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Answer key Midterm 1 - Fall 2003

(1) $Y = \beta_0 + \beta_1 X + U$ $n = 15$

$\Sigma Y^2 = 4500$

$\Sigma X^2 = 4305$

$\Sigma XY = 1655$

$\bar{Y} = 12$

$\bar{X} = 15$

$\bar{X} = \frac{1}{n} \Sigma X$

$15\bar{X} = \Sigma X$

$\Sigma X = 225$

$\Sigma Y = 180$

$$\Sigma (Y-\bar{Y})(X-\bar{X}) = \Sigma (Y-\bar{Y})X = \Sigma XY - \bar{Y}\Sigma X$$

$$1655 - 12(225)$$

$$1655 - 2700 = -1045$$

$$\Sigma (X-\bar{X})^2 = \Sigma (X-\bar{X})X$$

$$= \Sigma X^2 - \bar{X}\Sigma X$$

$$= 4305 - 15(225)$$

$$= 4305 - 3375$$

$$= 930$$

(i)

$$\hat{\beta}_1 = \frac{\Sigma(Y-\bar{Y})(X-\bar{X})}{\Sigma(X-\bar{X})^2} = \frac{-1045}{930} = -1.123$$

$\boxed{6 \text{ pts}}$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 12 - \left(\frac{-1045}{930}\right)15 = 12 - (-1.123)15$$

$$12 + 1.123(15)$$

$$12 + 16.845 = 28.854$$

$\boxed{4 \text{ pts}}$

(b)

$$ESS = \hat{\beta}_1^2 \sum (x-\bar{x})^2 = (930)(1.2626) = 1174.22$$
$$TSS = \sum (y-\bar{y})^2 = \sum y^2 - n\bar{y}^2 = 4500 - 15(144) = 2340$$

$$RSS = TSS - ESS \qquad R^2 = \frac{ESS}{TSS} = 1 - \frac{RSS}{TSS} = .501$$
$$= 2340 - 1174.22 = 1165.77$$

(c)
$$\widehat{V(\hat{\beta}_1)} = \frac{\frac{1}{n-2}\sum \hat{u}^2}{\sum (x-\bar{x})^2} = \frac{RSS}{(n-2)\sum (x-\bar{x})^2} = \frac{1165.77}{13(930)} = .096424 \quad \boxed{6\text{pts}}$$

$$s.e.(\hat{\beta}_1) = \sqrt{\widehat{V(\hat{\beta}_1)}} = .310523 \quad \boxed{2\text{pts}}$$

(e)
$$Y = \beta_0 + U \qquad \hat{\beta}_0 = \bar{Y} = 12 \qquad V(\hat{\beta}_0) = V(\bar{Y}) = V\left(\tfrac{1}{n}\sum(\beta_0 + U)\right)$$
$$V\left(\tfrac{1}{n}\sum \beta_0 + \tfrac{1}{n}\sum U\right)$$
$$V\left(\beta_0 + \tfrac{1}{n}\sum U\right)$$

constant

$$V(\hat{\beta}_0) = \frac{\sigma^2}{n}$$

$$V\left(\tfrac{1}{n}\sum U\right) = \frac{1}{n^2} V\left(\sum U\right)$$

$$\widehat{V(\hat{\beta}_0)} = \frac{RSS}{(n-1)n} = \frac{TSS}{(n-1)n} = \frac{2340}{15 \cdot 14} = 11.1429$$

$$= \frac{1}{n^2} \sum V(U) = \frac{n\sigma^2}{n^2} = \frac{\sigma^2}{n}$$

$$R^2 = \frac{ESS}{TSS} = \frac{0}{TSS} = 0 \qquad TSS = Y - \bar{Y}$$

$$R^2 = 1 - \frac{RSS}{TSS} = 1 - 1 = 0$$

$$Y = \hat{Y} + \hat{U}$$
$$Y - \hat{Y} = \hat{U}$$
$$Y - \bar{Y} = \hat{U}$$
$$(Y - \bar{Y})^2 = \hat{U}^2$$
$$TSS = \sum (Y - \bar{Y})^2 = \sum \hat{U}^2 = RSS$$

(2) $\hat{GPA} = .65 + .0020\,V + .0015\,M$ $n = 40$

$(.086) \quad (.0011) \quad (.0012)$

(i) interpret slope coefficients

2 pts if V ↑ 100 pts $\hat{GPA}$ ↑ .2 pts
2 pts M ↑ 100 pts $\hat{GPA}$ ↑ .15 pts

Both have the correct sign

(ii) Are coefficients signifigantly different from zero $df = 37$

$H_0: \beta_V = 0$
$H_1: \beta_V > 0$

$H_0: \beta_M = 0$
$H_1: \beta_M > 0$

$t_{\beta_V} = \dfrac{.0020}{.0011} = 1.8\overline{18}$

$t_{\beta_M} = \dfrac{.0015}{.0012} = 1.25$

Test @ 5% level

$c = 1.684$

3 pts fail to reject $H_0: \beta_M = 0$ @ 5% level of significance

3 pts reject $H_0: \beta_V = 0$ @ 5% level of significance

(iii) 99% confidence interval for β_M

$\hat{\beta}_M \pm c_{99} \cdot s.e.(\hat{\beta}_M)$

5 pts

$.0015 \pm 2.704 \cdot .0012 = \{$

$.0015 \pm .003245 = (-.001745, .004745)$

(iv) find p-value of $H_0: \beta_M = 0$
$H_1: \beta_M > 0$

$t_{\beta_M} = \frac{.0015}{.0012} = 1.25$ $df \approx 40$

p-value ≈ 12, 13% $\boxed{5 \text{ pts}}$

(v) if V & M are divided by 100 the estimated regression will have new slope parameters, but the fitted values, the fitted residuals, and the overall fit of the model will not change from the previous model. $\boxed{6 \text{ pts}}$

