Final Exam : Spring 2004

Question (1)

(c) $V(b_K) = \dfrac{\sigma^2}{\sum x_K^2 (1 - R^2_K)} = \dfrac{\sigma^2}{\sum x_K^2} \, VIF$

$\Rightarrow VIF = \dfrac{1}{1 - R^2_K}$

max $VIF = \infty$ when $R^2_K = 1$

min $VIF = 1$ when $R^2_K = 0$

$R^2_K = 1 \Rightarrow$ perfect multicollinearity

$R^2_K = 0 \Rightarrow$ regressors are uncorrelated

(a), (b) – lecture Notes

Question 2

Mobil : $G = \alpha_0 + \alpha_1 Y + \alpha_2 B + \alpha_3 E + \alpha_4 JC$

Chevy : $G = \beta_1 Y + \beta_2 B + \beta_3 E + \beta_4 JC + \beta_5 V$

(a) (i) $\alpha_2 = E(G \mid Y, B=1) - E(G \mid Y, V=1)$

(ii) $\beta_2 - \beta_5 = E(G \mid Y, B=1) - E(G \mid Y, V=1)$

$\alpha_2 = \beta_2 - \beta_5$

(b) (i) $H_0 : \alpha_2 = 0$; $H_1 : \alpha_2 \neq 0$

(ii) $H_0 : \beta_2 = \beta_5$; $H_1 : \beta_2 \neq \beta_5$

(c) Model : $G = \alpha_0 + \alpha_1 Y + \alpha_2 B + \alpha_3 E + \alpha_4 JC + \alpha_5 Y \cdot E + \alpha_6 Y \cdot JC$

$H_0 : \alpha_5 = \alpha_6 = 0$

(d) Model : $G = \beta_1 Y + \beta_2 B + \beta_3 E + \beta_4 JC + \beta_5 V + \beta_6 Y \cdot B + \beta_7 Y \cdot E + \beta_8 Y \cdot JC + \beta_9 Y \cdot Y$

$H_0 : \beta_6 = \beta_7 = \beta_8 = \beta_9 = 0$

Question ③

$$Y = \beta X + u$$

$$\ln L = -\frac{n}{2}\ln 2\pi - \frac{n}{2}\ln \sigma^2 - \frac{1}{2\sigma^2}\sum (Y - \beta X)^2$$

$$\frac{\partial \ln L}{\partial \beta} = \frac{1}{\sigma^2}\sum (Y - \beta X)X = 0$$

$$\Rightarrow \hat{\beta}_{ML} = \frac{\sum XY}{\sum X^2} \quad \Rightarrow \text{Var}(\hat{\beta}_{ML}) = \frac{\sigma^2}{\sum X^2}$$

$$\frac{\partial^2 \ln L}{\partial \beta^2} = -\frac{\sum X^2}{\sigma^2}$$

$$\left.\frac{\partial^2 \ln L}{\partial \beta^2}\right|_{\hat{\beta}_{ML}} = -\frac{\sum X^2}{\sigma^2} < 0 \Rightarrow \hat{\beta}_{ML} \text{ maximizes}$$

the likelihood fun'l

Now $\hat{\beta}_{ML}$ is MVUE if $\text{Var}(\hat{\beta}_{ML})$ = Cramer-Rao LB

$$CRLB = \left\{ E\left[-\frac{\partial^2 \ln L}{\partial \beta^2}\right]\right\}^{-1}$$

$$CRLB = \frac{\sigma^2}{\sum X^2}$$

$\therefore$ $\text{Var}(\hat{\beta}_{ML}) = CRLB \Rightarrow \hat{\beta}_{ML}$ is MVUE.

(b) OLS estimator of β is inconsistent if $\text{cov}(X, u) \neq 0$.

(c) If Z were an instrument for X.

$\Rightarrow$ 1) $\text{plim } \frac{1}{n} \sum Zu = 0$

2) $\text{plim } \frac{1}{n} \sum ZX$ finite

$$\hat{\beta}_{IV} = \frac{\sum Zy}{\sum ZX}$$

note equivalent to GLS when dimension of data matrix = dimension of instrument

$$\hat{\beta}_{IV} = \frac{\sum Z[\beta X + u]}{\sum ZX} = \beta + \frac{\sum Zu}{\sum ZX}$$

$$\Rightarrow \text{plim } \hat{\beta}_{IV} = \beta + \frac{\text{plim } \frac{1}{n} \sum Zu}{\text{plim } \frac{1}{n} \sum ZX}$$

$$\Rightarrow \text{plim } \hat{\beta}_{IV} = \beta \Rightarrow \text{consistent}$$

d) $Y = \beta x^* + u$

$x = x^* + v$

$\Rightarrow Y = \beta x + (u - \beta v)$

$$\hat{\beta}_{ols} = \frac{\sum xy}{\sum x^2}$$

$$\hat{\beta}_{ols} = \frac{\sum x\left[\beta x + (u - \beta v)\right]}{\sum x^2}$$

$$\Rightarrow \hat{\beta}_{ols} = \beta + \frac{\sum xu}{\sum x^2} - \beta \frac{\sum xv}{\sum x^2}$$

now $\operatorname{plim} \frac{1}{n} \sum x^2 = \sigma^2_{x^*} + \sigma^2_v$

$\operatorname{plim} \frac{1}{n} \sum xu = 0$

$\operatorname{plim} \frac{1}{n} \sum xv = \sigma^2_v$

$$\therefore \operatorname{plim} \hat{\beta}_{ols} = \beta + \frac{\operatorname{plim} \frac{1}{n} \sum xu}{\operatorname{plim} \frac{1}{n} \sum x^2} - \beta \frac{\operatorname{plim} \frac{1}{n} \sum xv}{\operatorname{plim} \frac{1}{n} \sum x^2}$$

$$\text{plim } \hat{\beta}_{ols} = \beta - \beta \frac{\sigma^2_v}{\sigma^2_{x^*} + \sigma^2_v}$$

degree of inconsistency $= \text{plim } (\hat{\beta}_{ols} - \beta) = - \frac{\beta \sigma^2_v}{\sigma^2_{x^*} + \sigma^2_v}$

