Econ 616
HW#1
problem 1

assume IID sample and this density function

$$f(x;\theta) = \begin{cases} \frac{2(\theta - x)}{\theta^2} & 0 < x < \theta \\ 0 & \text{otherwise} \end{cases}$$

Likelihood function is

$$\mathbb{X} = (X_1 \dots X_n)$$

$$L(\theta;\mathbb{X}) = \prod_{k=1}^{n} \frac{2(\theta - X_k)}{\theta^2} = \frac{2^n}{\theta^{2n}} \prod_{k=1}^{n} (\theta - X_k)$$

log likelihood function is

$$\ln L(\theta;\mathbb{X}) = n\ln 2 - 2n\ln\theta + \sum_{k=1}^{n} \ln(\theta - X_k)$$

FOC

$$\frac{\partial \ln L(\cdot)}{\partial \theta} = \frac{-2n}{\theta} + \sum_{k=1}^{n} \frac{1}{\theta - X_k} = 0$$

$$\frac{2n}{\theta} = \sum_{k=1}^{n} \frac{1}{\theta - X_k}$$

- seems to have no analytical solution
- should be solved numerically
- SOC and other conditions should be checked afterwards $\frac{\partial^2 \ln L(\cdot)}{\partial \theta^2} < 0$

assume IID sample and this density function

$$f(x;\theta) = \begin{cases} (\theta+1)x^{\theta} & \text{for } 0<x<1 \\ 0 & \text{otherwise} \end{cases}$$

likelihood function is (where $\mathbb{X} = (X_1 \dots X_n)$)

$$L(\theta;\mathbb{X}) = \prod_{k=1}^{n} (\theta+1) X_k^{\theta} = (\theta+1)^n \prod_{k=1}^{n} X_k^{\theta}$$

obtain log likelihood function

$$\ln L(\theta;\mathbb{X}) = n \ln(\theta+1) + \theta \sum_{k=1}^{n} \ln X_k$$

FOC

$$\frac{\partial \ln L(\theta;\mathbb{X})}{\partial \theta} = \frac{n}{\theta+1} + \sum_{k=1}^{n} \ln X_k = 0$$

$$\frac{n}{\theta+1} = -\sum_{k=1}^{n} \ln X_k$$

$$\frac{\theta+1}{n} = -\frac{1}{\sum_{k=1}^{n} \ln X_k}$$

$$\hat{\theta} = -\frac{n}{\sum_{k=1}^{n} \ln X_k} - 1$$

before claiming that $\hat{\theta}$ is ML estimator Second order condition should be checked

SOC $\frac{\partial^2 \ln L(\theta; X)}{\partial \theta^2} < 0$ for maximum

$$\frac{\partial^2 \ln L(\theta; X)}{\partial \theta^2} = n(-1)(\theta+1)^{-2} = \frac{-n}{(\theta+1)^2} < 0$$

thus $\hat{\theta} = \hat{\theta}_{ML}$

problem3

(a) $f(x;y) = f(y|x) \cdot f(x) = \frac{e^{-\beta x}(\beta x)^y}{y!} \cdot \theta e^{-\theta x} = \frac{e^{-\beta x - \theta x} \cdot (\beta x)^y}{y!}$

$$= \frac{e^{-(\beta+\theta)x} \cdot (\beta x)^y \cdot \theta}{y!}$$

(b) $f(y) = \int_0^\infty f(x;y)\,dx = \int_0^\infty \frac{e^{-(\beta+\theta)x}(\beta x)^y \theta}{y!}\,dx = \infty.$

now check out section 5.4.2b. in Greene's text book (p. 178)

the Gamma function: $\Gamma(p) = \int_0^\infty t^{p-1} e^{-t}\,dt$ (*)

