Answer Key to final exam

1. $Y_t = \beta_0 + \beta_1 X_t + U_t \quad t = 1, 2, \ldots, N \quad (1)$

$E(U_t) = 0$ & $V(U_t) = \sigma_t^2 \quad Cov(U_t, U_{t+z}) = 0$

a) Show that the model in (1) with the assumptions given that the OLS estimators of β_0 & β_1 are unbiased.

$$\hat{\beta}_1 = \frac{\sum_{i=1}^{N}(x_i-\bar{x})(y_i-\bar{y})}{\sum_{i=1}^{N}(x_i-\bar{x})^2} = \frac{\sum_{i=1}^{N}(x_i-\bar{x})(y_i)}{\sum_{i=1}^{N}(x_i-\bar{x})(x_i)} = \frac{\sum_{i=1}^{N}(x_i-\bar{x})(\beta_0+\beta_1 x_i+u_i)}{\sum_{i=1}^{N}(x_i-\bar{x})(x_i)}$$

$$= \frac{\beta_0\sum_{i=1}^{N}(x_i-\bar{x}) + \beta_1\sum_{i=1}^{N}(x_i-\bar{x})(x_i) + \sum_{i=1}^{N}(x_i-\bar{x})(u_i)}{\sum_{i=1}^{N}(x_i-\bar{x})(x_i)} = 0 + \beta_1 + \frac{\sum_{i=1}^{N}(x_i-\bar{x})(u_i)}{\sum_{i=1}^{N}(x_i-\bar{x})(x_i)}$$

$$E(\hat{\beta}_1) = \beta_1 + \frac{\sum(x_i-\bar{x})E(u_i)}{\sum_{i=1}^{N}(x_i-\bar{x})(x_i)} = \beta_1 \quad \text{so } \hat{\beta}_1 \text{ unbiased}$$ 6

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1\bar{X} = \beta_0 + \beta_1\bar{X} + \bar{U} - \hat{\beta}_1\bar{X}$$

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$$E(\hat{\beta}_0) = \beta_0 + \beta_1\bar{X} + 0 - E(\hat{\beta}_1)\bar{X} = \beta_0 + \beta_1\bar{X} - \beta_1\bar{X} = \beta_0 \quad \text{so } \hat{\beta}_0 \text{ unbiased}$$

b) if $E(u_t)=0$ but $V(u_t|X_t)=\sigma^2 X_t^2$ $t=1,2,\ldots,n$

4 $\hat{\beta}_0$ & $\hat{\beta}_1$ are still unbiased because the variance of u_t does not affect the parameter estimates only the standard errors. This can be seen from the proof in a) which is the same proof to show $\hat{\beta}_0$ & $\hat{\beta}_1$ unbiased when heteroscedasticity is present.

To estimate β_0 & β_1 more efficiently we would use WLS

1: Take original model $Y_t = \beta_0 + \beta_1 X_t + u_t$ where $V(u_t|X_t)=\sigma^2 X_t^2$
and create weights $\omega_t = \sqrt{X_t^2} = X_t$

2: Divide both sides of model by ω_t

$$Y_t^* = \frac{Y_t}{X_t} = \beta_0\left(\frac{1}{X_t}\right) + \beta_1 + \frac{u_t}{X_t} = \beta_0 X_t^* + \beta_1 + \varepsilon_t \quad V(\varepsilon_t|X_t)=\sigma^2$$

3: Perform OLS on this transformed model and $\hat{\beta}_0$ & $\hat{\beta}_1$ will be BLUE

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c) You suspect that $V(u_t|X_t)=\sigma_t^2$

(i) use white test

1: Run OLS on (1) and find $\hat{u}_t$

5 2: Square $\hat{u}_t$ and regress on all x's, cross products, and squared terms. That is $\hat{u}^2 = \delta_0 + \delta_1 x_1 + \delta_2 x_2^2 + v_t$

3: Use OLS on this regression and save $R_{\hat{u}}^2$.

