Hw # 1

① If X has density function

$$f(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & \text{elsewhere,} \end{cases}$$

find ~~the~~ mean and variance of X.

(Note: you answers should be expressed in terms of a and b.)

② If Z has distribution function

$$F(z) = \begin{cases} 0, & z \le 0 \\ z/8, & 0 < z < 2 \\ z^2/16, & 2 \le z < 4 \\ 1, & z \ge 4, \end{cases}$$

find ~~the~~ mean and variance of Z.

3. Suppose that the random variables Y_1 and Y_2 have joint probability density function $f(y_1, y_2)$ given by

$$f(y_1, y_2) = \begin{cases} 6(1-y_2), & 0 \le y_1 \le y_2 \le 1 \\ 0, & \text{elsewhere.} \end{cases}$$

a) Find the marginal density functions for Y_1 and Y_2.

b) Find $P(Y_2 \le 1/2 \mid Y_1 \le 3/4)$

c) Find the conditional density function of Y_1 given $Y_2 = y_2$.

d) Find ~~the~~ conditional density function of Y_2 given $Y_1 = y_1$.

e) Are Y_1 and Y_2 independent

f) Find $E(Y_1)$, $E(Y_2)$, $V(Y_1)$, $V(Y_2)$, $E(Y_1 - 3Y_2)$, $\text{Cov}(Y_1, Y_2)$.

(4) Suppose that the random variables Y_1 and Y_2 have means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively. Use the definition of the covariance of two random variables to show that:

a) $\mathrm{Cov}(Y_1, Y_2) = \mathrm{Cov}(Y_1, Y_2)$

b) $\mathrm{Cov}(Y_1, Y_1) = V(Y_1) = \sigma_1^2$

(5) Using the properties of the summation operator Σ and the formulas $\bar{x} = \frac{1}{n} \sum_{i=1}^{n} x_i$, $\bar{y} = \frac{1}{n} \sum_{i=1} y_i$,

Show that $\sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y})$ is equivalent to

(i) $\sum_{i=1}^{n} x_i y_i - n\bar{x}\bar{y}$

(ii) $\sum_{i=1}^{n} (x_i - \bar{x}) y_i$

(iii) $\sum_{i=1}^{n} (y_i - \bar{y}) x_i$

6) Use $\hat{\beta}_1 = \dfrac{\sum_i (X_i - \bar{X})(Y_i - \bar{Y})}{\sum_i (X_i - \bar{X})^2}$ to show that

(i) $\hat{\beta}_1 = \dfrac{\sum (X_i - \bar{X}) Y_i}{\sum (X_i - \bar{X})^2}$

(ii) $R^2 = \dfrac{SSE}{SST} = \dfrac{\hat{\beta}_1^2 \sum (X_i - \bar{X})^2}{\sum (Y_i - \bar{Y})^2}$

Also (iii) $R^2 = \dfrac{\left\{ \sum (X_i - \bar{X})(Y_i - \bar{Y}) \right\}^2}{\sum (X_i - \bar{X})^2 \sum (Y_i - \bar{Y})^2} = r_{YX}^2$

Define $k_i = \dfrac{X_i - \bar{X}}{\sum (X_i - \bar{X})^2}$ to show that

(iv) $\sum_i k_i = 0$, $\sum_i k_i^2 = \dfrac{1}{\sum (X_i - \bar{X})^2}$

$\sum k_i (X_i - \bar{X}) = \sum_i k_i X_i = 1$.

7) For the model $Y = \beta_0 + \beta_1 X + u$ we proved that $E(\hat{\beta}_1) = \beta_1$. Use $\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$ to prove that $E(\hat{\beta}_0) = \beta_0$.

Now consider the model $Y_i = \beta X_i + u_i$ (no intercept)

Show that the OLS estimator $\hat{\beta}$ is $\hat{\beta} = \dfrac{\sum X_i Y_i}{\sum X_i^2}$ and $E(\hat{\beta}) = \beta$.