Partners, Associates, and Shareholders:
Strategy-Proof Determination of Their Relative Returns

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Abstract:
When a firm is owned by its workers and other shareholders, and managed by a subset of the workers, a variation of a recently developed fair-division mechanism achieves a strategy-proof allocation of the net return among the recipients as workers and owners. The returns to ownership of the non-managing workers and non-working shareholders are strictly proportional to shares of ownership. The returns to ownership of the managing workers diverge slightly from proportionality, to preserve strategy-proofness and other valued properties.

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Journal of Economic Literature Classification Codes: D33, D72

Keywords: Fair division, aggregating opinions, strategy-proof, worker-ownership
1. INTRODUCTION

Consider a firm that is owned in specified shares by managing partners and non-managing associates who all work for the firm, and by other shareholders who do not work for the firm. After paying expenses, including salaries of the non-managing associates, the firm makes a positive profit. This profit is used to compensate each partner for his contributions to the firm, pay bonuses to the associates to reward them for exceptional contributions, and pay dividends for the shares of ownership in the firm held by partners, associates, and non-working shareholders. The partners’ general contributions and the associates’ exceptional contributions include services like general management and the provision of informal counsel that cannot be measured by an objective indicator like billable hours. Because the partners are chiefly responsible for the firm’s management and its future prosperity, it is their task to divide the profit in an acceptable way. An attractive division is one on which the partners agree unanimously. But how should they divide the profit if they find it too difficult and time consuming to reach unanimous agreement?

De Clippel et al. (2008) and Tideman and Plassmann (2008) offer a division rule for the special case when the partners divide the profit only among themselves. They suggest that such a division rule should have three intuitive properties: the rule should be strategy-proof—no partner should be able to affect his own share by the input that he provides, objective—the calculation of the shares should not depend on any partner’s statement about what he deserves relative to others, and consensual—if there is a set of shares that is consistent with all of the input that the partners provide, then the rule should assign those shares. De Clippel et al. (2008) prove that there is only one division rule that possesses all three properties if there are three partners, and that this rule has an extension to a class of rules if there are four or more partners. The current paper extends their analysis to the more general and more widely-applicable
situation described above, in which the partners not only determine their own compensation but must also assign shares of the firm’s profit to non-managing associates and to those who own equity in it. We refer the reader to Tideman and Plassmann (2008) for a discussion of the division rule’s general properties and for a comparison with other rules.

2. THE DIVISION RULE FOR PARTNERS, ASSOCIATES, AND NON-WORKING SHAREHOLDERS

Let \( N_P \geq 3 \) be the number of partners and \( N_A \geq 1 \) be the number of associates, so that the number of persons eligible to receive bonuses is \( N_B = N_P + N_A \) (for simplicity, we refer to the payments to partners as ‘bonuses’), and let \( N_S \geq 1 \) be the number of shareholders in the firm. The firm’s profit \( \pi \) must be divided among these \( N = N_B + N_S \) budgetary “positions.” Denote the bonus that partner or associate \( i \) receives by \( b_i \geq 0, \ i = 1, \ldots, N_B \), the number of shares of ownership that shareholder \( j \) holds by \( s_j \geq 0, \ j = 1, \ldots, N_S \), and the divided rate per share of ownership held by shareholder \( j \) by \( d_j, j = 1, \ldots, N_S \).

If the partners divide the entire profit without leftover so that \( \sum_{i} b_i + \sum_{j} d_j s_j = \pi \), then the proportions of profits paid to the \( N \) positions sum to 1, and this sum can be expressed as

\[
\sum_{k=1}^{N_B} \frac{b_k}{\sum_{i=1}^{N_B} b_i + \sum_{j=1}^{N_S} d_j s_j} + \sum_{k=1}^{N_S} \frac{d_k s_k}{\sum_{i=1}^{N_B} b_i + \sum_{j=1}^{N_S} d_j s_j} = \sum_{k=1}^{N_B} \frac{1}{\sum_{i=1}^{N_B} b_i + \sum_{j=1}^{N_S} d_j s_j} + \sum_{k=1}^{N_S} \frac{1}{\sum_{i=1}^{N_B} b_i + \sum_{j=1}^{N_S} d_j s_j} = 1. \tag{1}
\]

Thus each position’s share can be expressed as a function of the shares that all budgetary positions receive relative to that position’s share. For example, the share of partner 1’s bonus is
a function of the $N_B$ ratios of bonuses, $b_i/b_1$ and the $N_S$ ratios of returns to shares of ownership to partner 1’s bonus, $d_{sj}/b_1$. The division rule requires every partner to provide a proposal for each such ratio, and determines the share of $\pi$ that each of the $N$ positions receives by replacing each ratio in equation (1) with some generalized average of the partners’ proposals for that ratio, omitting the proposals of the partners mentioned in the ratio.

