Econometrics 1

Assignment # 1

1. If $X_1, X_2, \ldots, X_n$ are values of a random sample of size n from a population having the density function

$$f(x;\theta) = \begin{cases} \dfrac{2(\theta - x)}{\theta^2} & \text{for } 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

find the ML estimator of θ.

2. Given a random sample of size n from a population having the density function

$$f(x;\theta) = \begin{cases} (\theta+1)\,x^{\theta} & \text{for } 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

find the ML estimator of θ.

3. (This is a modification of Exercise 9.4 in your text.) Consider the joint distribution of two random variables, y, which is the number of failures of some component (disk drive) in a brand of computer per unit of time and x, the average lifetime of some different but related component (a chip). Note that y is a discrete random variable and x is a continuous random variable. Suppose that the conditional distribution of y is

$$f(y|x) = e^{-\beta x}(\beta x)^y / y!,\ y = 0,1,...,\ x \geq 0,\ \beta > 0,$$

while the marginal distribution of x is

$$f(x) = \theta e^{-\theta x},\ x \geq 0,\ \theta > 0.$$

Thus, conditioned on x, y has a Poisson distribution with parameter βx, while x, unconditionally, has an exponential distribution.

a. What is the joint distribution of these two random variables, $f(y,x)$? (Hint, Equation (3-61) is a fundamental result in distribution theory.)

b. Show that the unconditional density of y is $f(y) = \delta(1-\delta)^y$ where $\delta = \theta/(\beta+\theta)$ by using result (3-45).

c. Show that $E[x] = 1/\theta$ and $Var[x] = 1/\theta^2$. [Hint: See Section 3.4.5 of your text. (Note the typo in (3-39); $x \leq 0$ should be $x \geq 0$.) In case they are useful, gamma integrals which are used to obtain these results are discussed on pages 178-179 of your text.]

For a Poisson distribution with parameter α for some discrete random variable z, $f(z)=e^{-\alpha}\alpha^z/z!$, $E[z] = Var[z] = \alpha$. It follows then, that in our conditional distribution,

$$E[y|x] = Var[y|x] = \beta x.$$

Note that this "regression" model has a linear conditional mean function. You could obtain $E[y]$, $Var[y]$,and $Cov[y,x]$ from the marginal distribution $f(y)$ and the joint distribution $f(x,y)$ by summing and integrating using the definitions. But, there is a much easier way.

d. Using the fundamental results:

$$E[y] = E_x[E[y|x]],\quad Var[y] = E_x[Var[y|x]] + Var_x[E[y|x]],\ Cov[x,y] = Cov[x,E[y|x]],$$

show that $E[y] = \beta/\theta = \gamma$, $Var[y] = \beta/\theta + (\beta/\theta)^2 = \gamma(1+\gamma)$, and $Cov[x,y] = \beta/\theta^2 = \gamma/\theta$.

e. Now, we consider the conditional distribution of x. Show that the regression of x on y, $E[x|y]$, is given by the linear function

$$E[x|y] = \mu + \mu y \text{ where } \mu = 1/(\beta + \theta).$$

(note that the constant equals the slope). Hints: You have $f(y,x)$ and $f(y)$. Then, $f(x|y)$ is just $f(y,x)/f(y)$. In this ratio, cancel some terms, gather the remainder, and use the relationship between factorials and gamma integrals given on page 178. Now, show that the resulting conditional distribution is a gamma distribution using the results in Section 3.4.5. The mean of this distribution is the conditional mean you seek. This is also given in Section 3.4.5.

4. The *Poisson* distribution can be defined as the limit of a Binomial distribution as $n \to \infty$ and $\theta \to 0$ such that $n\theta = \lambda$ is a positive constant. For example, this could be the probability of a rare disease and we are random sampling a large number of inhabitants, or it could be the rare probability of finding oil and n is the large number of drilling sights. This discrete probability function is given by

$$f(X;\lambda) = \frac{e^{-\lambda}\lambda^X}{X!} \qquad X = 0,1,2,\ldots$$

For a random sample from this Poisson distribution

a. Show that $E(X) = \lambda$ and $\mathrm{var}(X) = \lambda$.

b. Show that the MLE of λ is $\hat{\lambda}_{mle} = \bar{X}$.

c. Show that the method of moments estimator of λ is also $\bar{X}$.

d. Show that $\bar{X}$ is unbiased [redacted] for λ.

[redacted]

e. Derive the Cramér-Rao lower bound for any unbiased estimator of λ. Show that $\bar{X}$ attains that bound.

5. The *Exponential* distribution is given by

$$f(X;\theta) = \frac{1}{\theta} e^{-X/\theta} \qquad X > 0 \text{ and } \theta > 0$$

This is a skewed and continuous distribution defined only over the positive quadrant.

a. Show that $E(X) = \theta$ and $\mathrm{var}(X) = \theta^2$.

b. Show that $\hat{\theta}_{mle} = \bar{X}$.

~~c. Show that the method of moments estimator of θ is also~~ $\bar{X}$.

c. Show that $\bar{X}$ is an unbiased [redacted] estimator of θ.

[redacted]

d. Derive the Cramér-Rao lower bound for any unbiased estimator of θ? Is $\bar{X}$ MVU for θ?