Econometrics 616
Answer Key: Exam I
Spring 2004

1. $x'x = \begin{bmatrix} 50 & 60 \\ 60 & 100 \end{bmatrix}$ $x'y = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$, $y'y = 20.75$, $n = 20$

$$b_2 = \frac{100 \cdot 10 - 60 \cdot 5}{50 \cdot 100 - 60^2} = \frac{700}{1400} = .5, \quad b_3 = \frac{50 \cdot 5 - 60 \cdot 10}{50 \cdot 100 - 60^2} = \frac{-350}{1400} = -.25$$

a)

$$\hat{V}(b_2) = \frac{\hat{\sigma}^2 \cdot 100}{50 \cdot 100 - 60^2} = \frac{\hat{\sigma}^2}{14}, \quad \hat{V}(b_3) = \frac{\hat{\sigma}^2 \cdot 50}{50 \cdot 100 - 60^2} = \frac{\hat{\sigma}^2}{28}$$

$$\hat{\sigma}^2 = \frac{e'e}{n-k} = \frac{y'y - bx'y}{n-k} = \frac{20.75 - (.5 \cdot 10 - .25 \cdot 5)}{20 - 3} = 1$$

$$\hat{V}(b_2) = \frac{1}{14}, \quad \hat{V}(b_3) = \frac{1}{28}$$

b) $t = \dfrac{b_2 + b_3}{s.e.(b_2 + b_3)} = \dfrac{.5 - .25}{\sqrt{1/14 + 1/28 + 2 \cdot (-6/140)}} = \dfrac{.25}{\sqrt{3/140}} = 1.70783$ which is slightly lower than 2 so we would fail to reject for a standard significance level.

c) $\hat{\theta}_0 = \overline{Y} - \hat{\beta}_2 \overline{\overline{X}} = 10 - .5 \cdot (20 - 25) = 10 - (-2.5) = 12.5 = \hat{E}(Y_0)$

d)

$$\hat{\delta} = \frac{(50 + 100 - 2 \cdot 60) \cdot 5 - (60 - 100)(10 - 5)}{(50 + 100 - 2 \cdot 60) \cdot 100 - (60 - 100)^2} = \frac{30 \cdot 5 - (-40) \cdot 5}{30 \cdot 100 - 40^2} = \frac{350}{1400} = .25$$

$$\hat{V}(\hat{\delta}) = \frac{\hat{\sigma}^2 \cdot (50 + 100 - 2 \cdot 60)}{1400} = \frac{3}{140}$$

$$t = \frac{\hat{\delta}}{s.e.(\hat{\delta})} = \frac{.25}{\sqrt{3/140}} = \frac{.25}{.146385} = 1.70783$$

we get the same value as is expected.

$$Y - \overline{Y} = \beta_1 - \left(\overline{Y} - \beta_2\overline{X}_2 - \beta_3\overline{X}_3\right) + \beta_2\left(X_2 - \overline{X}_2\right) + \beta_3\left(X_3 - \overline{X}_3\right) + u$$

$$\frac{Y - \overline{Y}}{S_y} = \frac{\beta_1 - \left(\overline{Y} - \beta_2\overline{X}_2 - \beta_3\overline{X}_3\right)}{S_y} + \frac{\beta_2\left(X_2 - \overline{X}_2\right)}{S_y} + \frac{\beta_3\left(X_3 - \overline{X}_3\right)}{S_y} + \frac{u}{S_y}$$

e) $$Y^* = \frac{\beta_1 - \left(\overline{Y} - \beta_2\overline{X}_2 - \beta_3\overline{X}_3\right)}{S_y} + \frac{\beta_2 S_{x2}\left(X_2 - \overline{X}_2\right)}{S_y S_{x2}} + \frac{\beta_3 S_{x3}\left(X_3 - \overline{X}_3\right)}{S_y S_{x3}} + \frac{u}{S_y}$$

$$Y^* = \beta_1^* + \beta_2^* X_2^* + \beta_3^* X_3^* + \frac{u}{S_y}$$

$$\hat{\beta}_2^* = \frac{b_2 S_{x2}}{S_y}, \quad \hat{\beta}_3^* = \frac{b_3 S_{x3}}{S_y}, \quad \hat{\beta}_1^* = \frac{b_1 - \left(\overline{Y} - b_2\overline{X}_2 - b_3\overline{X}_3\right)}{S_y} = \frac{b_1 - b_1}{S_y} = 0$$

$$\beta_2^* = \frac{.5 \cdot \sqrt{50}}{\sqrt{20.75}} = .776151, \quad \beta_3^* = \frac{-.25 \cdot 10}{\sqrt{20.75}} = -.548821$$

$$V\left(\hat{\beta}_2^*\right) = \frac{50}{14 \cdot 20.75} = .172117, \quad V\left(\hat{\beta}_3^*\right) = \frac{100}{28 \cdot 20.75} = .172117$$

$$t_{\beta_2^*} = \frac{.776151}{\sqrt{.172117}} = 1.87083, \quad t_{\beta_3^*} = \frac{-.548821}{\sqrt{.172117}} = -1.32288$$