The division rule has three steps. The first step is to ask each partner to supply a proposal for each of the $N$ positions (or as many of them as he can reasonably evaluate), subject to the constraint that the proposed dividend rate per share of ownership be the same for all owners. We denote partner $m$’s proposal for the $i$th bonus as $b_i^m$ and his proposal for the return per share of ownership as $d_i^m$, so that his proposal for the $j$th equity return is $d_i^m s_j$. (The division rule does not require a partner’s proposals for the $N$ positions to sum to $\pi$ because the rule uses only ratios of a partner’s proposals for returns to budget positions.)

The second step is to derive, for each ordered pair of budget positions, an aggregate proposed ratio of returns to the positions. Each aggregate proposed ratio is derived from the individual proposals for this ratio offered by all partners, except those partners who receive the returns described by the ratio. Objectivity requires that the calculation of shares not depend on any partner’s statement about what he deserves relative to others, so the aggregate proposal for, say, $b_m/b_i$, $m, i \leq N_P$, may not depend on either partner $m$’s or partner $i$’s proposal for this ratio. Strategy-proofness requires that a partner’s proposal not affect the share that he receives, so none of partner $m$’s proposed ratios can be used to determine the share of his bonus or his return to equity. A proposal that meets both criteria is “impartial,” and there are either $N_P$, $N_P - 1$, or $N_P - 2$ impartial individual proposals for any ratio of budget positions, depending on how many

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1 If there are pairs of positions for which no partner proposed a ratio, it may be possible to derive a ratio from other available data. See Tideman and Plassmann (2008, Section 5) for a discussion.
partners are mentioned in the ratio. (The division rule is not applicable if \( N_P = 2 \) because there would be no impartial individual proposals for ratios that consist of the bonuses of two partners.)

While there is a variety of permissible methods of calculating the aggregate proposed ratios (see De Clippel et al., 2008, and Tideman and Plassmann, 2008 for details), the “geometric median” is attractive because of its symmetry and its resistance to being affected by outliers. (The geometric median is the ordinary median for an odd number of proposals and the geometric average of the two medians for an even number of proposals.) In what follows we assume that each aggregate proposed ratio is the geometric median of the proposals for that ratio by partners not involved in the ratio.

The final step of the division rule is to use the aggregate proposed ratios in place of the ratios in equation 1. For example, for a firm with 3 partners, 1 associate, and no non-working shareholders, the share of partner 1’s bonus, \( b_1/\pi \), is determined as

\[
\frac{b_1}{\pi} = \frac{1}{1 + \frac{b_2^2}{b_1^3} + \frac{b_2^3}{b_1^5} + \left( \frac{d_2^2 s_1 d_3^3 s_1}{b_1^2 b_2^3} \right)^{\frac{1}{2}} + \frac{d_3^3 s_2}{b_1^3} + \frac{d_2^2 s_3}{b_1^2} + \left( \frac{d_2^2 s_4 d_3^3 s_4}{b_1^2 b_3^3} \right)^{\frac{1}{2}}},
\]

where the subscript 4 refers to the associate, and the square roots are the geometric medians of partners 2’s and 3’s proposed ratios.\(^2\) The calculation of \( b_1/\pi \) is strategy-proof because it uses no input from partner 1 (no proposed value has superscript 1), and it is objective because it uses no input from partners 2 and 3 in which they propose their own returns (no term has identical super- and subscripts). The same logic applies to the determination of the other shares by equations similar to equation (2). The division rule is consensual because the ratio of any two shares equals their proposed ratio if the partners propose consistent relative shares. For example, the equality

\(^2\) The geometric median and geometric mean are identical if there are only two proposals.
\[
\frac{b_2}{b_1} = \frac{1 + b_2^3 + b_1^2 + \left(\frac{b_4}{b_2} \right)^{3/2} + \left(\frac{d_1}{b_1} \right)^{3/2} + \frac{d_2}{b_1^3} + \frac{d_3}{b_1^3} + \left(\frac{d_4}{b_2} \right)^{3/2}}{1 + b_3^1 + b_3^1 + \left(\frac{b_4}{b_2} \right)^{3/2} + \left(\frac{d_1}{b_1} \right)^{3/2} + \frac{d_2}{b_1^3} + \frac{d_3}{b_1^3} + \left(\frac{d_4}{b_2} \right)^{3/2}} = \frac{b_3^1}{b_1^3},
\]

holds if \(b_i^k / b_j^k = b_i^m / b_j^m\) and \(d_i^k / b_j^k = d_i^m / b_j^m\), \(i, j \neq k, m\), so that the two individual proposed ratios in each pair of parentheses are identical, and the following four equalities hold,

\[
\frac{b_3^1 b_2^3}{b_2^1 b_1^3} = \frac{b_3^2 b_2^3}{b_2^1 b_1^3} = \frac{b_3^4 b_2^3}{b_2^1 b_1^3} = \frac{d_1 s_3 b_2^3}{b_2^1 b_1^3} = \frac{d_1 s_1 b_2^3}{b_2^1 b_1^3} = \frac{d_2 s_3}{b_2^1 b_1^3}, \quad \frac{d_2 s_1}{b_2^1 b_1^3} = \frac{d_2 s_4}{b_2^1 b_1^3},
\]

so that partner 2’s proposed ratios are consistent with the products of the corresponding proposed ratios of partners 1 and 3.