(vi) $\hat{GPA} = .65 + .002(500) + .0015(500) = .65 + 1 + .75$
$= 2.40$ $\boxed{3 \text{ pts}}$

$\boxed{2 \text{ pts}}$
- The student may have had a lower GPA but studied for the exam and was well prepared.
- The student may have had a lower GPA but off the test believing he/she would do very well.

③ (i) $\quad \text{I}: Y = \beta_0 + \beta_1 X + u$

$\text{II}: Y = \alpha_0 + \alpha_1 (X - \bar{X}) + u$

a) $\quad \text{I}: \hat{\beta}_1 = \dfrac{\sum (y - \bar{y})(x - \bar{x})}{\sum (x - \bar{x})^2}$

$\boxed{\text{3 pts}}$

$$\text{II}: \hat{\alpha}_1 = \frac{\sum (y - \bar{y})((x - \bar{x}) - (\overline{x - \bar{x}}))}{\sum ((x - \bar{x}) - (\overline{x - \bar{x}}))^2} = \frac{\sum (y - \bar{y})(x - \bar{x})}{\sum (x - \bar{x})^2} = \hat{\beta}_1$$

b) $\quad \text{I}: \hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$

$\boxed{\text{5 pts}}$

$$\text{II}: \hat{\alpha}_0 = \bar{Y} - \hat{\alpha}_1 (\overline{x - \bar{x}}) = \bar{Y} - \hat{\alpha}_1 (\overline{x - \bar{x}}) = \bar{Y}$$

$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = \hat{\alpha}_0 - \hat{\beta}_1 \bar{X} = \hat{\alpha}_0 - \hat{\alpha}_1 \bar{X}$$

(ii) $\quad Y = \beta_0 + \beta_1 X + U$

(a) $\quad \bar{\hat{Y}} = \bar{Y}$

$$\hat{Y} = \hat{\beta}_0 + \hat{\beta}_1 X$$
$$\hat{Y} = \bar{Y} - \hat{\beta}_1 \bar{X} + \hat{\beta}_1 X = \bar{Y} + \hat{\beta}_1 (X - \bar{X})$$
$$\bar{\hat{Y}} = \bar{Y} + \hat{\beta}_1 (\overline{X - \bar{X}}) = \bar{Y} \checkmark \quad \boxed{\text{2 pts}}$$

$$\sum (\hat{Y} - \bar{Y}) \hat{u} = 0$$

$\hat{Y} = \bar{Y} + \hat{\beta}_1 (x - \bar{x})$

$$\sum (\bar{Y} + \hat{\beta}_1 (x - \bar{x}) - \bar{Y}) \hat{u} = 0$$
$$\sum (\hat{\beta}_1 (x - \bar{x})) \hat{u} = 0$$
$$\hat{\beta}_1 \sum (x - \bar{x}) \hat{u} = 0 \Rightarrow \hat{\beta}_1 \Big[\underbrace{\sum x \hat{u}}_{0 \text{ by assumption}} - \bar{x} \underbrace{\sum \hat{u}}_{0} \Big] = 0 \checkmark$$

$\boxed{\text{3 pts}}$

(b) $\sum (Y-\bar{Y})(\hat{Y}-\bar{Y}) = \sum (\hat{Y}-\bar{Y})^2$

$\boxed{6 \text{ pts}}$

$Y = \hat{Y} + \hat{u}$

$$\sum (\hat{Y} + \hat{u} - \bar{Y})(\hat{Y}-\bar{Y}) = \sum (\hat{Y} - \bar{Y} + \hat{u})(\hat{Y}-\bar{Y})$$

$$= \sum (\hat{Y}-\bar{Y})^2 + \underbrace{\sum (\hat{Y}-\bar{Y})\hat{u}}_{=0 \text{ from above}} = \sum (\hat{Y}-\bar{Y})^2 \quad \checkmark$$

$$r^2_{Y\hat{Y}} = \frac{\{\sum (Y-\bar{Y})(\hat{Y}-\bar{Y})\}^2}{\sum (Y-\bar{Y})^2 \sum (\hat{Y}-\bar{Y})^2} = \frac{\{\sum (\hat{Y}-\bar{Y})^2\}^2}{\sum (Y-\bar{Y})^2 \sum (\hat{Y}-\bar{Y})^2}$$

$$r^2_{Y\hat{Y}} = \frac{\sum (\hat{Y}-\bar{Y})^2}{\sum (Y-\bar{Y})^2} = \frac{\sum (\hat{Y}-\bar{\hat{Y}})^2}{\sum (Y-\bar{Y})^2} = \frac{ESS}{TSS} = R^2$$

$\boxed{3 \text{ pts}}$