been proposed by others, as well as two new models, for the case of three candidates. We evaluate these twelve models with two sets of voting data. Our first data set was assembled by Nicolaus Tideman in 1987 and 1988, and it consists of individual ballot information for 87 elections that we transform into 883 three-candidate elections.[2] Our second data set consists of 913 three-candidate "elections" that we construct from the "thermometer scores" that are part of the surveys conducted by the American National Election Studies (ANES).[3] The results that we obtain from these two rather different data sets are very consistent and indicate that a spatial model describes the structure of the election-generating universe very well—much better than any of the models that have so far been used in theoretical analyses of voting events, and well enough that it may be difficult to devise an alternative model that would describe actual election data significantly better.

The spatial model is too complex to permit the type of theoretical analyses of the probabilities of voting events that theorists have undertaken so far (as an example, see Gehrlein, 2002). Thus we envisage using this model for Monte Carlo simulations to generate data that have the same characteristics as data from actual elections. Simulating several millions of such elections will permit us to estimate the probabilities of various voting events. Such simulations would be unnecessary if there were enough data from actual elections in which voters rank the candidates to determine the frequencies of rare voting events with a reasonable degree of accuracy. But there are not nearly enough ranking data for this task. For example, Brams and Fishburn (2001) and Saari (2001) use data from a single election—the 1999 election for

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[2] These data have been analyzed previously by Feld and Grofman (1990 and 1992), Felsenthal *et al*. (1993), Felsenthal and Machover (1995), Tideman and Richardson (2000), Regenwetter *et al*. (2002), and Tideman (2006).

[3] ANES survey data have been used in several previous analyses of voting. For example, Chamberlin and Featherston (1986) use scores from ANES surveys administered in 1972, 1974, 1976, and 1978 to construct combinations of four candidates. Regenwetter *et al.* (2002 and 2003) analyze the thermometer scores of the three major candidates in the four ANES surveys administered in 1968, 1980, 1992, and 1996 to construct combinations of three candidates. Our method of constructing three-candidate "elections" is the same as that in these earlier analyses.

president of the *Social Choice and Welfare Society*—to illustrate and analyze the properties of different voting methods. Chamberlin and Featherston (1986) analyze data from five presidential elections of the *American Psychological Association*, while Regenwetter *et al*. (2002) use data from 12 elections for positions in professional organizations. Tideman's data set of 87 elections is one of the largest data sets of elections with multiple candidates that voting analysts have used. Even our two data sets with 883 and 913 three-candidate elections cannot provide reliable information about the frequencies of rare voting events. But if we are able to infer the structure of the election-generating universe from these elections, then data simulated to conform to this structure can reveal the frequencies of voting events of interest.

The remainder of this paper is organized as follows: we introduce the twelve models of the structure of the election-generating universe in Section 2, and we explain our strategy for assessing the accuracy of these models in Section 3. We describe our data and report the results of our statistical analysis in Section 4, and conclude in Section 5.

2. Twelve Statistical Models of the Structure of the Election-Generating Universe

Consider an election with M candidates, in which each of N voters submits a strict ranking of the candidates. There are $M!$ possible strict rankings. Let the random variable N_r describe the number of votes for ranking r, $r = 1, \ldots, M!$, let p_r be the vote share for ranking r with $\Sigma p_r = 1$, and let $p = \{p_1, \ldots, p_{M!}\}$ be a vector of vote shares of length $M!$. The collection of feasible p forms the election-generating universe. The requirement that $\Sigma p_r = 1$ implies that the election-generating universe is contained in a unit $(M! - 1)$-simplex. Let P be a random vector of length $M!$ that is defined on the collection of feasible p. A statistical model of the structure of the election-generating universe describes the distribution of P over all elections.

Statistical models of the structure of the election-generating universe differ in the probabilities that they assign to different p as well as in range of P. It is straightforward to describe the differences among the probability structures, but less straightforward to illustrate the differences in ranges for general M. However, it is customary in the theoretical literature of voting events to restrict the number of candidates, and many theoretical results are available for elections with three candidates only. We continue this tradition and restrict our analysis in this paper to elections with $M = 3$; this permits us to represent the corresponding 5-simplex as a three-dimensional octahedron, which we use to derive intuitive graphical illustrations of the differences among the ranges of different models. In this octahedron, each of the six vertices represents a vector p with one probability equal to 1 and the remaining 5 probabilities equal to zero, each of the 15 edges (including the 3 virtual edges connecting pairs of opposite vertices) represents a vector p with 4 probabilities equal to zero while the remaining two p_r sum to 1, and each of the 20 (real and virtual) faces represents a vector p with 3 probabilities equal to zero while the remaining three p_r sum to 1. The point at the center of the octahedron represents the vector of equal probabilities with each $p_r = 1/6$. Although such a three-dimensional representation of a five-dimensional space cannot distinguish among all vectors of probabilities permitted by the five-dimensional space, it is nevertheless sufficient to illustrate the differences in the ranges of all but one of the models that we analyze. To simplify the exposition, we label the three candidates A, B, and C, and order the six rankings {ABC, ACB, CAB, CBA, BCA, BAC} so that ABC is ranking $r = 1$, ACB is ranking $r = 2$, and so on.

We classify the statistical models that describe the distribution of P into four categories. Models in the first category have a range of the entire 5-simplex and accept every feasible vector p as a potential source of elections. Models in categories 2 through 4 assign zero probability as the source of an election to all points in the simplex except for subsets of measure zero that contain the subspaces of permitted probability vectors. Models in category 2 are of zero dimensionality and consist of either a single point or a single set of symmetric points within the simplex. Models in categories 3 and 4 are of higher dimensionality; those in category 3 are specified by linear restrictions on the unit simplex, while those in category 4 impose non-linear restrictions on the unit simplex. We describe each model below and summarize the properties of all twelve models in Table 1.

2.1 Models whose ranges are the entire unit simplex

Our first model, the Impartial Anonymous Culture (IAC), was proposed in Kuga and Nagatani (1974) and Gehrlein and Fishburn (1976a). This model assumes that all points within the 5-simplex are equally likely. Figure 1a shows the range of IAC—the entire octahedron. Several voting theorists have used IAC to calculate the probability that certain voting events will occur,[4] but they generally emphasize that they do not necessarily believe that equally likely probabilities of strict rankings describe vote-casting.[5] We are not aware of any formal test of the statistical adequacy of this assumption.

Several variations on IAC were developed specifically to examine the probability of observing Condorcet's paradox, and we consider three of them. Our second model, IAC$_b$(k_b) was introduced in Gehrlein (2004). As with IAC, the range of IAC$_b$(k_b) is the entire simplex, but the probabilities of IAC$_b$(k_b) differ in a subtle way from those of IAC. Assume that one candidate, say *C*, is ranked last no more often than either *A* or *B*, so that the frequency with which *C* is ranked last is a real number k_b between 0 and 1/3. IAC$_b$(k_b) assumes that the probability of *ABC* is distributed uniformly on the interval [0, k_b], and that the probability of *BAC* equals k_b minus the probability of *ABC*. The probabilities of the four other rankings *ACB*, *BCA*, *CBA*, and *CAB* are distributed uniformly on the tetrahedron whose components sum to $1 - k_b$. Thus while the components of *P* are determined simultaneously under IAC, they are determined sequentially in two steps under IAC$_b$(k_b). Lepelley (1995) considers the case of this model when $k_b = 0$, so that the range of the model is limited to a subset of the unit simplex. We consider this limiting case of IAC$_b$(k_b) below as model 7.

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[4] For example, Saari (1990) uses this assumption to analyze the probability of strategic voting under different voting methods, Gehrlein (2002) uses it to analyze the probability of observing Condorcet's paradox, and Cervone *et al*. (2005) use it to analyze the probability that a Condorcet candidate, if it exists, will win the election.

[5] See, for example, Gehrlein (2002, p. 169) and Cervone *et al*. (2005, p.182).

