Exam 3 : Fall 1998

Question 1

$$Y_t = \beta_1 + \beta_2 X_{2t} + \beta_3 X_{3t} + u_t \quad , \quad t = 1, \ldots, 100$$

(a) $$\begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = (x'x)^{-1} x'y = \begin{pmatrix} 20 & 15 \\ 15 & 25 \end{pmatrix}^{-1} \begin{pmatrix} 35 \\ 40 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

(b) $$R^2 = \frac{ESS}{TSS} = \frac{b_2' x'y}{TSS} = 75/100 = 0.75$$

$$R_2^2 = \frac{ESS_2}{TSS_2}$$ from the regression →

$$X_2 = \alpha_0 + \alpha_1 X_3 + v$$

$$\Rightarrow \quad \hat{\alpha}_1 = \frac{\sum x_2 x_3}{\sum x_3^2} = 15/25 = 0.6$$

$$\Rightarrow \quad ESS_2 = \hat{\alpha}_1 \sum x_2 x_3$$

$$TSS_2 = \sum x_2^2$$

$$\Rightarrow \quad R_2^2 = \frac{0.6 \times 15}{20} = 0.45$$

Also $R_2^2 = R_3^2$

Since $R_k^2 < R^2$ for $k = 2, 3$

$\Rightarrow$ no serious multicollinearity

(c) $X_2 = m X_3$

$Y = \alpha_1 + \alpha_2 X_3 + u$

$\hookrightarrow \hat{\alpha}_2 = \dfrac{\sum x_3 y}{\sum x_3^2}$

$\hookrightarrow ESS = \hat{\alpha}_2 \sum x_3 y$

$\hookrightarrow TSS = \sum y^2$

$R^2_{YX_3} = \dfrac{(\sum x_3 y)^2}{\sum x_3^2 \sum y^2}$

$Y = \gamma_1 + \gamma_2 X_2 + u$

$\Rightarrow Y = \gamma_1 + \gamma_2 (m X_3) + u$

$\hat{\gamma}_2 = \dfrac{\sum m x_3 y}{\sum (m x_3)^2} = \dfrac{1}{m} \dfrac{\sum x_3 y}{\sum x_3^2}$

$ESS = \hat{\gamma}_2 \sum m x_3 y$

$TSS = \sum y^2$

$R^2_{YX_2} = \dfrac{\sum x_3 y}{\sum x_3^2 \sum y^2}$

$\Rightarrow R^2_{YX_3} = R^2_{YX_2}$ $\Rightarrow$ R^2 invariant to change in scale

Question (2)

$$Y_t^* = \alpha + \beta X_t^*$$

$$Y_t = Y_t^* + u_t \quad / \quad X_t = X_t^* + v_t$$

a)

$$\Rightarrow Y_t - u_t = \alpha + \beta(X_t - v_t)$$

$$\Rightarrow Y_t = \alpha + \beta X_t + u_t - \beta v_t$$

For consistency, we need $\text{cov}(X_t, \text{error term}) = 0$

Now

$$\text{cov}(X_t, u_t - \beta v_t) = \text{cov}(X_t, u_t) - \beta\,\text{cov}(X_t, v_t)$$

$$= 0 - \beta\,\text{cov}(X_t, v_t)$$

$$= -\beta \sigma_v^2 \neq 0$$

$\Rightarrow$ Inconsistent estimates using OLS.

b)

$$b = \frac{\sum xy}{\sum x^2} = \frac{\sum x\left[\beta x + (u_t - \beta v_t)\right]}{\sum x^2}$$

$$= \beta + \frac{\sum xu}{\sum x^2} - \beta \frac{\sum xv}{\sum x^2}$$

$$\text{plim}(b) = \beta + \frac{\text{plim}\ 1/n \sum xu}{\text{plim}\ 1/n \sum x^2} - \beta \frac{\text{plim}\ 1/n \sum xv}{\text{plim}\ 1/n \sum x^2}$$

Now $\text{plim}\ 1/n \sum xu = 0$

$\text{plim}\ 1/n \sum x^2 = \sigma^2_{x^*} + \sigma^2_v$

$\text{plim}\ 1/n \sum xv = \sigma^2_v$

$\therefore \text{plim}(b) = \beta - \beta\left(\frac{\sigma^2_v}{\sigma^2_{x^*} + \sigma^2_v}\right)$