$p > 0$ and $\Gamma(p) = (p-1)!$ if p is integer (**)

lets make use of the above formulas

set $c = \beta + \theta \Rightarrow \frac{\theta \beta^y}{y! c^y} \int_0^\infty x^y e^{-cx}\,dx$

$$\Rightarrow \frac{\theta \beta^y}{y!\, c^y} \int_0^\infty x^y c^y e^{-cx}\,dx = \frac{\theta \beta^y}{y!\, c^y} \int_0^\infty (cx)^y e^{-cx}\,dx = \dots$$

now set $cx = t$ and thus $\frac{dt}{c} = dx$

$$\ldots = \frac{\theta \beta^y}{y! \, c^y \cdot c} \int_0^\infty t^y \cdot e^{-t} \, dt$$ ... now notice resemblance of this integral and the integral marked $(*)$ and $p-1 = y \Rightarrow p = (y+1)$

$$\ldots = \frac{\theta \beta^y}{y! \, c^{y+1}} \cdot \Gamma(y+1) = \ldots$$ apply the equation marked $(**)$

$$\ldots = \frac{\theta \beta^y}{y! \, c^{y+1}} \cdot y! = \frac{\theta \beta^y}{c^{y+1}} = \frac{\theta \beta^y}{(\theta+\beta)^y \cdot (\theta+\beta)}$$

$$= \left[\frac{\theta}{\theta+\beta}\right]\left[\frac{\beta}{\beta+\theta}\right]^y$$

take a rest now - you deserve it...

(c) $E(X) = \int x f(x) dx$

$E(X) = \int_0^\infty x \theta e^{-\theta x} dx$ this integral (I think)

can be solved by parts:

define $u = \theta x$ and $v = e^{-\theta x}/\theta$

$\frac{du}{dx} = \theta$ $dv = e^{-\theta x} dx$

$du = \theta dx$ and

$uv - \int dv$

$-xe^{-\theta x} - \int_0^\infty -e^{-\theta x} dx =$ where $u = -\theta x$ $\frac{du}{-\theta} = \frac{-\theta dx}{-\theta}$

$= -xe^{-\theta x}\Big|_0^\infty - \frac{1}{\theta}\int e^{u} du =$

$= -xe^{-\theta x}\Big|_0^\infty - \frac{1}{\theta} e^{-\theta x}\Big|_0^\infty = (0-0) -$

$-(0 - 1/\theta) = 1/\theta = E(X)$

$Var(X) = E(X^2) - E(X)^2 = E(X^2) - \left(\frac{1}{\theta}\right)^2$

$E(X^2) = \int_0^\infty x^2 \theta e^{-\theta x} dx$

integration by parts: $u = x^2\theta$ $dv = e^{-\theta x} dx$

$du = 2\theta x dx$ $v = \frac{-e^{-\theta x}}{\theta}$

$$= -x^2 e^{-\theta x} - \int_0^\infty -2x e^{-\theta x}\,dx + \frac{2}{\theta}\int_0^\infty x\theta e^{-\theta x}\,dx =$$

$$= \frac{2}{\theta}E(X) - x^2 e^{-\theta x}\Big|_0^\infty =$$

$$= \frac{2}{\theta}\cdot\frac{1}{\theta} - 0 = \frac{2}{\theta^2}$$

$$Var(X) = \frac{2}{\theta^2} - \frac{1}{\theta^2} = \frac{1}{\theta^2}$$

(d) $E(Y) = E_X[E(Y|X)] = \ldots$

$E(Y|X) = \beta X$ (see 3c)

$\ldots = E_X(\beta X) = \beta E_X(X) = \beta\cdot\frac{1}{\theta} = \frac{\beta}{\theta}$

$Var(Y) = E_X[Var(Y|X)] + Var_X[E(Y|X)] = \ldots$

$Var(Y|X) = E(Y|X) = \beta X$ (see 3c)

$\ldots = E_X(\beta X) + Var_X(\beta X) = \beta E_X(X) + \beta^2 Var_X(X) =$

$= \frac{\beta}{\theta} + \beta^2\frac{1}{\theta^2} = \frac{\beta}{\theta}\left(1 + \frac{\beta}{\theta}\right) = \ldots$

