ECON 466 Ans KEY

Spring 07 Midterm Exam I

1. $Y = \beta_0 + \beta_1 X + u$

a) $\Sigma \hat{u}_i = 0$

$\Sigma \hat{u}_i = \Sigma (Y_i - \hat{Y}_i) = \Sigma (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)$

from minimizing sum of squared residuals to find $\hat{\beta}_0$

$$\frac{\partial \Sigma \hat{u}_i^2}{\partial \hat{\beta}_0} = -2 \Sigma (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i) = 0$$

$$\Rightarrow \Sigma (Y_i - \hat{\beta}_0 - \beta_1 X_i) = 0$$

$$\Rightarrow \Sigma \hat{u}_i = 0$$

b) $\Sigma X_i \hat{u}_i = 0$

also from minimizing sum of squared residuals to find $\hat{\beta}_1$

$$\frac{\partial \Sigma u_i^2}{\partial \hat{\beta}_1} = 2 \Sigma (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)(-X_i) = 0$$

$$\Rightarrow \Sigma (Y_i - \hat{\beta}_0 - \hat{\beta}_1 X_i)(X_i) = 0$$

$$\Rightarrow \Sigma \hat{u}_i X_i = 0$$

c) $\Sigma \hat{Y}_i \hat{u}_i = 0$

$$\Sigma \hat{Y}_i \hat{u}_i = \Sigma (\hat{\beta}_0 + \hat{\beta}_1 X_i) \hat{u}_i = \Sigma \hat{\beta}_0 \hat{u}_i + \Sigma \hat{\beta}_1 X_i \hat{u}_i$$

$$= \hat{\beta}_0 \Sigma \hat{u}_i + \hat{\beta}_1 \Sigma X_i \hat{u}_i = 0$$

d $\bar{\hat{Y}} = \frac{\sum \hat{Y}_i}{n} = \frac{1}{n}\sum(\hat{\beta}_0 + \hat{\beta}_1 X_i) = \hat{\beta}_0 + \hat{\beta}_1 \bar{X}$ ~~$=\bar{Y}$~~

$= \bar{Y} - \hat{\beta}_1 \bar{X} + \hat{\beta}_1 \bar{X}$

$= \bar{Y}$

e $\sum \hat{y}_i \hat{u}_i = \sum(\hat{Y}_i - \bar{Y})\hat{u}_i = \sum \hat{Y}_i \hat{u}_i - \bar{Y}\sum \hat{u}_i = 0$

Ques 2

$n = 10$, $\sum X = 40$, $\sum Y = 8$, $\sum X^2 = 200$, $\sum Y^2 = 26$,
$\sum XY = 20$, $\bar{X} = \frac{\sum X}{n} = 4$, $\bar{Y} = \frac{\sum Y}{n} = 0.8$

a) $\hat{\beta}_1 = \frac{\sum XY - n\bar{X}\bar{Y}}{\sum X^2 - n\bar{X}^2} = \frac{20 - 10(4)(0.8)}{200 - 10(4)^2} = \frac{-12}{40}$

$= -0.3$

$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X} = 0.8 - (-0.3)(4)$

$= 2$

$\hat{Y} = 2 + (-0.3)X \Rightarrow \hat{Y} = 2 - 0.3X$

$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum(X_i - \bar{X})^2} = \frac{\sigma^2}{\sum X_i^2 - n\bar{X}^2} = \frac{\sigma^2}{200 - 10(4)^2}$

$= \frac{\sigma^2}{40}$

$se(\hat{\beta}_1) = \hat{\sigma}\sqrt{\frac{1}{40}}$

$\hat{\sigma}^2 = \frac{1}{n-2} \sum \hat{u}_i^2$

Ques 2 part a) cont'd

$$Var(\hat{\beta}_0) = \frac{\sigma^2 \Sigma x_i^2}{n \Sigma (x_i - \bar{x})^2} = \frac{\sigma^2 \Sigma x_i^2}{n(\Sigma x_i^2 - n\bar{x}^2)}$$

$$= \frac{\sigma^2 \, 200}{10(200 - 10(4)^2)} = \frac{\sigma^2 \, 20}{40} = 0.5\sigma^2$$

$$se(\hat{\beta}_0) = \hat{\sigma}\sqrt{0.5}$$

b) Total variation in Y

$$SST = \Sigma(Y_i - \bar{Y})^2 = \Sigma Y^2 - n\bar{Y}^2 = 26 - 10(0.8)^2 = 19.6$$

$$SSE = \hat{\beta}_1^2 \, \Sigma(x_i - \bar{x})^2 = \hat{\beta}_1^2(\Sigma x_i^2 - n\bar{x}^2)$$