$\Rightarrow \text{plim}(b - \beta) = -\beta\left(\frac{\sigma^2_v}{\sigma^2_{x^*} + \sigma^2_v}\right)$

(c) $Y_t = \beta_1 X_{1t} + \beta_2 X^*_{2t} + u_t$

Instrumental variable:

- find a variable W_t such that

a) $\text{plim}\left(1/n \sum W'X\right)$ is finite ; $X = [X_1, X_2]$

b) $\text{plim}\ 1/n \sum W_t v_t = \text{plim}\ 1/n \sum W_t u_t = 0$

then use GLS - -

Question ③

$Y = \alpha + \beta X + u \qquad H_0: \beta = 1 \qquad \sigma^2$ known

$$\ln L_u = -\tfrac{n}{2}\ln(2\pi) - \tfrac{n}{2}\ln\sigma^2 - \tfrac{1}{2\sigma^2}\sum(Y - \alpha - \hat{\beta}X)^2$$

$$= \text{const} - \tfrac{1}{2\sigma^2}\sum(Y - \hat{\alpha} - \hat{\beta}x)^2 \qquad \text{where } \hat{\alpha} = \bar{Y} - \hat{\beta}\bar{X}$$

$$\ln L_R = \text{const} - \tfrac{1}{2\sigma^2}\sum(Y - \hat{\alpha} - x)^2$$

For the restricted model

$$Y = \alpha + \beta X + u \Rightarrow Y - X = \alpha + u$$

$$\Rightarrow \hat{\alpha} = \bar{Y} - \bar{X}$$

$$\Rightarrow \ln L_R = \text{const} - \tfrac{1}{2\sigma^2}\sum(y - x)^2$$

$$\ln L_u = \text{const} - \tfrac{1}{2\sigma^2}\sum(y - \hat{\beta}x)^2$$

Then $LR = -2[\ln L_R - \ln L_u]$

$$= \tfrac{1}{\sigma^2}\left\{\sum(y - x)^2 - \sum(y - \hat{\beta}x)^2\right\}$$

$$= \tfrac{1}{\sigma^2}\left\{\frac{(\sum xy)^2}{\sum x^2} - 2\sum xy + \sum x_2\right\}$$

$$\Rightarrow LR = \frac{1}{\sigma^2}\frac{\{\Sigma xy - \Sigma x^2\}^2}{\Sigma x^2}$$

$$W = \frac{(\hat{\beta} - 1)^2}{Var(\hat{\beta})} = \frac{\left(\frac{\Sigma xy}{\Sigma x^2} - 1\right)^2}{\sigma^2 / \Sigma x^2}$$

$$\Rightarrow W = \frac{1}{\sigma^2}\frac{\{\Sigma xy - \Sigma x^2\}^2}{\Sigma x^2}$$

$$LM = S(\theta)\, I^{-1}(\theta)\, S(\theta)\Big|_{\theta = \tilde{\theta}} \quad , \tilde{\theta} = \begin{pmatrix} \hat{\alpha} \\ 1 \end{pmatrix} \quad , \hat{\alpha} = \bar{y} - \bar{x}$$

$$S(\tilde{\theta}) = \begin{bmatrix} \frac{1}{\sigma^2}\Sigma(y - \hat{\alpha} - x) \\ \frac{1}{\sigma^2}\Sigma(y - \hat{\alpha} - x)x \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ \frac{1}{\sigma^2}\Sigma xy - \Sigma x^2 \end{bmatrix}$$

$$I(\hat{\theta}) = \begin{bmatrix} n/\sigma^2 & \Sigma x/\sigma^2 \\ \Sigma x/\sigma^2 & \frac{\Sigma x^2}{\sigma^2} \end{bmatrix}$$

$$\Rightarrow I^{-1}(\hat{\theta}) = \frac{\sigma^4}{n \Sigma x^2} \begin{bmatrix} \frac{\Sigma x^2}{\sigma^2} & -\Sigma x/\sigma^2 \\ -\Sigma x/\sigma^2 & n/\sigma^2 \end{bmatrix}$$

$$\Rightarrow LM = \frac{1}{\sigma^4}\left\{\Sigma xy - \Sigma x_2\right\}^2 \cdot \frac{\sigma^4}{n \Sigma x^2} \cdot n/\sigma^2$$

$$\Rightarrow LM = \frac{\left\{\Sigma xy - \Sigma x_2\right\}^2}{\sigma^2 \Sigma x^2}$$

$$\Rightarrow \boxed{LR = LM = W}$$