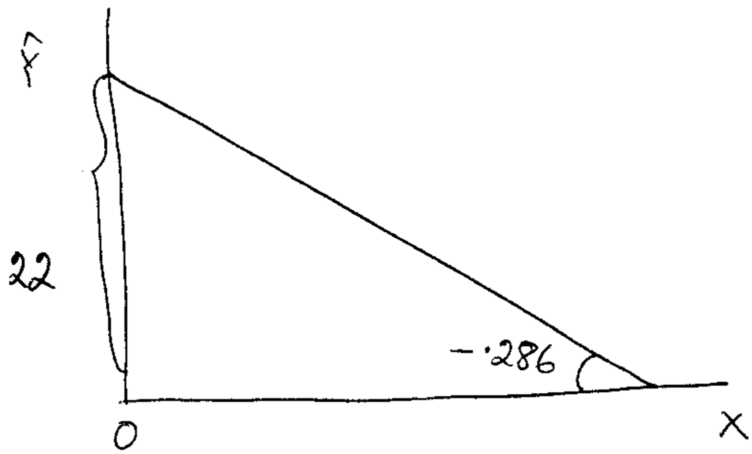
Key to Midterm 1, F 2002

1. (a) $\hat{\beta}_1 = \dfrac{\Sigma(X-\bar{X})(Y-\bar{Y})}{\Sigma(X-\bar{X})^2} = -\dfrac{100}{350} = -.286$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 12 - (-.286)35 = 22$$



$\hat{\beta}_0 = \hat{Y}$ when $X = 0$. Thus the intercept gives an estimate of demand for natural gas when electricity cost is zero.

$\hat{\beta}_1 = \dfrac{d\hat{Y}}{dX}$ gives an estimate of change in demand when electricity cost is changed by a dollar. That is, if electricity cost is increased by a \$, demand for natural gas (estimated) will decrease by .286 thousand cubic feet.

Fitted line: $\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$

$$\Rightarrow \bar{\hat{Y}} = \underbrace{\bar{Y}} = \underbrace{\hat{\beta}_0 + \hat{\beta}_1 \bar{X}}$$

i.e., $12 = 14 - 2 = 12$

b. $SSE = \sum(\hat{Y}-\bar{Y})^2 = \sum\{\hat{\beta}_1(X-\bar{X})\}^2 = \hat{\beta}_1^2 \sum(X-\bar{X})^2$

$= (-.286)^2(350) = 28.57$

$SSR = SST - SSE = 60 - 28.57 = 31.43$

$$R^2 = \frac{28.57}{60} = .476$$

$$r_{XY}^2 = \frac{\{Cor(X,Y)\}^2}{V(X)\,V(Y)} = \frac{\left(\sum(X-\bar{X})(Y-\bar{Y})\right)^2}{\sum(X-\bar{X})^2\sum(Y-\bar{Y})^2} = .476.$$

c. $$\widehat{Var}(\hat{\beta}_1) = \hat{\sigma}^2/\sum(X-\bar{X})^2 = \frac{SSR/n-2}{\sum(X-\bar{X})^2} = \frac{(31.43)/58}{350}$$

$= .00155$

Std error of $\hat{\beta}_1 = \sqrt{.00155}$

d. $Y^* = Y/10$ when Y^* is in 10,000 cubic feet.

Start from $Y = \beta_0 + \beta_1 X + u$

$$\Rightarrow \frac{Y}{10} = \underbrace{\frac{\beta_0}{10}} + \underbrace{\frac{\beta_1}{10}} X + \underbrace{\frac{u}{10}}$$

$\Rightarrow Y^* = \beta_0^* + \beta_1^* X + u^*$ (new regression).

$$\hat{\beta}_1^* = \frac{\sum(X-\bar{X})(Y^*-\bar{Y}^*)}{\sum(X-\bar{X})^2} = \frac{\frac{1}{10}\sum(X-\bar{X})(Y-\bar{Y})}{\sum(X-\bar{X})^2} = \frac{1}{10}\hat{\beta}_1$$

$\Rightarrow$ the slope is decreased by a factor of 10.

$$\hat{\beta}_0^* = \bar{Y}^* - \hat{\beta}_1^* \bar{X} = \frac{1}{10}\bar{Y} - \frac{1}{10}\hat{\beta}_1 \bar{X} = \frac{1}{10}(\bar{Y} - \hat{\beta}_1 \bar{X})$$

$$= \frac{1}{10}\hat{\beta}_0$$

$\Rightarrow$ the intercept is decreased by a factor of 10.

