Spring 2007

1. No regression $\frac{R^2/(K-1)}{(1-R^2)/n-K} \sim F_{K-1,\, n-K}$.

(i)

Model 1: $\frac{.956/2}{(1-.956)/23} = 249.86$

Model 3: $\frac{.75/1}{(1-.75)/24} = 72$

(ii) $\frac{(e_1'e_1 - e'e)/\#\text{rest}}{e'e/\text{d.f}} \sim F_{\#\text{rest},\, \text{d.f.}}$

$\frac{(2.37-1.82)/1}{(1.82)/23} = 6.95$ Compare with $F_{1,23}$

(iii) $\frac{(R^2_{ur} - R^2_r)/\#\text{rest}}{(1-R^2_{ur})/\ \text{df}} = \frac{(.956-.94)/1}{(1-.956)/23} = 8.36$

Compare with $F_{1,23}$

Two tests should be identical. Something wrong with the numbers.
(rounding error)

$$\frac{e_1'e_1 - e'e}{e'e} = \frac{\frac{e_1'e_1}{y'y} - \frac{e'e}{y'y}}{e'e/y'y} = \frac{1-R^2_r - (1-R^2_{ur})}{1-R^2_{ur}} = \frac{R^2_{ur} - R^2_r}{1-R^2_{ur}}.$$

(iv) NO. The dependent variables in the restricted and unrestricted models should be the same. This is not the case in (3). That is, one cannot compare (3) with (1).
You can do a t-test $= \frac{b_2+b_3-1}{se(b_2+b_3)} = \frac{.76+.24-1}{\sqrt{V(b_2)+V(b_3)+2\,\text{cov}(b_2,b_3)}}$

$\text{Cov}(b_2,b_3)$ not given. But the numerator is zero. So the t value $=0$. Compare this with $t_{23,\alpha}$. will reject.

2. $\underset{2n\times 1}{y^*} = \begin{pmatrix} Y \\ -Y \end{pmatrix}$, $\underset{2n\times(K+1)}{X^*} = \begin{pmatrix} 1 & X \\ 1 & -X \end{pmatrix}$

(i)

$$\underset{(K+1)\times 1}{b^*} = \begin{pmatrix} b_0 \\ \underset{K\times 1}{b} \end{pmatrix} \text{ scalar}$$

$$b^* = \begin{pmatrix} b_0 \\ b \end{pmatrix} = (X^{*\prime}X^*)^{-1}X^{*\prime}Y^* = \begin{bmatrix} 2n & 0 \\ 0 & 2X'X \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 2X'Y \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2n} & 0 \\ 0 & \frac{1}{2}(X'X)^{-1} \end{bmatrix} \begin{pmatrix} 0 \\ 2X'Y \end{pmatrix} = \begin{pmatrix} 0 \\ (X'X)^{-1}X'Y \end{pmatrix} = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

$\Rightarrow$ The slope coes of the reg Y^* on X^* are nothing but the reg. coe of Y on X (without an intercept).

(ii)

$$\hat{\sigma}^2 = \frac{e^{*\prime}e^*}{2n-(K+1)} = \frac{2e'e}{2n-(K+1)} = 2\left(\frac{e'e}{n-K}\right)\cdot\frac{n-K}{2n-(K+1)}$$

$$\because \; e^* = Y^* - X^*b^* = \begin{bmatrix} Y \\ -Y \end{bmatrix} - \begin{bmatrix} Xb \\ -Xb \end{bmatrix} = \begin{pmatrix} e \\ -e \end{pmatrix}.$$

$$V(b) = \sigma^2 \cdot \frac{1}{2}(X'X)^{-1}$$

$$\Rightarrow \hat{V}(b) = \hat{\sigma}^2 \cdot \frac{1}{2}(X'X)^{-1} = s^2(X'X)^{-1}\frac{n-K}{2n-(K+1)} = V\left(\underset{\text{of } Y \text{ on } X}{\text{OLS-estimator}}\right)\cdot\frac{n-K}{2n-(K+1)}$$

3.

$$b_R = (X_1'X_1)^{-1}X_1'Y = (X_1'X_1)^{-1}X_1'(X_1b_1 + X_2b_2 + e)$$

$$= b_1 + Pb_2 + \underbrace{(X_1'X_1)^{-1}X_1'e}_{0}.$$

$$= b_1 + Pb_2.$$

direct effect

indirect effect through X_2

4.