4: use F or LM statistic to test $H_0: \delta_1 = \delta_2 = 0$ against
$H_1: H_0$ false

$$F_{\hat{u}} = \frac{R_{\hat{u}}^2/2}{(1-R_{\hat{u}}^2)/(n-3)} \qquad LM_{\hat{u}} = nR_{\hat{u}}^2$$

$$F \sim F(2, n-3) \qquad LM_{\hat{u}} \sim \chi_2^2$$

ii) Assuming that we find evidence of Heteroscedasticity we can still obtain heteroscedasticity corrected standard errors of the OLS estimators $\hat{\beta}_0$ & $\hat{\beta}_1$

$$s.e.(\hat{\beta}_1) = \sqrt{\frac{\sum_{i=1}^{N}(x_i-\bar{x})^2(\hat{u}_i)^2}{\left(\sum_{i=1}^{N}(x_i-\bar{x})^2\right)^2}}$$ 3

For the standard error of β_0 corrected for heteroscedasticity we do the following procedure.

1: run OLS to $Y=\beta_0+\beta_1 X+u$ and keep fitted residuals $\hat{u}$.
2: run OLS to $1=\delta_0 X+v$ and keep fitted residuals $\hat{v}$.
3: Find $s.e.(\hat{\beta}_0)$ as $\sqrt{\frac{\sum_{i=1}^{N}\hat{v}_i^2\hat{u}_i^2}{\sum_{i=1}^{N}\hat{v}_i^2}}$ 2

d). let the model be $Y_t=\beta_0+\beta_1 X_t+u_t$ where Y_t is a dummy variable

Therefore $Y=1$ w/ probability p
$Y=0$ w/ probability $1-p$

$E(y)=p$ $\quad$ $E(y_t)=\beta_0+\beta_1 X_t$
$V(y)=E((y-p)^2) = E(y^2)-(E(y))^2 = p-p^2 = p(1-p)$ 6
$V(y_t)=E(\beta_0+\beta_1 X_t+u_t-(\beta_0+\beta_1 X_t))^2 = E(u_t^2) = p(1-p)$
Since we do not know p we estimate it. Therefore $E(y_t)=\hat{p}_i=\hat{\beta}_0+\hat{\beta} x_i$

Thus $v(u_i) = \hat{p}(1-\hat{p}) = (\hat{\beta}_0 + \hat{\beta}_1 X_i)(1-\hat{\beta}_0 - \hat{\beta}_1 X_i)$ which is not constant.

1: To estimate this model we estimate the model by OLS and find the fitted values. $\hat{p}_i = \hat{\beta}_0 + \hat{\beta}_1 \hat{X}_i$

2: we create weights equal to $w_i = \sqrt{\hat{p}_i(1-\hat{p}_i)}$

3: we weight the model as $\frac{Y_i}{w_i} = \beta_0/w_i + \beta_1 \frac{X_i}{w_i} + \frac{U_i}{w_i}$

4: we estimate this model using OLS, where our error term satisfies all of the Gauss Markov assumptions. 4

e) Consider the same model (1) but now $U_t = \rho U_{t-1} + \varepsilon_t$ $|\rho| < 1$ and ε_t is iid$(0, \sigma^2)$.

Using the same argument in a) $\hat{\beta}_0$ & $\hat{\beta}_1$ are unbiased 2

• If ρ was known then we estimate β_0 & β_1 by adjusting the model as $Y_t^* = \beta_0^* + \beta_1 X_t^* + \varepsilon_t$

$$Y_t^* = Y_t - \rho Y_{t-1}$$
$$\beta_0^* = \beta_0(1-\rho)$$
$$X_t^* = X_t - \rho X_{t-1}$$
$$\varepsilon_t = U_t - \rho U_{t-1}$$

If we run OLS on this model we get $\hat{\beta}_0^*$ & $\hat{\beta}_1$. To recover $\hat{\beta}_0$ we use $\hat{\beta}_0 = \frac{\hat{\beta}_0^*}{1-\rho}$. 5

1: If ρ is not known then we estimate model (1) by OLS and keep the residuals $\hat{u}_t$.