$$= 0.09(200 - 10(4^2)) = 3.6$$

$$R^2 = 3.6/19.6 = 0.1837 \text{ or } 18.37\%$$

c) confidence interval for β_1 at 95%

$\hat{\beta}_1 \pm c \cdot se(\hat{\beta}_1)$ $\qquad df = 10 - 2 = 8$

$c = 2.306$

$$-0.3 \pm 2.306\left(\hat{\sigma}\sqrt{\frac{1}{40}}\right)$$

$$= -0.3 \pm 2.306\left(\frac{\Sigma \hat{u}_i^2}{n-2}\right)^{1/2}\left(\frac{1}{40}\right)^{1/2}$$

$$= -0.3 \pm 2.306\left(\frac{15}{8}\,\frac{1}{40}\right)^{1/2} = -0.3 \pm 2.306\frac{\sqrt{3}}{8}$$

$$= -0.3 \pm 0.4993$$

$$= (-0.7993, 0.1993)$$

Ques 2 part d cont'd

d) $H_0 : \beta_1 = 1$

$H_1 : \beta_1 \neq 1$

$$\frac{\hat{\beta}_1 - 1}{se(\hat{\beta}_1)} \sim t_{5\%, 8}$$

$$\frac{-0.3 - 1}{0.2165} = \frac{-1.3}{0.2165} = -6.005$$

$$t_{5\%;8} = 2.306$$

So we reject the null hypothesis

e) $H_0 : \beta_1 = 0$

$H_1 : \beta_1 \neq 0$

$$\frac{\hat{\beta}_1}{Se(\hat{\beta}_1)} \sim t_{5\%;8}$$

$$\frac{-0.3}{0.2165} = 1.386 < t_{5\%;8}$$

So we fail to reject the null hypothesis

Ques 3

a) For a one percent increase in the prison population per capita, there is a 1.06 percent increase in the crime rate in the 50 states and DC.

b) Note that if prison population have a crime deterring effect, then β_1 should be negative

$H_0: \beta_1 = 0$

$H_1: \beta_1 < 0$

c) This is a one tailed test.

$$t = \frac{\hat{\beta}_1}{se(\hat{\beta}_1)} = \frac{1.0693}{0.1093} = 9.783$$

this value obviously lies inside the region where we fail to reject because it is positive.

C.I.

$\hat{\beta}_1 \pm c \cdot se(\hat{\beta}_1)$

$1.0693 \pm 2.009575(0.1093)$

$(0.849653, 1.288947)$

Ques 3 part d cont'd

d) 1.0893 represents the estimated percentage increase in crime rates when prison population per capita increases by 1 percent, given that per capita income is held fixed.

Given that prison population per capita is held fixed, a 1 percent increase in per capita income reduces the estimated crime rate by 0.4 percent

Expect the sign on $\hat{\beta}_2$ to be negative because of the presumption that the more per capita income that someone has the less likely he or she is to commit a crime.

e) $H_0: \beta_1 = 0, \beta_2 = 0$
H_1: null is not true

Note that this is a joint test

$$F = \frac{(SSR_R - SSR_{UR})/q}{SSR_{UR}/n-(k+1)} = \text{test statistic}$$

$q = 2$; $df = 51 - 3 = 48$

Since values already are provided in this question, we need $F_{2,48} = 5.0767$

$$F = \frac{R^2_{ur}/q}{(1-R^2_{ur})/n-(k+1)} = \frac{0.6627/2}{(1-0.6627)/48}$$

$$= \frac{0.33135}{0.007027}$$

$$= 47.15 > F_{2,48} = 5.0761$$

So we reject the null hypothesis that prison population and per capita income has no effect on crime rates

f) The estimated coefficient on ln PI will not change.

$$\ln Y = \beta_0 + \beta_1 \ln X + \beta_2 \ln PI + u$$

$$PI^* = \frac{PI}{1000} \Rightarrow PI = 1000\,PI^*$$

$$\ln Y = \beta_0 + \beta_1 \ln X + \beta_2 \ln(PI * 1000) + u$$

$$\ln Y = \beta_0 + \beta_2 \ln 1000 + \beta_1 \ln X + \beta_2 \ln PI + u$$

g) $\ln \hat{Y} = 0.544 + 1.0893(-5) + (-0.4132)(3)$

$= -6.1421$

to construct a confidence interval we also need the $se(\ln \hat{Y})$

C.I : $-6.1421 \pm 2 \cdot se(\ln \hat{Y})$

To get std error, regress ln Y on (ln X + 5) and (ln PI − 3). The std error of the intercept is the std error of the predicted $\ln \hat{Y}$