$\gamma = \beta/\theta$

$\ldots = \gamma(1+\gamma)$

$$\mathrm{Cov}(X;Y) = \mathrm{Cov}(X; E(Y|X)) = \ldots$$

$$E(Y|X) = \beta$$

$$\ldots = \mathrm{Cov}(X;\beta X) = E(XY) - E(X)\cdot E(Y) =$$

$$= E(\beta X^2) - E(X)\cdot E(\beta X) = \beta E(X^2) - \frac{\beta}{\theta^2} =$$

$$= \frac{2\beta}{\theta^2} - \frac{\beta}{\theta^2} = \beta/\theta^2 = \ldots$$

$$\gamma = \beta/\theta$$

$$\ldots = \frac{\gamma}{\theta}$$

3e) $$f(X|Y) = \frac{f(Y|X)\cdot f(X)}{f(X;Y)} = \frac{e^{-\beta X}(\beta X)^Y \theta e^{-\theta X}}{Y!\left(\frac{\theta}{\theta+\beta}\right)\left(\frac{\beta}{\theta+\beta}\right)^Y} =$$

$$= \frac{\theta e^{-X(\theta+\beta)}(X\beta)^Y}{Y!\left(\frac{\theta}{\theta+\beta}\right)\left(\frac{\beta}{\theta+\beta}\right)^Y} = \ldots$$

set $C = \theta+\beta$

$$\ldots = \frac{\theta e^{-XC}(\beta X)^Y}{Y!\left(\frac{\theta}{C}\right)\left(\frac{\beta}{C}\right)^Y} = \frac{e^{-XC}(X)^Y}{Y!\left(\frac{1}{C}\right)\left(\frac{1}{C}\right)^Y} =$$

$$= \frac{e^{-xc} x^{y} c^{y+1}}{y!}$$

$$E(X|Y) = \int_0^\infty x \cdot f(x|y)\, dx = \ldots$$

set $t = xc \Rightarrow \frac{dt}{dx} = c \Rightarrow dx = \frac{1}{c} dt$

$$\ldots = \int_0^\infty \frac{xc \cdot t^{y} \cdot e^{-t}}{Y!} dx = \int_0^\infty \frac{t^{Y+1} \cdot e^{-t}}{Y!} dx =$$

$$= \frac{1}{c \cdot Y!} \int_0^\infty t^{Y+1} \cdot e^{t} dt = \ldots$$

go back to page 178 in Greene or 36
the integral became a Gamma function

$$\ldots = \frac{1}{c \cdot Y!} \cdot \Gamma(Y+2) = \frac{1}{c \cdot Y!} \cdot (y+1)! =$$

because of $Y+1 = P-1$

$$= \frac{1}{c \cdot Y!} \cdot Y!(y+1) = \frac{Y+1}{c} = \frac{Y+1}{\theta+\beta} =$$

$$= \frac{Y}{\theta+\beta} + \frac{1}{\theta+\beta} = \ldots$$

set $\mu = \frac{1}{\theta+\beta}$

$$\ldots = Y\mu + \mu.$$

problem 4

(a) X is a discrete random variable

$$E(X)=\sum_{k=1}^{n} x_k f_X(x)=\sum_{k=1}^{n} x_k \frac{e^{-\lambda}\cdot\lambda^{x_k}}{x_k!}=$$

$$=\sum_{k=1}^{n}\frac{x_k e^{-\lambda}\lambda^{x_k}}{x_k(x_k-1)!}=\sum_{k=1}^{n}\frac{e^{-\lambda}\lambda^{(x_k-1)}\cdot\lambda}{(x_k-1)!}=$$

$$=\lambda\sum_{k=1}^{n}\frac{e^{-\lambda}\lambda^{(x_k-1)}}{(x_k-1)!}=\lambda\cdot 1=\lambda$$