$$\widehat{Var}(\hat{\beta}_1^*) = \frac{\hat{\sigma}^{*2}}{\Sigma(X-\bar{X})^2}$$

here $\sigma^{*2} = var(u^*)$

$$= \frac{1}{10^2} \cdot var(u) = \frac{\sigma^2}{100}$$

$$= \frac{\frac{1}{100}\hat{\sigma}^2}{\Sigma(X-\bar{X})^2}$$

$$\Rightarrow \hat{\sigma}^{*2} = \frac{\hat{\sigma}^2}{100}$$

$$= \frac{1}{100}\widehat{Var}(\hat{\beta}_1)$$

$R^2 = \frac{SSE}{SST}$ is not changed.

e. Let $X^* = X + 2$

Go back to $Y = \beta_0 + \beta_1 X + u$

$$\Rightarrow Y = \beta_0 + \beta_1(X+2) - \beta_1 \cdot 2 + u$$

$$= (\beta_0 - 2\beta_1) + \beta_1 X^* + u$$

$$= \beta_0^* + \beta_1 X^* + u$$

Follow the procedure in part d to find that there will be <u>no</u> change in $\hat{\beta}_1$

$$\hat{\beta}_0^* = \hat{\beta}_0 - 2\hat{\beta}_1$$

f. $\tilde{\beta}_1 = \dfrac{\Sigma xy}{\Sigma x^2}$

$$= \frac{(-100 + 60(35)(12))}{350 + 60(35)^2}$$

$$= \frac{25100}{73850} = .3399$$

Note: $\Sigma(x-\bar{x})^2 = \Sigma x^2 - n\bar{x}^2$

$\Rightarrow \Sigma x^2 = \Sigma(x-\bar{x})^2 + n\bar{x}^2$

Similarly $\Sigma(y-\bar{y})^2 = \Sigma y^2 - n\bar{y}^2$

$\Rightarrow \Sigma y^2 = \Sigma(y-\bar{y})^2 + n\bar{y}^2$

and $\Sigma(x-\bar{x})(y-\bar{y}) = \Sigma xy - n\bar{x}\bar{y}$

$\Rightarrow \Sigma xy = \Sigma(x-\bar{x})(y-\bar{y}) + n\bar{x}\bar{y}$

$\widehat{Var}(\tilde{\beta}_1) = \tilde{\sigma}^2 / \Sigma x^2$ where $\tilde{\sigma}^2 = \dfrac{\Sigma \tilde{u}^2}{n-1}$

and $\Sigma \tilde{u}^2 = \Sigma(y - \tilde{\beta}_1 x)^2$

$= \Sigma y^2 - 2\tilde{\beta}_1 \Sigma xy + \tilde{\beta}_1^2 \Sigma x^2$

$= 8700 - 2(.3399)(25100)$

$+ (.3399)^2(73850)$

$= 169.06$

$\Rightarrow \tilde{\sigma}^2 = 2.865$

$$\Rightarrow \widehat{Var}(\tilde{\beta}_1) = \frac{2.865}{73850} = .000039$$

g. $\sum \tilde{u} = \sum(Y - \tilde{\beta}_1 X) = \sum Y - \tilde{\beta}_1 \sum X$

$= n(\bar{Y} - \tilde{\beta}_1 \bar{X}) = 60[12 - (.3399)35] = 6.21 \neq 0$

2. (a) True

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 0 - \hat{\beta}_1(0) = 0.$

(b) False, $\hat{\beta}_1 = \frac{d\hat{Y}}{dX}$.

(c) True, $\sum \hat{u} = \sum(Y - \hat{Y}) = \sum(Y - \hat{\beta}_0 - \hat{\beta}_1 X)$

Since $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

$= \sum(Y - (\bar{Y} - \hat{\beta}_1 \bar{X}) - \hat{\beta}_1 X)$

$= \sum\{(Y - \bar{Y}) - \hat{\beta}_1 (X - \bar{X})\}$

$= \underbrace{\sum(Y - \bar{Y})}_{=0} - \hat{\beta}_1 \underbrace{\sum(X - \bar{X})}_{=0}$

$= 0.$

d. $\mathrm{Cor}(X,Y) = 0 \Rightarrow \sum(X - \bar{X})(Y - \bar{Y}) = 0$

$\hat{\beta}_1 = \frac{\sum(X - \bar{X})(Y - \bar{Y})}{\sum(X - \bar{X})^2} = 0.$