Gehrlein (2006) describes two variations on $IAC_b(k_b)$ that he calls $IAC_t(k_t)$ and $IAC_c(k_c)$. $IAC_t(k_t)$ assumes that one candidate is ranked first no more often than the other two candidates, while $IAC_c(k_c)$ assumes that one candidate is ranked in the middle no more often than the other two candidates.[6] In both models, the probabilities of the six rankings are determined analogously to those in $IAC_b(k_b)$. We analyze $IAC_t(k_t)$ and $IAC_c(k_c)$ as models three and four. The exact values of p in models 2 – 4 depend on the unknown parameters k_b, k_t, and k_c. In the appendix we describe how we calibrate these parameters to our two data sets.

2.2 Models whose ranges are composed of 0-dimensional subsets of the unit simplex

The most restrictive of the models that restrict the range to a proper subset of the unit simplex is the impartial culture (IC), which assumes that the range of P is a single point at the center of the simplex where all rankings are equally likely, or $p_r = 1/6$. Figure 1b shows this range as the octahedron's center. Its computational simplicity made IC popular in early Monte Carlo studies (see, for example, Campbell and Tullock, 1965), but there is now considerable empirical evidence that IC does not describe the structure of the election-generating universe (see Regenwetter *et al.*, 2006, for a summary).

Our sixth model, which was proposed in Chamberlin and Featherston (1986) and which we call Unique Unequal Probabilities (UUP), assumes that in every election, each candidate occupies a specifiable ranking niche (first, second, etc.), and that for each possible ranking of the candidates described by these niches, there is a constant probability that this ranking will be used in an election. UUP restricts the range of P to a set of 6 points, one for each permutation of the rankings. Figure 1c shows the range of UUP—six points in symmetric locations in the octahedron. Unlike IC, UUP does not specify the values of the six probabilities, so it has 5 unknown parameters that need to be estimated (see the appendix).

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[6] $IAC_b(k_b)$ measures the proximity of voter preferences to the case of single-peakedness (which occurs at $k_b = 0$), while the other two representations measure the proximities to "single-troughness" and "single-centeredness."

2.3 Models whose ranges are more than 0-dimensional and are specified by linear restriction on the unit simplex

Our seventh model is the limiting case of $IAC_b(k_b)$ with $k_b = 0$ that was proposed in Lepelley (1995). We refer to this model as SPP because it ensures single-peaked (group) preferences, making a voting cycle impossible. The range of SPP consists of all vectors p with either $p_4 = p_5 = 0$ (A is never ranked last), $p_2 = p_3 = 0$ (B is never ranked last), or $p_1 = p_6 = 0$ (C is never ranked last). The pairs of restrictions imply that each case could be viewed as a tetrahedron; if we could see in five dimensions, then the range of SPP would be three tetrahedrons of which any two tetrahedrons share an edge where the appropriate four probability restrictions hold. Figure 1d depicts this range in three dimensions.

Our eighth model, the *Dual Culture* (DC) proposed in Gehrlein (1978), assumes that the probabilities of opposite rankings are equal, that is, $p_1 = p_4$, $p_2 = p_5$, and $p_3 = p_6$. In contrast, our ninth model, the *Uniform Culture* (UC) proposed in Gehrlein (1982), assumes that the probabilities of neighboring rankings are equal, that is, $p_1 = p_2$, $p_3 = p_4$, and $p_5 = p_6$. Neither model specifies the probabilities of any of the three pairs, which thus need to be estimated (see the appendix). Each set of three equalities specifies a plane in the 5-simplex. Both the range of DC and the range of UC can be represented by a triangle determined by the midpoints of three edges of the octahedron. These triangles appear to be coplanar in the octahedron (see Figure 1e), but in five dimensions their only common point is their centers where all probabilities equal 1/6. Thus both models include IC as limiting case.

2.4 Models whose ranges are curved subsets of the unit simplex

As tenth and eleventh model of the election-generating universe, we investigate two models for which the Borda voting method and the Condorcet voting method, respectively, are maximum likelihood estimators. The “Borda model” supposes that there is a “best candidate” and evaluates the evidence that each candidate is best. Such evidence is measured by the number of times a candidate is ranked second plus twice the number of times the candidate is ranked first.

Conitzer and Sandholm (2005) showed that this measure of the evidence is optimal only if the probabilities for both rankings with the best candidate first are equal, the probabilities for both rankings with the best candidate second are equal, and the probabilities for both rankings with the best candidate third are equal. That is, $p_1 = p_2$, $p_3 = p_6$, and $p_4 = p_5$ if A is best, $p_1 = p_4$, $p_2 = p_3$, and $p_5 = p_6$ if B is best, and $p_1 = p_6$, $p_2 = p_5$, and $p_3 = p_4$ if C is best. In addition, the probabilities for the pairs of rankings must follow a geometric sequence, so that

$$p_r = c_1 e^{\beta w_r} \tag{1}$$

where w_r denotes the position that the "best candidate" occupies in ranking r, β is a constant, and

$$c_1 = 1/\left(5\sum_{m=0}^{5} e^{m\beta}\right) \tag{2}$$

to ensure $\Sigma p_r = 1$. Because the Borda model does not distinguish between rankings that rank a given candidate in the same position but differ in the positions they assign to the other candidates, its range includes the midpoints of the three edges between rankings that list a given candidate first (where $\beta = \infty$), but the range includes none of the vertices. The three curved lines in Figure 1e that start at the center of the octahedron (where $\beta = 0$; here the Borda model nests IC) and end midway between the pairs of vertices where a given candidate is first ($\beta = \infty$) depict the range of the Borda model for $0 < \beta < \infty$. The differences among the three equality restrictions imply that these three curved lines lie in three different planes.

Analogous to Conitzer and Sandholm's (2005) derivation of the Borda model, we define our eleventh model so that the Condorcet voting method is a maximum likelihood estimator of the ranking that is most favorable in terms of the statistical model that we assume has generated the election data (the "correct ranking"). While the Borda voting method assigns a score to each of the M candidates, the Condorcet voting method assigns a score to each of the $M!$ possible rankings.[7] Let r^* be the correct ranking and let n_{rr^*} be the number of pairs of candidates that are ranked the same in ranking r and ranking r^*. The Condorcet model specifies the components of p as

$$p_r = c_2 \, e^{\gamma n_{rr^*}} \tag{3}$$

where γ is a constant and

$$c_2 = 1 \Big/ \left(\sum_{m=0}^{3} f(m,3) e^{m\gamma} \right) \tag{4}$$

to ensure $\Sigma p_r = 1$, where $f(m, 3)$ is the frequency distribution of Kendall's τ.[8]

The six corkscrew-shaped lines in Figure 1g that start at the center of the octahedron (where the Condorcet model nests IC at $\gamma = 0$) and end at the six vertices ($\gamma = \infty$) depict the range of the Condorcet model for $0 < \gamma < \infty$. The actual range in the 5-simplex is six elongated corkscrews that each span a three-space specified by two opposite vertices (for example, *ABC* and *CBA*) and the two points that are midway between pairs of vertices whose orderings both differ from one of the opposite vertices by a permutation of one pair of adjacent candidates (for example, *BAC* and *ACB* both differ from *ABC* by such permutations).

None of the 11 models discussed so far are based on the belief that the associated distributions of P might actually describe rankings in actual elections. IAC, IC, UUP, DC, and UP assume that various components of p are equally likely, for the sake of algebraic tractability. $\text{IAC}_b(k_b)$, $\text{IAC}_t(k_t)$, $\text{IAC}_c(k_c)$ and SSP seek to describe rankings that have meaningful interpretations for the problem of defining probabilities of observing Condorcet's paradox. The Borda and Condorcet models are rationalizations of claims about how one ought to determine the winner in an election.

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[7] Condorcet's explanation of his method (Condorcet, 1785) was opaque and contained errors; Kemeny (1959) proposed the same voting method in the twentieth century, and Young (1988) explained how Condorcet's intention could be understood despite his errors.

[8] See Kendall and Gibbons (1990, pp. 91 – 92).