De Clippel et al. (2008) show that, for three partners, the shares will add up to 1 if the proposals are consistent, and to something that is never more than 1 if the proposals are not consistent, so that there is never a deficit. Tideman and Plassmann (2008) show that the surplus, when there is one, is generally very close to 0. Any surplus can be disposed of while maintaining the rule’s strategy-proofness by dividing the surplus among the associates and non-working shareholders.

The returns to equity of all non-partners are strictly proportional to the non-partners’ respective shares of ownership. To see this, consider the ratio of the shares of person \(k\) and person \(m\)’s returns to equity,

\[
\frac{d_k s_k}{d_m s_m} = \frac{\sum_{j=1}^{N_B} \left(\frac{b_j}{d_m} \right)^{M} \frac{1}{s_m} + \sum_{j=1}^{N_K} \left(\frac{d_j}{d_m} \right)^{M} \frac{s_j}{s_m}}{\sum_{i=1}^{N_K} \left(\frac{b_i}{d_k} \right)^{M} \frac{1}{s_k} + \sum_{i=1}^{N_B} \left(\frac{d_i}{d_k} \right)^{M} \frac{s_i}{s_k}},
\]
where the superscript $M$ indicates the aggregate proposal for a ratio (the geometric medians of the available impartial proposals for this ratio). For notational simplicity, assume that the $N_P$ partners are the first $N_P$ shareholders, so that $s_j, j \leq N_P$, denotes the number of shares owned by partner $j$.

If neither $k$ nor $m$ is a partner, then the aggregate proposed ratios $(b_i/d_m)^M$ and $(b_i/d_k)^M$ are determined from the same individual proposed ratios. A partner’s proposals for all ratios of a bonus to a return per share of ownership have the same denominator because every partner proposes only one return per share of ownership. Thus each of the first $N_B$ terms in the numerator of equation (4) has the same geometric median as the corresponding term in the denominator, or $(b_i/d_m)^M = (b_i/d_k)^M \forall k, m > N_P$. In addition, because every partner proposes just one return per share of ownership, all aggregate proposals for relative returns per share of ownership equal 1, or $(d_i/d_m)^M = (d_i/d_k)^M = 1$. It follows that the ratio of every term in the numerator of equation (4) to the corresponding term in the denominator is $s_k/s_m$, which implies that the relative returns to equity of any two non-partners $k$ and $m$ are strictly proportional to their relative shares of ownership.

If $k$ and/or $m$ are partners, then some of the first $N_P$ aggregate proposed ratios in the numerator are determined from the individual proposed ratios of different partners than their corresponding aggregate proposed ratios in the denominator. For example, if $m$ is a partner, then the aggregate proposed ratio $(b_i/d_m)^M$ is the geometric median of the $N_P - 1$ individual proposed ratios of all partners except partner $m$. If $k$ is a partner, then the corresponding aggregate

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3 They are determined from $N_P - 1$ individual proposed ratios if $i$ is a partner and from all $N_P$ individual proposed ratios if $i$ is an associate.

4 Because $(b_i/d_m)^M$ and $(b_i/d_k)^M$, $k, m > N_P$, are determined from the same individual proposed ratios, this result holds not only for the geometric median but for all aggregation methods that are permissible (see deClippel et al, 2008).
proposed ratio in the denominator, \((b_i/d_k)^M\), is the geometric median of the \(N_P - 1\) individual proposed ratios of all partners except partner \(k\). If \(k\) is not a partner, then \((b_i/d_k)^M\) is the geometric median of the individual proposed ratios of all \(N_P\) partners. The ratio of some of the first \(N_P\) terms in the numerator and their corresponding terms in the denominator may therefore differ from \(s_k/s_m\), which implies that the relative returns to equity of partner \(k\) and person \(m\) are not necessarily strictly proportional to their relative shares of ownership. Still, they are likely to be close to proportional because the removal of one element from a set changes the geometric median by only (approximately) half the difference between the median element and a neighboring one. Thus while variations in returns per share of ownership will occur, they should not lead to equity concerns.

We emphasize that the division rule does not replace discussions among the partners about the relative contributions made by partners or associates and the appropriate return per share of equity. Such discussions help the partners to either obtain or recall relevant information on which to base their proposals. However, if the partners believe that they themselves are the best arbiters, they consider it too time consuming to continue their discussion until they reach consensus, and they value strategy-proofness, objectivity, and consensuality, then the division rule described here is their best choice.
REFERENCES