2. $Y = \beta_1 + \beta_2 X + u$

$$\overline{Y} = 10, \quad \overline{X} = 20, \quad \sum XY = 6000$$

$$\sum X^2 = 12{,}000, \quad \sum Y^2 = 4{,}000, \quad n = 25$$

$$\hat{\beta}_1 = \overline{Y} - \hat{\beta}_2\overline{X} = 10 - .5 \cdot 20 = 0$$

a) $$\hat{\beta}_2 = \frac{6000 - 25 \cdot 10 \cdot 20}{12000 - 25 \cdot 20^2} = \frac{1000}{2000} = .5$$

$$E(Y_0) = .5 \cdot 25 = 12.5$$

b) $$V\left[\hat{E}(Y_0)\right] = \frac{\hat{\sigma}^2}{25}\left[\begin{bmatrix}1 & 25\end{bmatrix}\begin{bmatrix}6 & -.25 \\ -.25 & 1/80\end{bmatrix}\begin{bmatrix}1 \\ 25\end{bmatrix}\right] = \frac{43.4783}{25} \cdot \frac{105}{80} = 2.28261$$

$$\hat{\sigma}^2 = \frac{1500 - .5 \cdot 1000}{23} = 43.4783$$

c) To compute the variance of $\hat{\theta}_0$, use the above formula but change X to (X-25) which is the new regressor in (5). $V\left(\hat{\theta}_0\right) = 43.4783 \cdot \frac{21}{400} = 2.28261$

d) There is more than one way to skin a cat. We can use brute force or a simple reparameterization to obtain the information that we want.

3.

	Number	Wage
Male	n_1	$\overline{Y}_1$
Female	n_2	$\overline{Y}_2$
Total	n	$\overline{Y}$

Note: $\overline{Y} = \left(n_1\overline{Y}_1 + n_2\overline{Y}_2\right)/n$

Model A: $Y = \beta_1 + \beta_2 D_F + u$ (1) where D_F is the female dummy variable.

Model B: $Y = \gamma_1 D_M + \gamma_2 D_F + v$ (2) where D_M is the male dummy and there is no intercept in (2).

a) First set up the X and then the $X'X$ matrix (2x2). Then solve for the intercept and the slope coefficients.

$$\hat{\beta}_1 = \overline{Y} - \hat{\beta}_2\overline{D}_F = \overline{Y} - \left(\overline{Y}_2 - \overline{Y}_1\right)\frac{n_2}{n} = \overline{Y}_1$$

$$\hat{\beta}_2 = \frac{n \cdot n_2(\overline{Y}_2 - \overline{Y})}{n \cdot n_2 - n_2^2} = \frac{n_1 \cdot n_2\left(\overline{Y}_2 - \overline{Y}_1\right)}{n_1 \cdot n_2} = \left(\overline{Y}_2 - \overline{Y}_1\right)$$

The slope coefficient is the difference, **on average**, between the wages of males and females. The intercept is just the **average** wage for a male, which is the best that we can say about a males wage if we know nothing else about him.

b) $$\begin{bmatrix} \hat{\gamma}_1 \\ \hat{\gamma}_2 \end{bmatrix} = \begin{bmatrix} 1/n_1 & 0 \\ 0 & 1/n_2 \end{bmatrix}^{-1} \begin{bmatrix} n_1\overline{Y}_1 \\ n_2\overline{Y}_2 \end{bmatrix} = \begin{bmatrix} \overline{Y}_1 \\ \overline{Y}_2 \end{bmatrix}$$

The male slope coefficient is the wage, **on average,** of being a male and the slope coefficient for the female is the wage, **on average**, of being a female compared to a male. The difference between the two represents the expected difference between wages of males and females.

c) The two models give the same results. In both models we say a male gets his average wage and that a female gets her average wage. They may look the same but they produce the same results.

d) The interpretation of the slope coefficient from (1) will not change although its value most likely will. That coefficient will still represent the average disparity between the wage paid to a man and paid to a woman.