In contrast, our final model of the election-generating universe is based on a model of voter behavior, the spatial model of voting. As in other spatial models of voting, our spatial model assumes that voters care about the "attributes" of candidates; these attributes form a multi-dimensional "attribute space."[9] Every voter has an indifference map in attribute space, which contains an "ideal point" that describes the quantities of each attribute that the voter's ideal candidate would possess. Actual candidates also possess specifiable quantities of each attribute and therefore have locations in attribute space. We assume that attribute space has at least 2 dimensions and the candidates are in "general position," where any slight change in the position of any one candidate does not change the dimensionality of the space that they span, so that the positions of the 3 candidates in attribute space span an two-dimensional "candidate plane" that is a subspace of attribute space.[10] Voters' indifference maps are defined in candidate space through their definition in attribute space.

We follow Good and Tideman (1976) and assume that the positions of voters' ideal points in attribute space follow a spherical multivariate normal distribution, which implies that the distribution of "relative" ideal points in candidate space is bivariate normal.[11] We further assume that every voter's utility loss from the choice of a particular candidate is the same increasing function of the distance between the candidate's location in candidate space and the voter's relative ideal point in candidate space, so that every voter's indifference surfaces are concentric spheres centered on the voter's ideal point.[12]

Suppose there is a set of candidates for which every voter submits a truthful ranking that reflects his ideal point, his indifference surfaces, and the positions of the candidates. To

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[9] See Davis *et al.* (1970), and Enelow and Hinich (1984 and 1990).

[10] The case when all candidates' attributes lie in a single line requires special treatment because not all of the 6 possible rankings of the candidates occur, but it does not pose conceptual difficulties. See Good and Tideman (1976, pp. 380 – 381) for a description of the general case with $M > 3$.

[11] A copy of Good and Tideman (1976) is available at http://bingweb.binghamton.edu/~fplass/GoodTideman.pdf.

[12] None of these assumptions is conceptually necessary and each could be replaced—at a cost of more complex calculations—if there is evidence that it does not represent election data sufficiently well. See Good and Tideman (1976) for a discussion.

determine the vote share of each ranking, consider the triangle in the candidate plane that is formed by the locations of the three candidates, *A*, *B*, and *C*. We divide the candidate plane into six sectors by drawing the perpendicular bisectors of the three sides of this triangle. These bisectors intersect at the triangle's circumcenter, *T*. For the voters' ideal points in each sector, the distances to the locations of the three candidates have a unique rank order. These rank orders are indicated in Figure 2, together with the mode of the circular bivariate normal distribution at *O*. The integral of the density function of this distribution over each sector is the expected value of the fraction of the voters who rank the candidates in the order corresponding to the sector's rank order.[13] These six integrals determine the probabilities p_r of the six rankings. Note that even though sectors that are opposite each other have the same angle, they do not have the same integral of the density function (and therefore do not imply the same p_r), unless *O* is inside either of the sectors and the two lines that form the sectors come equally close to *O*. If *O* is exactly at the triangle's circumcenter *T*, then the spatial model coincides with our eighth model DC that assumes $p_1 = p_4$, $p_2 = p_5$, and $p_3 = p_6$.

The range of the spatial model in the 5-simplex is sufficiently complex to make it difficult to represent it in two or three dimensions, but we can offer some insights. Every vertex and every edge of the hexateron is in the range of the spatial model. Of the 20 faces of the hexateron, each specified by three vertices, 18 are in the range of the spatial model. The two faces that are not in the range of the spatial model are the ones that are defined by rankings that form a voting cycle: *ABC*, *BCA*, *CAB* and *CBA*, *BAC*, *ACB*. There are 15 cells in the hexateron, each specified by four vertices and spanning a three-space on the surface of the hexateron. Of these 15 cells, nine are in the range of the model and six are not. The nine that are in the range are the ones for which the two vertices that are not among the four in the defining set differ by a permutation of the positions of two candidates.

[13] We use the algorithm described in DiDonato and Hageman (1980) to compute the integral of the bivariate normal distribution over each sector.

Within the hexateron, the range of the spatial model consists of six curved hyper-cells (four-spaces), that is, one curved hyper-cell for each ordering of the candidates. Figure 3 offers some insights into the shape of these hyper-cells. The figure assumes fixed shares for 3 rankings (here $p_{\text{ABC}} = p_{\text{ACB}} = p_{\text{CAB}} = 1/6$), and shows the combinations of values for the remaining shares that the spatial model allows as the curve within their 2-simplex (formed by $p_{\text{CBA}} + p_{\text{BCA}} + p_{\text{BAC}} = 1/2$). We then increase p_{CAB} to 1/3 and plot the range of the spatial model for $p_{\text{CBA}} + p_{\text{BCA}} + p_{\text{BAC}} = 1/3$ as the curve in the mid-size triangle. Finally, we increase p_{CAB} to 1/2 and plot the range of the spatial model for $p_{\text{CBA}} + p_{\text{BCA}} + p_{\text{BAC}} = 1/6$ as the curve in the smallest triangle. The combination of the three curves suggests that, if we increased p_{CAB} continuously from 0 to 2/3, we would be looking at a saddle-shaped figure, from the front of the saddle. The saddle we show is symmetric because $p_{\text{CBA}} = p_{\text{BCA}}$, and it tilts to one side if $p_{\text{CBA}} \neq p_{\text{BCA}}$. The spatial model nests DC (when the locations of *T* and *O* coincide) and thus IC, and we confirmed empirically that it also nests the Borda model but not the Condorcet model.

3. Model evaluation

For models whose range is less than the 5-simplex, we assess how closely each model is able to match the six vote shares that we observe in actual three-candidate elections, taking account of the fact that the models differ in their degrees of freedom. To assess models whose range is the entire 5-simplex and which can therefore match any observed vector *p*, we undertake Monte Carlo simulations and assess whether the accuracy on the simulated data of the models whose range is a subset of the 5-simplex is comparable to their accuracy on the observed data.

3.1 *Evaluation of models whose range is the entire 5-simplex*

For each such model, we simulate 1,000,000 "elections." To assess whether these simulated data have the same statistical properties as observed election data, we calculate, for each model whose range is a proper subset of the simplex, the mean (multi-dimensional Euclidean) distance from an observed vector of vote shares to the vector that the model describes as the source of the election. We measure the mean distance σ as the mean square root of the sum of the squared differences,

$$\sigma = \frac{1}{E}\sum_{e=1}^{E}\sqrt{\sum_{r=1}^{6}(p_{re}-s_{re})^2}\ , \qquad (5)$$

where E is the number of elections, s_{re} is the observed vote share of ranking r in election e, and p_{re} is the corresponding vote share predicted by the respective model We compare the σ computed from the simulated data with the σ computed from actual elections. If the two measures of distance differ significantly from each other, then this is evidence that the simulated data differ in significant ways from the observed data and that the generating model of the simulated data is unlikely to be a good description of the statistical process that has generated the observed data.

3.2 *Evaluation of models whose range is a proper subset of the 5-simplex*

We assess the relative accuracy of these models through their likelihood functions, which we derive by linking each model's vector p to the number of votes for each of the 6 rankings, the N_r. We consider two statistical models that provide intuitive descriptions of such a link. Our first model assumes that the vector $\{N_r,\ r = 1, \ldots, 6\}$ follows a multinomial distribution with density function

$$f(N_1,\ldots,N_6;N,p_1,\ldots,p_6) = \frac{N!}{\prod_{r=1}^{6} N_r!}\prod_{r=1}^{6} p_r^{\,N_r}\ , \qquad (6)$$

whose first two moments are $E[N_r] = Np_r$, $Var[N_r] = Np_r(1 - p_r)$, and $Cov[N_r, N_s] = - Np_rp_s$.

The multinomial distribution assumes that the p_r are deterministic and implies that the sources of elections will be confined to the subspaces of measure zero predicted by these models. Because this assumption is unrealistic, we consider an alternative statistical model that assigns positive probabilities to points outside the sets of measure zero that the models permit, with higher probability densities assigned to points closer to the sets predicted by the strict models.