2: we run the auxilary regression $\hat{u}_t = \rho \hat{u}_{t-1} + \varepsilon_t$ and find $\hat{\rho}$

3: we construct $Y_t^* = Y_t - \hat{\rho} Y_{t-1}$

$X_t^* = X_t - \hat{\rho} X_{t-1}$ and regress Y_t^* on X_t^* and an intercept

4: we take the residuals from the regression in 3 and go to step 2:
run these steps until $\hat{\rho}$ from consecutive iterations is less than some prespecified level.

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f) let X_t be measured with error so that $X_t = X_t^* - v_t$ and the true model is $Y_t = \beta_0 + \beta_1 X_t^* + u_t = \beta_0 + \beta_1 X_t + \beta_1 v_t + u_t = \beta_0 + \beta_1 X_t + \varepsilon_t$

$$\hat{\beta}_1 = \beta_1 + \frac{\sum_{i=1}^{N}(x_i - \bar{x})\varepsilon_i}{\sum_{i=1}^{N}(x_i-\bar{x})^2} \Rightarrow E(\hat{\beta}_1) = \beta_1 + \frac{\sum_{i=1}^{N} E[(x_i-\bar{x})\varepsilon_i]}{\sum_{i=1}^{N}(x_i-\bar{x})^2}$$

if $E((x_i-\bar{x})\varepsilon_i) = 0$ Then $\hat{\beta}_1$ is unbiased. However $E(x_i \varepsilon_i) \neq 0$
so $E((x_i - \bar{x})\varepsilon_i) \neq 0$. $E(x_i \varepsilon_i) = E((X_t^* - v_t)(\beta_1 v_t + u_t)) = E(X_t^* \beta_1 v_t) - E(\beta_1 v_t^2) + E(X_t^* u_t) - E(u_t x_t) \neq 0$

since $-E(\beta_1 v_t^2) = -\beta_1^2 Var(v_t) = \alpha \sigma^2 \neq 0$

So $\hat{\beta}_1$ is biased when there is measurement error. 8

2. $Y = \alpha_1 + \alpha_2 D1 + \alpha_3 X + \alpha_4 (D1 * X) + u$

$D1 = 1$ when state legislature is run by Democrats; 0 otherwise

$Y = \beta_1 + \beta_2 D_2 + \beta_3 X + \beta_4 (D2 * X) + v$

$D2 = 1$ when state legislature is not run by Democrats; 0 otherwise

Obtain estimates of the α's from estimates of the $\hat{\beta}$'s

$Y = \hat{\beta}_1 + \hat{\beta}_2 D_2 + \hat{\beta}_3 X + \hat{\beta}_4 \; D2*X$

when D1 on D2 off

$\hat{\alpha}_1 + \hat{\alpha}_2 = \hat{\beta}_1$ so $\hat{\beta}_1 + \hat{\beta}_2 + \hat{\alpha}_2 = \hat{\beta}_1$

$\hat{\alpha}_1 = \hat{\beta}_1 + \hat{\beta}_2$ $\hat{\alpha}_2 = \hat{\beta}_1 - \hat{\beta}_1 - \hat{\beta}_2 = -\hat{\beta}_2$

$$\boxed{\begin{array}{ll} \hat{\alpha}_1 = \hat{\beta}_1 + \hat{\beta}_2 & 2 \\ \hat{\alpha}_2 = -\hat{\beta}_2 & 2 \\ \hat{\alpha}_3 = \hat{\beta}_3 + \hat{\beta}_4 & 2 \\ \hat{\alpha}_4 = -\hat{\beta}_4 & 2 \end{array}}$$

$\hat{\alpha}_3 = \hat{\beta}_3 + \hat{\beta}_4$

$\hat{\alpha}_3 + \hat{\alpha}_4 = \hat{\beta}_3 \rightarrow \hat{\beta}_3 + \hat{\beta}_4 + \hat{\alpha}_4 = \hat{\beta}_3$

