Equivalence of 2SLS and IV estimators

Consider <u>one</u> particular equation of the system and write it as

$$\underset{n\times 1}{y} = \underbrace{Y_1}_{n\times g-1}\beta + \underset{n\times k}{X_1}\gamma + u \qquad \cdots (1)$$

total # endo variables $= g$ (y and Y_1).

In (1) there are g endo and k exog. variables. The equation in (1) is identified $\Rightarrow$ # variables (endo + exog) absent from (1) $\geq G-1 \Rightarrow (G-g)+(K-k) > G-1$
($G =$ # endo variables, $K =$ # exog variables in the <u>system</u>).

$\Rightarrow K-k > g-1$ (# exog variables <u>absent</u> from the eqn $\geq$ # endo variables <u>included</u> -1).

<u>2SLS</u>

Step 1: Regress Y_1 on X and get $\hat{Y}_1 = X(X'X)^{-1}X'Y_1$

Step 2: Replace Y_1 in (1) by $\hat{Y}_1$ and Regress y on $\hat{Y}_1$ and X_1.

For this we rewrite (1) as

$$y = \underbrace{\begin{bmatrix} \hat{Y}_1 & X_1 \end{bmatrix}}_{\hat{W}} \begin{bmatrix} \beta \\ \gamma \end{bmatrix} + \text{error}$$

$$\Rightarrow \begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} = (\hat{W}'\hat{W})^{-1}\hat{W}'y \quad \text{or} \quad (\hat{W}'\hat{W})^{-1}\begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} = \hat{W}'y$$

$$\Rightarrow \begin{pmatrix} \hat{Y}_1'\hat{Y}_1 & \hat{Y}_1'X_1 \\ X_1'\hat{Y}_1 & X_1'X_1 \end{pmatrix}\begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} \hat{Y}_1'y \\ X_1'y \end{pmatrix} \qquad \cdots (2)$$

From Step 1, write $Y_1 = \hat{Y}_1 + V_1 \Rightarrow \hat{Y}_1' V_1 = 0,\ X_1' V_1 = 0$

$$\Rightarrow \hat{Y}_1'\hat{Y}_1 = \hat{Y}_1'(Y_1 - V_1) = \hat{Y}_1' Y_1 =$$

$$\text{and } \hat{Y}_1' X_1 = (Y_1 - V_1)' X_1 = Y_1' X_1 .$$

Thus (2) becomes
$$\begin{bmatrix} \hat{Y}_1 Y_1 & \hat{Y}_1' X_1 \\ X_1' \hat{Y}_1 & X_1' X_1 \end{bmatrix} \begin{pmatrix} \hat{\beta} \\ \hat{\gamma} \end{pmatrix} = \begin{pmatrix} \hat{Y}_1' y \\ X_1' y \end{pmatrix} \quad \cdots (3)$$

IV estimator: Write (1) as $y = [Y_1 \ \ X_1]\begin{pmatrix} \beta \\ \gamma \end{pmatrix} + u \equiv W\delta + u$

The instrument matrix $Z = [\hat{Y}_1 \ \ X_1]$

$$\Rightarrow \hat{\delta}_{IV} = \begin{pmatrix} \beta \\ \gamma \end{pmatrix}_{IV} = (Z'W)^{-1} Z' y \Rightarrow (Z'W)\hat{\delta}_{IV} = Z'y$$

$$\Rightarrow \begin{bmatrix} \hat{Y}_1' Y_1 & \hat{Y}_1' X_1 \\ X_1' \hat{Y}_1 & X_1' X_1 \end{bmatrix} \begin{bmatrix} \hat{\beta}_{IV} \\ \hat{\gamma}_{IV} \end{bmatrix} = \begin{bmatrix} \hat{Y}_1' y \\ X_1' y \end{bmatrix} \quad \cdots \quad (4)$$

(2) and (4) are the same, which proves that the 2SLS and the IV estimators are the same.