Instead of assuming that each election is generated by a deterministic vector p as in equation (6), we assume that this vector is a draw from a vector of random variables Q. A natural assumption is that Q follows a Dirichlet distribution with parameter vector δp, where p is the vector allowed by a strict model and the parameter δ is inversely proportional to the variances of Q. An election is then modeled by compounding the multinomial distribution with the Dirichlet distribution, yielding the multivariate Pólya distribution with density function

$$f(N_1,...,N_{M!};N,p_r,...,p_{M!}) = \frac{N!}{\prod_{r=1}^{M!} N_r!} \frac{\Gamma(\delta)}{\Gamma(N+\delta)} \prod_{r=1}^{M!} \frac{\Gamma(N_r + \delta p_r)}{\Gamma(\delta p_r)}, \quad (7)$$

whose first two moments are $E[N_r] = Np_r$, $Var[N_r] = Np_r(1 - p_r)\psi$, and $Cov[N_r, N_s] = - Np_r p_s \psi$, where $\psi = (N + \delta) / (1 + \delta)$.[14] As δ approaches infinity, ψ approaches 1, the variances of Q approach 0, and the multivariate Pólya distribution converges to the multinomial distribution. Thus δ is an indicator of how well a strict model describes actual elections: a larger fitted value of δ implies that less additional variation in the parameters of the model (that is, variation not explained by the strict model) is necessary to fit the model to actual elections.

For a set of elections whose outcomes are independent of each other, the likelihood function is proportional to the product over all elections of density function (6) or (7). For each model, we estimate the unknown parameters (if any) by maximizing the likelihood function over the observed election data. We describe the parameterization of each model and its degrees of freedom in the appendix. For nested models (for example, the Borda model nested in the spatial model or DC nested in the spatial model), a likelihood ratio test indicates which model yields a better fit of the data, given their different degrees of freedom. Nested as well as non-nested model can be compared by the Akaike and Bayesian information criteria (AIC and BIC) that

[14] See Mosimann (1962, pp. 67 – 68).

account for differences in the degrees of freedom.[15] Models with lower values of AIC or BIC use degrees of freedom more efficiently in describing the data than models with higher AIC or BIC.

4. Empirical Evaluation of the Twelve Models

4.1. *The data*

Our first data set consists of 84 elections that were administered by the Electoral Reform Society (ERS) and tabulated by Nicolaus Tideman in 1987 and 1988 and three elections that he included from another source. For each election, we have individual ballot information about the strict ranking of candidates provided by each voter (the ballots did not permit ties). The number of voters in these elections ranges from 9 to 3,422, with a mean of 410.5, and the number of candidates ranges from 3 to 29, with a mean of 8.7. Most voters ranked only some of the candidates in these elections. We use these ballots to construct all possible combinations of three candidates within an election, for a total of 20,087 three-candidate combinations, with between 1 and 1,957 voters. We treat each combination as one election with three candidates. We found that elections with too few voters contain mostly random noise and do not provide much information about the structure of the election-generating universe. We therefore limit our analysis to elections with more than 350 voters, and our ERS data set contains 883 three-candidate elections with between 350 and 1,957 voters, with a mean of 716.4 voters.[16]

[15] These criteria are determined as AIC = $-2\ln(L) + 2d$ and BIC = $-2\ln(L) + d\,\overline{\ln(N_e)}$, where d is the total number of degrees of freedom, L is the maximum value of the likelihood function, and $\overline{\ln(N_e)}$ is the mean value of the log of the number of voters in an election. Thus BIC imposes a heavier penalty for the use of degrees of freedom than AIC.

[16] Our somewhat arbitrary choice of elections with 350 or more voters reflects our subjective tradeoff between reducing the noise from elections with too few voters and keeping a reasonable number of observations.

The data from these 883 three-candidate elections are not independent, but this lack of independence is unlikely to affect our conclusions.[17] It is possible that such three-candidate combinations that are derived from rankings of more than three candidates are qualitatively different from rankings of elections with exactly three candidates—for example, because it is often simpler to rank three candidates than a larger number of candidates. However, we have individual ballot data for only eight genuine three-candidate elections, which is not sufficient to draw reliable statistical inference about the election-generating universe.

Because no reasonable voting method is immune to strategic voting, it is possible that the rankings in our data set reflect voters' strategic considerations. We therefore examine a second ranking data set that is derived from survey data rather than election data, because the strategic considerations of survey respondents are likely to differ from those of voters. We assemble our second data set from the "thermometer" scores that are part of 18 surveys conducted by the American National Election Studies (ANES) between 1970 and 2004. These surveys are conducted every two years, and participants are asked to rate politicians on a scale from 0 to 100 (the thermometer). We refer to these persons as "candidates."[18]

The number of respondents in a survey ranges from 1,212 in 2004 to 2,705 in 1974, and the number of candidates included in the surveys ranges from 3 in 1986 and 1990 to 12 in 1976. As before, we construct all possible combinations of three candidates within a year, for a total of 913 three-candidate combinations from all 18 surveys, with between 759 and 2,521 responses and a mean of 1,566.8 responses. For simplicity, we will refer to the survey respondents as "voters" and to the three-candidate surveys as "elections." For each response, we rank the three

[17] Dependence among the elections requires that the likelihood function be calculated from the conditional, rather than the marginal, distributions of the six vote-share vectors over all elections. However, because we use the likelihood functions to compare the accuracy of each model with that of the other models, ignoring the dependence is unlikely to affect these relative assessments.

[18] In the surveys conducted before 1970, a candidate whom the survey respondent did not know received a score of 50 on the participant's answer sheet, while such a candidate was coded as "unknown" in the surveys from 1970 onward. To avoid ambiguities between unknown candidates and candidates evaluated at 50, we restrict our analysis to the 18 surveys conducted from 1970 to 2004.

candidates according to their thermometer scores, thereby eliminating any information about the intensity of the voter's preferences. If a response yields a strict ranking of candidates, then we count it as one vote for this ranking. Voters are allowed to assign equal scores to different candidates, and we adopt the following intuitive rule of accommodating ties: If all candidates are tied, then we count the response as 1/6 vote for each ranking, and if two candidates are tied, then we count the response as half a vote for each of the two possible strict rankings that break the tie. Thus our adjusted data set consists of the total number of votes for each of the six strict rankings in each of the 913 three-candidate elections.

Table 2 shows, for both data sets, the number of elections with different numbers of candidates and the average number of voters in the original data sets for elections with 350 or more voters, the number of three-candidate elections we could have extracted from each of these elections, the number of three-candidate elections that had 350 or more voters that we did extract, and the average number of voters in each of these three-candidate elections with 350 or more voters.

It is notable that both data sets have voting cycles—the 913 ANES surveys have 4 cycles (0.44 percent), and the 20,087 ERS elections have 476 cycles (2.37 percent). However, there are only 101 voting cycles (1.45 percent) among the 6,794 ERS elections with 21 or more voters, and only 6 voting cycles (0.68 percent) among the 883 ERS elections with 350 and more voters. Thus the frequency of voting cycles falls fairly quickly as the number of voters increases.

4.2. *Assessment of the eight models whose ranges are subsets of the 5-simplex*

SPP predicts that in every election, one of the candidates will never be ranked last. The fact that our two data sets contain predominantly elections in which every candidate is ranked last by some voters is conclusive evidence against the empirical relevance of this model (at least for our two data sets). We therefore do not consider SPP in our further analysis. Table 3 reports the log-likelihood values and the AIC and BIC as well as our estimates of the Dirichlet δs for the remaining seven models whose ranges are subsets of the 5-simplex. For both data sets, the three measures of accuracy agree about the relative ranking of these models. IC, UUP, and DC have the smallest log-likelihoods, the largest values of AIC and BIC, the smallest values of δ, and thus the lowest accuracy. The Borda model consistently has the fourth highest accuracy, while UC and the Condorcet model are the third and second most accurate.