$$Var(X)=E(X^2)-E(X)^2$$

lets deal with $E(X^2)$ first

$$E(X^2)=\sum_{k=1}^{n} x_k^2\cdot\frac{e^{-\lambda}\lambda^{x_k}}{x_k!}=\sum_{k=1}^{n}\frac{x_k e^{-\lambda}\cdot\lambda^{x_k-1}\cdot\lambda}{(x_k-1)!}=$$

$$=\lambda\sum_{k=1}^{n}\frac{(x_k-1+1)e^{-\lambda}\lambda^{x_k-1}}{(x_k-1)!}=\lambda\left(1+\sum_{k=1}^{n}\frac{e^{-\lambda}\lambda^{x_k-2}\cdot\lambda}{(x_k-2)!}\right)=$$

$$=\lambda[1+\lambda]=\lambda+\lambda^2$$

$$Var(X)=\lambda^2+\lambda-(\lambda)^2=\lambda$$

(4b) $f(x;\lambda) = \frac{e^{-\lambda}\lambda^x}{x!}$ given density

assume random sample $\mathbb{X} = (X_1 \ldots X_n)$

likelihood is $L(\lambda;\mathbb{X}) = \prod_{k=1}^{n} \frac{e^{-\lambda}\lambda^{x_k}}{x_k!} =$

$$= \frac{e^{-\lambda n} \cdot \prod_{k=1}^{n} \lambda^{x_k}}{\prod_{k=1}^{n} x_k!}$$

log of likelihood function is

$$\ln L(\lambda;\mathbb{X}) = -n\lambda + \sum_{k=1}^{n} x_k \ln\lambda - \sum_{k=1}^{n} \ln(x_k!)$$

FOC $\frac{\partial \ln L(\cdot)}{\partial \lambda} = 0$ gives

$$-n + \frac{\sum_{k=1}^{n} x_k}{\lambda} = 0$$

$$\hat{\lambda} = \frac{1}{n}\left(\sum_{k=1}^{n} x_k\right) = \bar{x}$$

SOC $\frac{\partial^2 \ln L(\cdot)}{\partial \lambda^2} = \frac{(-1)\left(\sum_{k=1}^{n} x_k\right)}{\lambda^2} < 0$

thus $\hat{\lambda} = \hat{\lambda}_{ML} = \bar{x}$

(4c) $E(X) = \lambda$ apply moment matching principle $\Rightarrow \frac{1}{n}\sum_{k=1}^{n} X_k = \hat{\lambda}_{MM}$

(4d) $E(\hat{\lambda}) = E\left(\frac{1}{n}\sum_{k=1}^{n} X_k\right) = \frac{1}{n}\sum_{k=1}^{n} E(X_k) =$

$= \frac{1}{n}\sum_{k=1}^{n} \lambda = \frac{1}{n}\cdot n \cdot \lambda = \lambda$

thus unbiased

(4e) $CRLB = [I(\lambda)]^{-1} = \left[-E\left(\frac{\partial^2 \ln L(\lambda)}{\partial \lambda^2}\right)\right]^{-1}$

from SOC in 4b:

$CRLB = \left[-E\left(-\frac{\sum_{k=1}^{n} x_k}{\lambda^2}\right)\right]^{-1} = \left[+\frac{n\cdot\lambda}{\lambda^2}\right]^{-1} = \frac{\lambda}{n}$

and

$Var(\hat{\lambda}) = Var(\bar{X}) = \frac{1}{n^2}\cdot n\lambda = \lambda$

thus $Var(\hat{\lambda}) = CRLB$: (efficient)

problem 5

(a) $E(X) = \int_0^\infty x\cdot f(x)dx = \int_0^\infty x\frac{1}{\theta}\cdot e^{-\frac{x}{\theta}}dx = \ldots$

integration by parts:

$u = x/\theta \qquad dv = e^{-x/\theta}dx$

$du = \frac{1}{\theta}dx \qquad v = -\theta e^{-x/\theta}$

$$\circ\circ\circ = -xe^{-x/\theta} - \int_0^\infty -e^{-x/\theta}dx = -xe^{-x/\theta} - \int_0^\infty \theta e^{-u} du =$$

$$= -xe^{-x/\theta} - \theta\int_0^\infty e^{-u} du = -xe^{-x/\theta} - \theta e^{-x/\theta}\Big|_0^\infty = \theta$$

$$Var(x) = E(x^2) - [E(x)]^2$$

lets concentrate on $E(x^2)$

$$E(x^2) = \int_0^\infty x^2 f(x)dx = \circ\circ\circ$$

again integration by parts

$$u = x^2/\theta \qquad dv = e^{-x/\theta}dx$$

$$du = \frac{2x}{\theta}dx \qquad v = -\theta e^{-x/\theta}$$

$$\circ\circ\circ = -x^2 e^{-x/\theta} + 2\theta\int_0^\infty \frac{xe^{-x/\theta}}{\theta}dx =$$

$$= -x^2 e^{-x/\theta} + 2\theta\left(\frac{xe^{-x/\theta}}{\theta}\Big|_0^\infty\right) =$$

$$= -x^2 e^{-x\theta} + x^2 e^{-x\theta} + 2\theta^2 = 2\theta^2$$

$$Var(x) = 2\theta^2 - (\theta)^2 = \theta^2$$

(5b) likelihood function assuming a random sample is: $X = (X_1 \dots X_n)$

$$L(\theta; X) = \prod_{k=1}^{n} \frac{1}{\theta} e^{-X_k/\theta} = \left(\frac{1}{\theta}\right)^n e^{-1/\theta \sum_{k=1}^{n} X_k}$$

$$\ln L(\theta; X) = -n \ln(\theta) - \frac{1}{\theta} \sum_{k=1}^{n} X_k$$

FOC gives: $\frac{-n}{\theta} + \frac{1}{\theta^2} \sum_{k=1}^{n} X_k = 0$

$$\Rightarrow \hat{\theta} = \frac{\sum_{k=1}^{n} X_k}{n} = \overline{X}$$

SOC gives: $\frac{\partial^2 \ln L(\cdot)}{\partial \lambda^2} = \frac{n}{\theta^2} - \frac{2\sum_{k=1}^{n} X_k}{\theta^3} =$

$$= \frac{1}{\theta^2}\underbrace{\left(n - \frac{2\sum_{k=1}^{n} X_k}{\theta}\right)}_{<0} < 0$$

thus $\hat{\theta} = \hat{\theta}_{ML} = \overline{X}$

(5c) $E(\hat{\theta}) = E\left(\frac{1}{n} \sum_{k=1}^{n} X_k\right) = \frac{1}{n} \sum_{k=1}^{n} E(X_k) = \frac{1}{n} \cdot n \cdot \theta = \theta$

unbiased

5d $\quad CRLB = [I(\theta)]^{-1} = \left[-E\left(\frac{\partial^2 \ln L(\theta;x)}{\partial \theta^2}\right)\right]^{-1} = \ldots$

$\ldots$ from sol of 5b

$$\ldots = \left[-E\left(\frac{n}{\theta^2} - \frac{2}{\theta^3}\sum_{k=1}^{n} x_k\right)\right]^{-1} = \left[-\frac{n}{\theta^2} + \frac{2}{\theta^3} \cdot n\theta\right]^{-1} =$$

$$= \left[\frac{n}{\theta^2}\right]^{-1} = \frac{\theta^2}{n}$$

$$Var(\hat{\theta}) = Var(\bar{X}) = \frac{\theta^2}{n}$$

$Var(\hat{\theta}) = CRLB$ ∴ thus $\hat{\theta}$ is minimum variance estimator.