(a) $Y = (\iota \; X)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + u$

$$b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{pmatrix} n & \iota' X \\ X'\iota & X'X \end{pmatrix}^{-1} \begin{pmatrix} \iota' Y \\ X' Y \end{pmatrix} = \begin{bmatrix} n & \Sigma X \\ \Sigma X & \Sigma X^2 \end{bmatrix}^{-1} \begin{bmatrix} \Sigma Y \\ \Sigma XY \end{bmatrix}$$

$$= \frac{1}{D}\begin{bmatrix} \Sigma X^2 & -\Sigma X \\ -\Sigma X & n \end{bmatrix} \begin{bmatrix} \Sigma Y \\ \Sigma XY \end{bmatrix} = \begin{bmatrix} \dfrac{\Sigma Y \cdot \Sigma X^2 - \Sigma X \, \Sigma XY}{D} \\ \dfrac{n \, \Sigma XY - \Sigma X \, \Sigma Y}{D} \end{bmatrix}$$

where $D = n\Sigma X^2 - \Sigma X \, \Sigma Y = n\Sigma x^2$

$$\Rightarrow \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \begin{bmatrix} \dfrac{\bar{Y}\Sigma X^2 - \bar{X}\,\Sigma XY}{\Sigma x^2} \\ \dfrac{\Sigma XY - n\bar{X}\bar{Y}}{\Sigma x^2} \end{bmatrix} = \begin{bmatrix} \dfrac{\bar{Y}(\Sigma x^2 + n\bar{x}^2) - \bar{x}(\Sigma xy + n\bar{x}\bar{Y})}{\Sigma x^2} \\ \dfrac{\Sigma xy}{\Sigma x^2} \end{bmatrix}$$

$$= \begin{pmatrix} \bar{Y} - \bar{X}\cdot b_2 \\ b_2 \end{pmatrix} \text{ where } b_2 = \frac{\Sigma xy}{\Sigma x^2}.$$

$$V\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \sigma^2 \begin{bmatrix} n & \Sigma X \\ \Sigma X & \Sigma X^2 \end{bmatrix}^{-1} = \sigma^2 \begin{bmatrix} \dfrac{\Sigma X^2}{D} & -\dfrac{\Sigma X}{D} \\ -\dfrac{\Sigma X}{D} & \dfrac{n}{D} \end{bmatrix}$$

$$\Rightarrow V(b_2) = \frac{\sigma^2}{\Sigma x^2}, \quad V(b_1) = \frac{\Sigma X^2}{n\Sigma x^2}, \quad \mathrm{Cor}(b_1, b_2) = \frac{-\bar{X}}{\Sigma x^2}.$$

(b) $X_0' = (1 \;\; c)$

$$\Rightarrow X_0'(X'X)^{-1}X_0 = (1 \;\; c)\begin{bmatrix} \dfrac{\Sigma X^2}{D} & -\dfrac{\Sigma X}{D} \\ -\dfrac{\Sigma X}{D} & \dfrac{n}{D} \end{bmatrix}\begin{pmatrix} 1 \\ c \end{pmatrix}$$

$$= \frac{\Sigma X^2}{D} - 2c\frac{\Sigma X}{D} + \frac{nc^2}{D} = \frac{\Sigma X^2}{n\Sigma x^2} - \frac{2c\bar{X}}{\Sigma x^2} + \frac{c^2}{\Sigma x^2}$$

$$= \frac{1}{n} + \frac{n\bar{X}^2}{n\Sigma x^2} - \frac{2c\bar{X}}{\Sigma x^2} + \frac{c^2}{\Sigma x^2} = \frac{1}{n} + \frac{(c-\bar{X})^2}{\Sigma x^2}.$$