All our measures of accuracy indicate that the spatial model describes the observed data much better than any of the other seven models. Likelihood ratio tests of the spatial model and the nested IC, DC, and Borda models indicate that the improvement in the likelihood justifies the spatial model's additional degrees of freedom. For all models, AIC and BIC also suggest that, by a wide margin, the spatial model provides the best description of the ERS elections and the ANES surveys, despite its much larger use of degrees of freedom. The δ of the spatial model is close to being infinite, which indicates that the strict spatial model provides a very good explanation of almost all of the variation among the observed vote shares, and that perturbing the predicted vote-shares in the manner described by the Dirichlet process provides no significant improvement in the spatial model's fit.[19] In contrast, the δs of the other models are very small in comparison, meaning that adding unexplained variation through the Dirichlet process improves the fit of these models considerably.[20]

[19] Although the strict spatial model yields an extremely good fit of both data sets, it would be surprising if it explained *all* the variation in the data. We were unable to obtain reliable estimates of δ for the spatial model because rounding errors that occurred with extreme values of δ (beyond 100,000,000) led to convergence problems in our numerical evaluation of the spatial model. A generalized Dirichlet distribution that uses additional parameters to describe the variations in the six vote-shares is likely to be a more appropriate extension of the strict spatial model (see Kotz *et al.*, 2000, pp.512-514 for a discussion of such generalized Dirichlet distributions). However, the non-explained variation is so small that the strict spatial model is already a very, very close approximation of the structure of the election-generating universe.

[20] We calculate AIC and BIC only from the multinomial log-likelihood functions because, for the six models other than the spatial model, the extra variation introduced by the Dirichlet process is not a part of the strict models and therefore should not affect the judgment of their relative accuracies.

4.3. *Assessment of the four models whose range is the entire 5-simplex*

We analyze next whether the election data support any of the four models whose range is the entire 5-simplex, IAC, IAC$_b$(k_b), IAC$_t$(k_t), and IAC$_c$(k_c). We used the ERS data to calibrate the three parameters—the shares of last, first, and middle ranks of the candidates with the fewest last, first, and middle ranks—as k_b = 0.182, k_t = 0.187, and k_c = 0.248.[21] The fact that these values differ notably from the values predicted by IAC (= 1/3) constitutes some evidence against the hypothesis that IAC has generated the observed data. For each model, we draw one million samples of probability vectors from the appropriate distribution on the unit 5-simplex, and evaluate these samples by σ, the mean Euclidean distance to the nearest vector permitted by each of the seven models whose range is a subset of the 5-simplex (see Section 3.1). If any of the four models has generated the observed data, then we would expect that, for the data simulated with that model, the mean distances to the nearest vectors permitted by the other seven models to be similar to the mean distances that we obtained from the observed data.

Columns 1 and 2 of Table 4 show the σs that we determined from the observed data, and Columns 3 to 6 show the corresponding values of σ that we determined from our simulations. The notable differences between the σs from the observed elections and the simulated data indicate that the ability of any of the other seven models to describe the simulated vote shares is very different from (and also much worse than) its ability to describe the observed elections. We obtained the same result with other measures of distance.[22] Our results indicate that the vectors of vote shares in the ERS and ANES data are much more clustered than what is permitted by any

[21] We describe our calibration of the frequency parameters k_b, k_t, and k_c in the appendix. The ANES data yield the parameters values k_b = 0.218, k_t = 0.201, and k_c = 0.226. The two sets of values lead to very similar simulation results, and we only report the results with the parameters values calibrated from the ERS data. We also considered parameter values closer to zero, and the simulations results were again very similar to the ones reported.

[22] We weighted the mean σ by its standard deviation and we also evaluated the ratios of predicted and observed likelihood functions.

of the four models that assign equal probabilities to either all possible vectors (in case of IAC) or vectors in which two probabilities sum to no more than a value less than 1/3 (in case of $IAC_b(k_b)$, $IAC_t(k_t)$, and $IAC_c(k_c)$). Specifically, our estimate of an effectively infinite δ for the spatial model implies that the vote-share vectors that generate elections are clustered extremely closely around the vote share vectors predicted by the strict spatial model. Thus the hypothesis that either the ERS data or the ANES data were generated by IAC, $IAC_b(k_b)$, $IAC_t(k_t)$, or $IAC_c(k_c)$ cannot be sustained.

5. Conclusion

The starting point of our analysis is the observation that no theoretical analysis of probability structures can tell us anything about the probability of observing vectors of rankings in actual elections. This is clearly not a new discovery. It parallels the mundane observation that no analysis in theoretical econometrics provides any information about the particular estimates that one obtains from analyzing specific data. The theoretical econometrics literature investigates the properties of different models that can describe data, but to analyze a specific data set, one needs to identify the model or the set of models that is most likely to have generated these data.

The relationship between our work and the theoretical literature on voting is like that between theoretical and empirical econometrics. The existing literature on voting is largely theoretical, in the sense that it does not seek to identify systematic patterns in ranking data from actual elections. The large literature on the probabilities of finding Condorcet cycles and the Condorcet efficiency of different voting rules has established that the results vary greatly across different models of the structure of the election-generating universe. But which of these models best describes the distribution of rankings in actual elections? The theoretical literature on voting cannot answer this question.

Our results suggest that a spatial model describes the structure of the election-generating universe much better than any other model that has been proposed so far, and so well that it may be difficult to find a model whose accuracy is significantly higher. We consider our result to be encouraging, but more work needs to be done. For example, we draw our current conclusions on the basis of two data sets, one compiled from elections and the other from surveys. Our analyses suggest that the two data sets have somewhat different properties, but it is not clear whether these differences stem from their different sources or from the fact that the average ANES "election" has almost twice the number of voters than the average ERS election. Analyses of additional election data are necessary to answer this question and to determine the robustness of our results. We also focus exclusively on three-candidate elections, partly because this simplifies the exposition and makes it easier to relate our analysis to the previous literature, and partly because we are currently only able to evaluate the spatial model for three candidates.[23] Extending our analysis to elections with more than three candidates will provide important insights about the general relative accuracy of the different models.

Our analysis applies to all inquiries into the frequency of rare voting events: the probability that strategic voting will alter the outcome of an election, the existence of dominant candidates, or the frequencies of voting paradoxes. Our framework makes it possible to incorporate realistic models of the structure of the election-generating universe into such analyses and thereby to improve significantly the accuracy of their predictions for actual elections. Such new inquiries into old questions are likely to yield interesting new insights.

[23] The computation of the vote share vector p requires the numerical computation of the integrals of $M!$ areas under an $(M - 1)$-variate normal distribution. We are currently working on methods of computing such integrals for normal distributions with more than two dimensions.

APPENDIX: MODEL PARAMETERIZATIONS AND PARAMETER ESTIMATION:

Two of our twelve models, IC and IAC, do not have any parameters that need to be estimated to either fit the model to observed voting data or to undertake Monte Carlo simulations. We do not calibrate SPP because the model assigns zero probability to the possibility that every candidate will be ranked last, and we observed numerous elections in both data sets in which every candidate is ranked last. This appendix describes our strategy for calibrating the parameters of the remaining nine models. We report the calibrated values for the ERS and ANES data sets in Table A-1.

(1) $IAC_b(k_b)$, $IAC_t(k_t)$, and $IAC_c(k_c)$: The range of these models is the entire 6-simplex and each model can be calibrated to describe any set of observed vote shares by setting $p_r = q_r$. To simulate elections with these models, one needs to determine one unknown parameter for each model—the frequency with which that candidate is ranked last (first, middle) who is ranked last (first, middle) the least. We assume that each parameter follows a beta distribution over the range [0, 1/3] whose mean and variance coincides with those that we observe in the actual elections (we report the means in Table A-1). We draw a share k from this beta distribution, determine p_1 as a draw from a uniform distribution on $[0, k]$, set $p_2 = k - p_1$, draw p_3, p_4, p_5, and p_6 from the unit 4-simplex, and rescale these 4 values so that they sum to $1 - k$. We then use these six shares p to draw the number of votes for each of the six rankings from density function (6) or (7).