(e) $Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t \quad |\beta_1| < 1$

(i) $u_t \sim iid(0, \sigma^2)$ $\Rightarrow$ contemporaneous correlation between Y, u

$$b_1 = \frac{\sum y_t y_{t-1}}{\sum y_{t-1}^2} = \beta_1 + \frac{\sum y_{t-1} u_t}{\sum y_{t-1}^2}$$

for iid $\Rightarrow \text{plim } 1/n \sum y_{t-1} u_t = 0$

$\therefore \text{plim } b_1 = \beta_1 \Rightarrow$ consistency

(ii) $u_t = \rho u_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim iid(0, \sigma^2)$

We can write $Y_t = \frac{\beta_0}{1-\beta_1} + u_t + \beta_1 u_{t-1} + \beta_1^2 u_{t-2} + \ldots$

$$\Rightarrow y_{t-1} = u_{t-1} + \beta_1 u_{t-2} + \beta_1^2 u_{t-3} + \beta_1^3 u_{t-4} + \cdots$$

$$\Rightarrow \text{plim } 1/n \sum y_t u_t$$

$$= \text{plim } 1/n \sum u_{t-1} u_t + \beta_1 \text{ plim } 1/n \sum u_{t-2} u_t$$

$$+ \beta_1^2 \text{ plim } 1/n \sum u_{t-3} u_t + \cdots$$

$$= \rho \sigma_u^2 + \beta_1 \rho^2 \sigma_u^2 + \beta_1^2 \rho^3 \sigma_u^2 + \cdots$$

$$= \frac{\rho \sigma_u^2}{1 - \beta_1 \rho} \neq 0$$

$$\therefore \text{plim } b_1 \neq \beta_1 .$$

(ii) Estimation for case (ii):

$$Y_t = \beta_0 + \beta_1 Y_{t-1} + u_t \quad \cdots \quad (1)$$

$$\rho Y_{t-1} = \rho \beta_0 + \beta_1 \rho Y_{t-2} + \rho u_{t-1} \quad \cdots \quad (2)$$

$$(1) - (2) \Rightarrow Y_t = \beta_0 (1-\rho) + (\beta_1 + \rho) Y_{t-1} - \beta_1 \rho Y_{t-2} + \varepsilon_t$$

Question (4)

Demand: $Q = \alpha_1 P + \alpha_2 Z_1 + u_1$

Supply: $Q = \beta_1 P + \beta_2 Z_2 + u_2$

(a) $$\begin{bmatrix} 1 & -\alpha_1 \\ 1 & -\beta_1 \end{bmatrix} \begin{bmatrix} Q \\ P \end{bmatrix} - \begin{bmatrix} \alpha_2 & 0 \\ 0 & \beta_2 \end{bmatrix} \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

For eqn (1) : order condition $\Rightarrow R \geq G-1$; $R = 1, G = 2$ ✓

rank condition $\Rightarrow$ rank $(A\phi) = G - 1 = 1$

$$A = \begin{bmatrix} 1 & -\alpha_1 & -\alpha_2 & 0 \\ 1 & -\beta_1 & 0 & -\beta_2 \end{bmatrix}, \quad \phi = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad A\phi = \begin{bmatrix} 0 \\ -\beta_2 \end{bmatrix}$$

and rank$(A\phi) = 1$ ✓

→ Repeat steps using eqn (2).

(b) $$P = \frac{1}{\alpha_1 - \beta_1}\left[\alpha_2 Z_1 + \beta_2 Z_2\right] + \frac{u_2}{\alpha_1 - \beta_1} - \frac{u_1}{\alpha_1 - \beta_1}$$

$$Q = \frac{\alpha_1 \alpha_2}{\alpha_1 - \beta_1} Z_1 + \frac{\alpha_1 \beta_2}{\alpha_1 - \beta_1} Z_2 + \frac{\alpha_1 u_2}{\alpha_1 - \beta_1} - \frac{\beta u_1}{\alpha_1 - \beta_1}$$

Intuition: In reduced form the regressors are all exogenous

$\Rightarrow$ Cov $(Z_1, \text{error}) = 0$

Cov $(Z_2, \text{error}) = 0$

Hence consistency !!

(c) using $Q = \alpha_1 P + \alpha_2 Z_1 + u_1$

$cov(Z_1, u_1) = 0$ **but** $cov(P, u_1) \neq 0$

inconsistent estimates via ols

<u>IV Estimator</u> using Z_2 and Z_1 as instruments:

Note the dimension of the instrument list is the <u>same</u> as that of the data matrix

$\Rightarrow$ applying GLS <u>after</u> premultiplying the demand function by the set of instruments, is equivalent to

$$\hat{\alpha}_{IV} = (W'X)^{-1} W'Q$$

where $W = [Z_2 \ Z_1]$

$X = [P \ Z_1]$