$\rightarrow \hat{\alpha}_4 = -\hat{\beta}_4$

$$\begin{aligned} Y_t &= \hat{\alpha}_1 + \hat{\alpha}_2 D_1 + \hat{\alpha}_3 X + \hat{\alpha}_4 \; D1\,X \\ &= \hat{\alpha}_1 + \hat{\alpha}_2 (1 - D2) + \hat{\alpha}_3 X + \hat{\alpha}_4 (1 - D_2) X \\ &= (\hat{\alpha}_1 + \hat{\alpha}_2) - \hat{\alpha}_2 D_2 + (\hat{\alpha}_3 + \hat{\alpha}_4) X - \hat{\alpha}_4 D_2 X \\ &= \hat{\beta}_1 + \hat{\beta}_2 D_2 + \hat{\beta}_3 X + \hat{\beta}_4 D_2 X \end{aligned}$$

3.

a) Model D — Q, Y, P are all in logs

D82 = D86 = 0 → 1960-1981 Price elasticity = −.371 2
Income elasticity = .732 2

D82 = 1 D86 = 0 1982-1985 Price elasticity = −.371 + .288 = −.083 2
Income elasticity = .732 − 2.602 = −1.87 2

D82 = D86 = 1 1986-1988 Price elasticity = −3.71 + .288 = −.083 2
Income Elasticity = .732 − 2.602 + 3.298 = 1.428 2

b) Find R^2 for Models A-D

$$\bar{R}^2 = 1 - \frac{(1-R^2)(n-1)}{n-k-1} \Rightarrow 1 - \bar{R}^2 = \frac{(1-R^2)(n-1)}{n-k-1}$$

$$\frac{(1-\bar{R}^2)(n-k-1)}{(n-1)} = 1 - R^2 \Rightarrow R^2 = 1 - \frac{(1-\bar{R}^2)(n-k-1)}{(n-1)}$$

For all models A-D $n = 29$

Model A: $\bar{R}^2 = .921$ $\quad R^2 = 1 - \frac{(1-.921)(20)}{28} = 1 - \frac{(.079)(20)}{28} = 1 - \frac{1.58}{28}$

$k = 8$

$R^2 = 1 - .0564 = .9436$ 2

Model B: $\bar{R}^2 = .859$ $\quad R^2 = 1 - \frac{(1-.859)(22)}{28} = 1 - \frac{(.141)(22)}{28} = 1 - \frac{3.102}{28}$

$k = 6$

$R^2 = 1 - .1108 = .8892$ 2

Model C: $\bar{R}^2 = .852$ $\quad R^2 = 1 - \frac{(1-.852)(24)}{28} = 1 - \frac{(.148)(24)}{28} = 1 - \frac{3.552}{28}$

$k = 4$

$R^2 = 1 - .1269 = .8731$ ②

Model D: $\bar{R}^2 = .921$ $\quad R^2 = 1 - \frac{(1-.921)(21)}{28} = 1 - \frac{(.079)(21)}{28} = 1 - \frac{1.659}{28}$

$k = 7$

$R^2 = 1 - .0593 = .9407$ ②

c) i) test the null that price & income elasticities for 1960-1981, 1982-1985, & 1986-1988 are the same

The restricted model is C the unrestricted model is A

$H_0: \beta_5 = \beta_6 = \beta_8 = \beta_9 = 0$

$H_1: H_0$ false

$$F = \frac{(.04195 - .0186)/4}{.0186/(29-9)} = \frac{.02335/4}{.0186/20} = \frac{.005838}{.00093} = 6.2774$$

This test statistic is distributed $F(4, 20)$. If we look for the 1% critical value for a $F(4,20)$ distribution, call F_c, then if $F > F_c$ we reject H_0 and conclude that the price elasticities and income elasticities are not the same across periods.

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ii) Test that the income elasticities are constant over time but the price elasticities are not

The restricted model is B the unrestricted is A

H_0: $\beta_5 = \beta_6 = 0$
H_1: H_0 false

$$G = \frac{(.0364 - .0186)/2}{.0186\ /(24-9)} = \frac{.0178/2}{.0186/20} = \frac{.0089}{.00093} = 9.5699$$

$G \sim F(2,20)$. If we look at the 1% critical value for a $F(2,20)$ distribution and see that $G > F_c$ then we reject the null and claim that both income elasticities as well as price elasticities are differing over time.

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