(2) UUP: This model has 5 parameters—five of the six vote shares (that sum to 1) that describe the expected ranking (the same for all elections). We calibrate these parameters by identifying the vector p that maximizes the likelihood function (6) or (7) over all elections. The maximum value of the likelihood function also determines the fit of UUP. Note that the 5 parameters of UUP are constant across elections, while the parameters of the remaining six models are calibrated separately for each election.

(3) DC and UC: Both models have 2 parameters each—two of the 3 pairs of probabilities of opposite (for DC) and neighboring (for UC) rankings. We fit these models to the data by using, for each election, the average value of the two observed vote shares of each pair of rankings as the predicted vote shares for that pair.

(4) Borda and Condorcet: Each model has one parameter. For the Borda model the parameter β is the increase in log probability associated with an increase by one in the rank that a voter assigns to the "best" candidate. For the Condorcet model the parameter γ is the increase in log probability associated with a reduction of one adjacent-pair permutation in the difference between the "best" ranking and the one that a voter reports. We fit these models to the data by calibrating, for each election, the values of β and γ in equations (1) and (3), respectively, so that the resulting vector p maximizes the likelihood function (6) or (7) for the observed vote shares.

(5) Spatial model: This model has 4 parameters; Figure A-1 shows one way of using the four degrees of freedom (as in Good and Tideman, 1976). The intersection of the perpendicular bisectors T is placed at the origin of a Cartesian coordinate system. The fact that the vote shares are independent of rotations around the mode of the distribution of voters' ideal points, O, permits us to rotate the coordinate system so that O is located on its horizontal axis. The first degree of freedom then specifies the distance between T and O. The remaining degrees of freedom specify the angles β_1, β_2, and β_3 formed by the line $\overline{TO}$ and the three perpendicular bisectors. Thus any feasible set of values of the four degrees of freedom corresponds to a set of p_r. We calibrate the spatial model by placing, for each election, the borders between pairs of adjacent rankings and the distance $\overline{TO}$ in such a way as to create sectors that match the six probabilities p_r (the integrals over the triangular-shaped slices under the bivariate normal distribution) as closely as possible to the six observed vote shares, q_r.

Table A1: Parameter values calibrated from ERS and ANES data

Model → / Data set ↓	IAC$_b$(k_b)	IAC$_t$(k_t)	IAC$_c$(k_c)	Borda	Condorcet	
	k_b	k_t	k_c	β	γ	
ERS	0.182	0.187	0.248	0.517	0.513	
	(0.002)	(0.002)	(0.001)	(0.009)	(0.008)	
ANES	0.218	0.201	0.226	0.358	0.360	
	(0.002)	(0.002)	(0.002)	(0.006)	(0.006)	
Model →	Spatial model					
		Angles of the perpendicular bisectors with the line $\overline{TO}$				
Data set ↓	$\overline{TO}$	β_1	β_2	β_3		
ERS	0.597	0.548	1.549	2.560		
	(0.009)	(0.011)	(0.011)	(0.011)		
ANES	0.445	0.556	1.550	2.592		
	(0.008)	(0.012)	(0.015)	(0.012)		
		Corresponding angles between pairs of bisectors:				
		A$_1$	A$_2$	A$_3$		
ERS		1.130	1.000	1.011		
		(0.006)	(0.006)	(0.006)		
ANES		1.105	0.994	1.042		
		(0.010)	(0.011)	(0.010)		
Model →	UUP					
Data set ↓	p_1	p_2	p_3	p_4	p_5	p_6
ERS	0.294	0.167	0.127	0.108	0.120	0.184
ANES	0.233	0.150	0.140	0.148	0.160	0.169

Notes:

(1) The values in parentheses are standard errors.

(2) All values were calibrated from the multinomial model (the values calibrated from the Pólya model are very similar).

(3) The entries for UUP are the shares, calibrated over all elections in the respective data set, which minimize the multinomial likelihood function. The entries for the other 6 models are the means of the values that we calibrated for each election in the respective data set.

(4) For DC and UC, aggregation across elections is meaningless because of the arbitrariness of labeling a candidate *A*, *B*, or *C*, so we do not show any calibrated values for these two models.

Table 1. Comparison of the twelve models of the structure of the election-generating universe

#	Model	Number of dimensions in subspace(s)	Parameters per election to be calibrated to fit the model to this election	Parameters to be calibrated to simulate data from this model	First proposal (of which we are aware) as a description of the election-generating universe
A. Range is the entire 5-simplex:					
1.	IAC	5	5	0	Kuga & Nagatani (1974) Gehrlein & Fishburn (1976)
2.	$IAC_b(k_b)$	5	5	1	Gehrlein (2004)
3.	$IAC_t(k_t)$	5	5	1	Gehrlein (2006)
4.	$IAC_c(k_c)$	5	5	1	Gehrlein (2006)
B. Range is one or more 0-dimensional subspaces of the 5-simplex:					
5.	IC	0	0	0	Campbell and Tullock (1965)
6.	UUP	5	5 (for all elections)	5	Chamberlin & Featherston (1986)
C. Range is one or more more-than-0-dimensional subspaces, defined by linear restrictions:					
7.	SPP ($IAC_b(0)$)	5	4	0	Lepelley (1995)
8.	DC	3	2	2	Gehrlein (1978)
9.	UC	3	2	2	Gehrlein (1982)
D. Range is one or more more-than-0-dimensional subspaces, defined by nonlinear restrictions:					
10.	Borda	1	1	1	Conitzer & Sandholm (2005)
11.	Condorcet	1	1	1	This paper
12.	Spatial model	4	4	4	This paper

Notes:

(1) The four models in A can describe any set of observed vote-shares and the parameters equal the observed shares. We describe our strategy for simulating data under these models in the appendix.

(2) The 5 parameters of UUP are calibrated from all elections simultaneously, and the model's fit to any individual elections is assessed on the basis of these parameter values. The parameters of SPP, DC, UC, the Borda model, the Condorcet model, and the spatial model are calibrated for each election individually.

(3) The 5 parameters of UUP are constants in simulations from UUP. To simulate from any of the other models with unknown parameters, we assign distributions to all parameters and draw pseudo random numbers from these distributions that we use as inputs into the density functions (6) and (7).

Table 2. Properties of the original ERS and ANES data sets and of the extracted three-candidate elections

	Number of candidates in the original elections (1)	Number of elections with 350 or more voters (2)	Average number of voters per election (3)	Potential number of three-candidate elections (with any number of voters) (4)	Number of three-candidate elections with 350 and more voters (5)	Average number of voters per three-candidate election (6)
ERS data:						
	3	0				
	4	6	614	24	4	1,005
	5	3	1,238	30	10	1,859
	6	2	770	40	20	414
	7	3	714	105	37	390
	8	3	541	168		
	9	2	450	168		
	10	5	679	600	120	665
	11	2	913	330	108	404
	12 – 20	9	986	3,883	584	531
	Total	35		5,348	883	
ANES data:						
	3	2	2,078	2	2	1,599
	4	0				
	5	2	1,396	20	20	1,248
	6	2	1,607	40	40	1,403
	7	5	1,858	175	175	1,540
	8	3	2,184	168	168	1,699
	9	2	1,561	168	168	1,228
	10	1	2,257	120	120	1,826
	11	0				
	12	1	2,248	220	220	1,663
	Total	18		913	913	

Table 3. Assessment of seven models whose range is a subset of the 5-simplex

		ERS elections					ANES surveys				
Number of elections:		883					913				
Mean number of voters:		716.4					1566.7				
	DF used for ERS; ANES	LLH Multi-nomial	LLH Pólya	δ	AIC	BIC	LLH Multi-nomial	LLH Pólya	δ	AIC	BIC
Model:	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
IC	0; 0	-113,551	-23,084	15.93	227,102	227,102	-195,310	-27,811	19.29	390,621	390,621
UUP	5; 5	-72,581	-22,133	25.51	145,172	145,204	-171,991	-27,586	21.42	343,992	344,019
DC	2,649; 2,739	-93,902	-23,028	17.22	193,102	210,036	-127,510	-26,915	29.70	260,498	278,099
UC	2,649; 2,739	-51,295	-21,091	43.92	107,888	124,822	-91,278	-26,000	45.23	188,034	205,635
Borda model	883; 913	-54,255	-21,215	40.16	110,276	115,921	-122,264	-26,679	32.52	246,354	251,218
Condorcet model	883; 913	-32,614	-19,588	93.49	66,994	72,639	-93,229	-25,988	44.94	188,284	193,148
Spatial model	3,532; 3,652	-14,267	-14,267	> 1.E08	35,598	58,177	-16,428	-16,428	> 1.E08	40,160	59,614

Notes:
(1) We calculated the AIC and BIC in Columns (5), (6), (10), and (11) using the multinomial log-likelihood functions.
(2) For the spatial model, rounding errors prevented us from determining the exact values of the Pólya log-likelihood functions, but the differences between the multinomial and Pólya log-likelihood function values are less than 0.5.

Table 4. Assessment of four models whose range is the entire 5-simplex

	Analyses of observed data		Analyses of simulated data			
Data source:	ERS	ANES	IAC	$IAC_b(k_b)$ $k_b = 0.182$	$IAC_t(k_t)$ $k_t = 0.187$	$IAC_c(k_c)$ $k_c = 0.248$
Number of voters:	716.4	1566.7	716.4	716.4	716.4	716.4
Model:	(1)	(2)	(3)	(4)	(5)	(6)
IC	0.226	0.197	0.330	0.339	0.336	0.304
	(0.0030)	(0.0021)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
UUP	0.135	0.153	0.278	0.282	0.281	0.252
	(0.0018)	(0.0017)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
DC	0.203	0.148	0.248	0.258	0.255	0.242
	(0.0028)	(0.0023)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
UC	0.139	0.126	0.248	0.258	0.249	0.226
	(0.0023)	(0.0020)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Borda model	0.144	0.149	0.270	0.276	0.263	0.244
	(0.0023)	(0.0018)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Condorcet model	0.092	0.121	0.220	0.194	0.193	0.198
	(0.0013)	(0.0015)	(0.0001)	(0.0001)	(0.0001)	(0.0001)
Spatial model	0.017	0.008	0.061	0.064	0.065	0.069
	(0.0005)	(0.0002)	(0.0001)	(0.0001)	(0.0001)	(0.0001)

Note: For the observed data, the entries are the mean Euclidean distances between the observed vote shares and the vote shares (see equation 5), among those vectors permitted by the respective model, that are most likely to have generated the data. For the simulated data, we undertook the same calculation with simulated vote shares substituted for observed vote shares. The values in parentheses are standard errors.

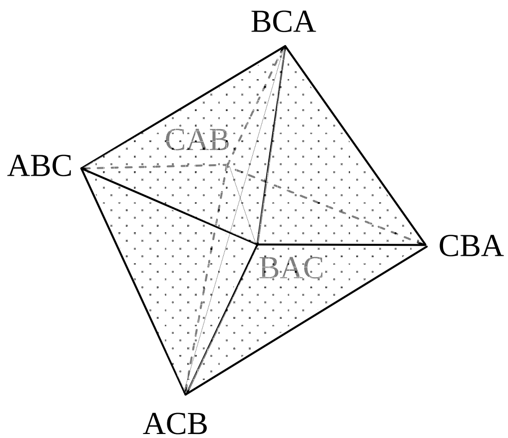

Figure 1a. Three-dimensional representation of the range of IAC and variations

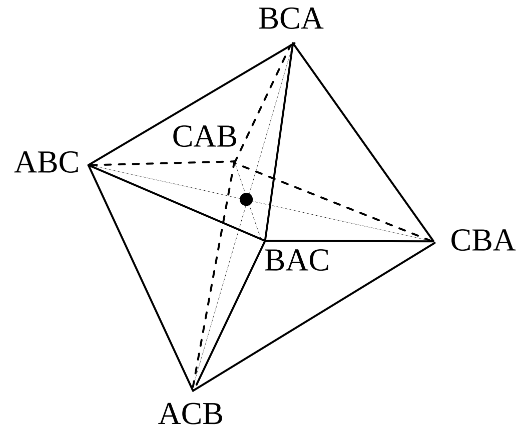

Figure 1b. Three-dimensional representation of the range of IC

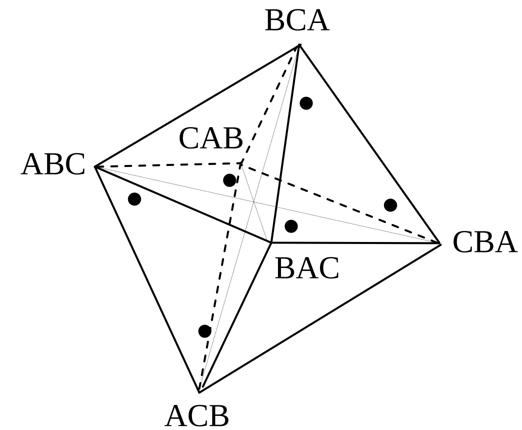

Figure 1c. Three-dimensional representation of the range of UUP

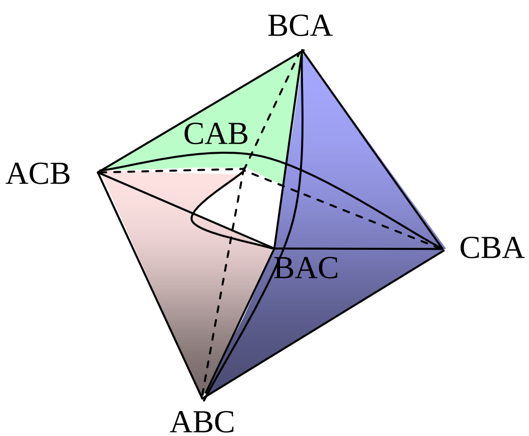

Figure 1d. Three-dimensional representation of the range of SSP

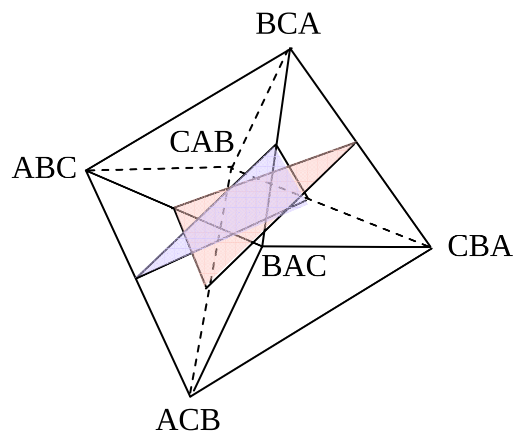

Figure 1e. Three-dimensional representations of the ranges of DC and UC

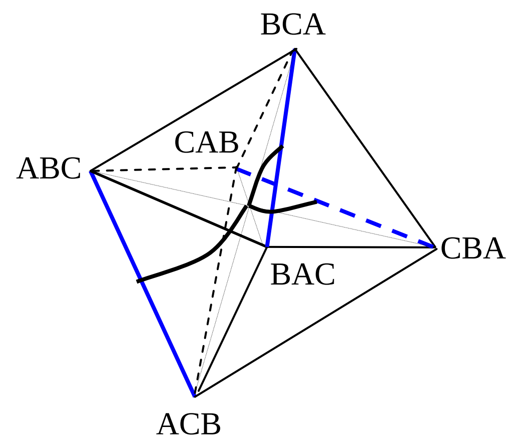

Figure 1f. Three-dimensional representation of the range of the Borda model

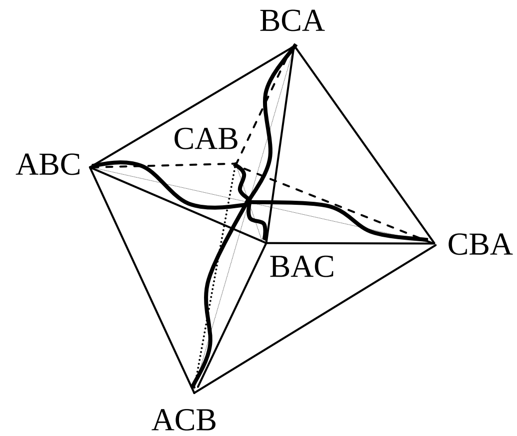

Figure 1e. Three-dimensional representation of the range of the Condorcet model

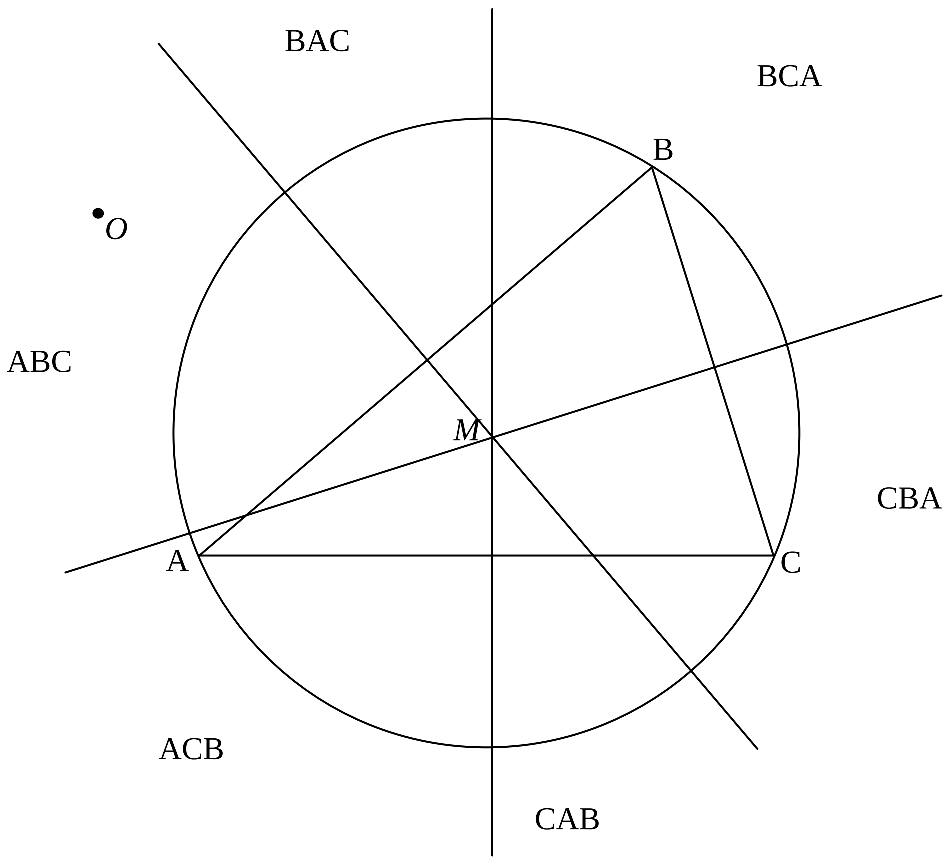

Figure 2. Division of the candidate plane into six sectors by drawing the perpendicular bisectors of the three sides of the triangle formed by the candidates' locations, and the associated rank orders of the sectors. (The figure is taken from Good and Tideman, 1976, p. 372.)

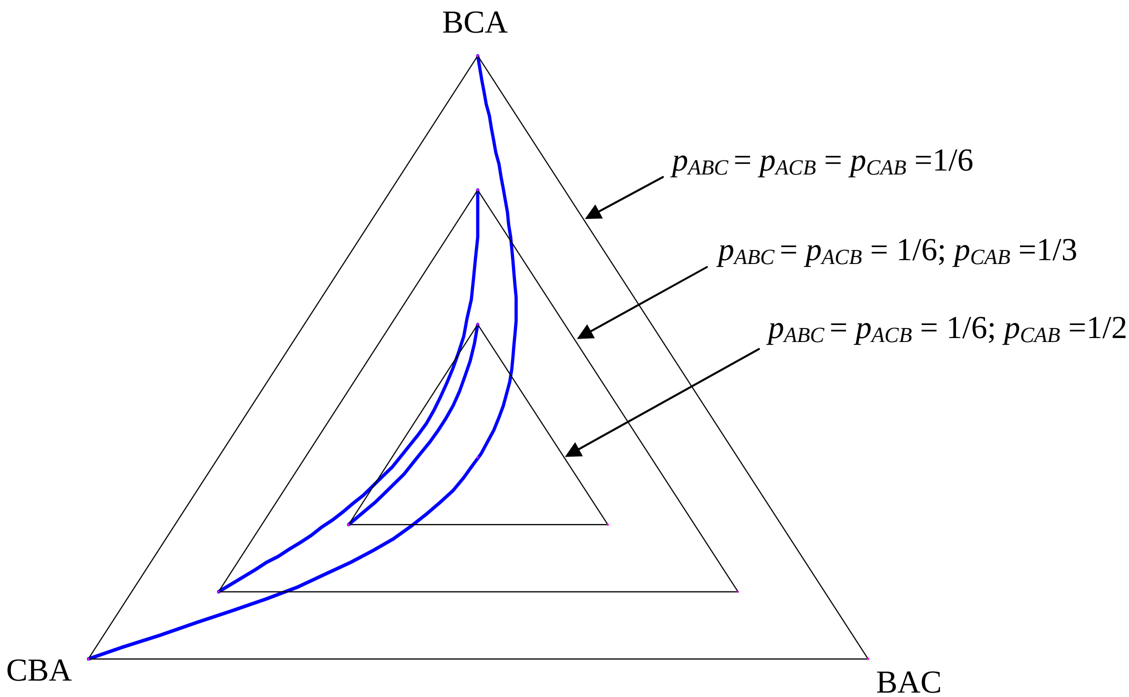

Figure 3. Two-dimensional representation of part of the range of the spatial model for the three shares CBA, BCA, and BAC.

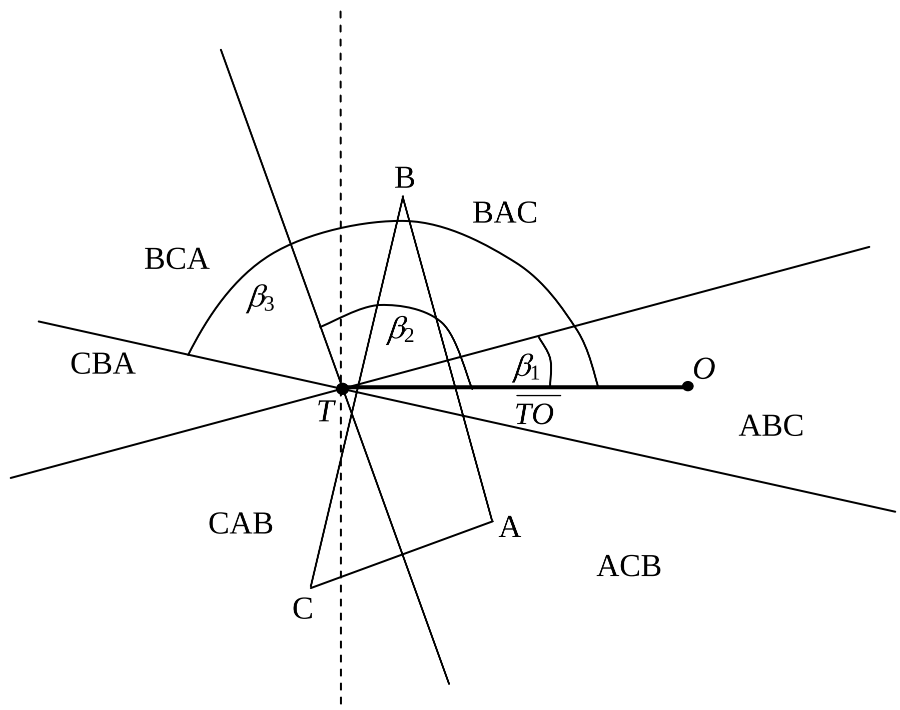

Figure A-1. The four parameters $\overline{TO}$, β_1, β_2, and β_3 that define a spatial model observation.

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