Econ616 HW#2

before you proceed with studying this answerkey check pages p32 (matrix inverses) and pages 231-233

problem1

model1: $Y = X\beta + \varepsilon \Rightarrow \hat\beta = (X'X)^{-1}X'Y \Rightarrow Var(\hat\beta) = Var((X'X)^{-1}X'Y) =$
$= \sigma^2 (X'X)^{-1}$ this is a matrix

model2: $Y = (X \;\; Z)\begin{pmatrix}\beta_X \\ \beta_Z\end{pmatrix} + \varepsilon \Rightarrow \hat\beta = \begin{pmatrix}\hat\beta_X \\ \hat\beta_Z\end{pmatrix} = \begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z\end{bmatrix}^{-1}\begin{pmatrix}X'Y \\ Z'Y\end{pmatrix} \Rightarrow$

$\Rightarrow Var(\hat\beta) = Var\begin{pmatrix}\hat\beta_X \\ \hat\beta_Z\end{pmatrix} = \sigma^2 \begin{bmatrix} X'X & X'Z \\ Z'X & Z'Z\end{bmatrix}^{-1}$

consider now only $\hat\beta_X = (X'M_Z X)^{-1} X'M_Z Y$

where M_Z is a "residual maker" $M_Z = (I - Z(Z'Z)^{-1}Z')$

$Var(\hat\beta_X) = Var((X'M_Z X)^{-1}X'M_Z Y) = \sigma^2 (X'M_Z X)^{-1}$

this is a matrix too

- assume that variance of ε in both models is the same
- If $A - B$ is PSD matrix then $A^{-1} - B^{-1}$ is NSD matrix

$\Rightarrow [Var(\hat\beta)]^{-1} - [Var(\hat\beta_X)]^{-1} = 1/\sigma^2 \, Z(Z'Z)^{-1}Z'$

$1/\sigma^2 > 0$ and $q'Z(Z'Z)^{-1}Z'q$ is a quadratic form (q is arbitrarily choosen vector), is positive

thus $Z(Z'Z)^{-1}Z'$ is PSD.

thus $Var(\hat\beta_X) - Var(\hat\beta)$ is PSD.

problem 2

model 1: $Y = X\beta + \varepsilon \Rightarrow$ "a residual maker" $M_X = I - X(X'X)^{-1}X'$

$\Rightarrow$ vector of residuals $e_X = M_X Y$

model 2: $Y = Z\gamma + \varepsilon \Rightarrow$ "a residual maker" $M_Z = I - Z(Z'Z)^{-1}Z'$

$\Rightarrow$ vector of residuals $e_Z = M_Z Y$

$Z = XP \rightarrow$ substitute this into e_Z

$$e_Z = M_Z Y = \left(I - (XP)((XP)'(XP))^{-1}(XP)'\right)Y =$$

$$= \left(I - (XP)(P'X'XP)^{-1}(P'X')\right)Y = \left(I - XPP^{-1}(X'X)^{-1}(P')^{-1}P'X'\right)Y$$

$$= (I - X(X'X)^{-1}X')Y = e_X$$

conclusion: changing units of measurement of X has no effect on the fit of the model.

problem 3

model $Y = X_1\beta_1 + X_2\beta_2 + \varepsilon$ where $E(\varepsilon) = X_1\gamma$

define a new error term $V = \varepsilon - E(\varepsilon) \Rightarrow \varepsilon = V + E(\varepsilon)$

V is uncorrelated with $(X_1\ X_2)$

$$Y = X_1\beta_1 + X_2\beta_2 + V + E(\varepsilon) = X_1\beta_1 + X_2\beta_2 + V + X_1\gamma =$$

$$= X_1(\beta_1 + \gamma) + X_2\beta_2 + V = X\beta + V \text{ where}$$

$\beta = \begin{pmatrix} \beta_1 + \gamma \\ \beta_2 \end{pmatrix}$ $X = (X_1\ X_2)$ to make long story short as

$$E(\hat\beta) = E\begin{pmatrix} \widehat{(\beta_1+\gamma)} \\ \hat\beta_2 \end{pmatrix} \neq \begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}$$

problem 4

model $Y = X\beta + \varepsilon$ where $X = [i \; X_*]$ ↳ in deviation form $X_* = M_0 Z$

$$\hat{\beta} = \begin{pmatrix} \hat{\beta}_i \\ \hat{\beta}_{X_*} \end{pmatrix} = \begin{bmatrix} i'i & i'X_* \\ X_*'i & X_*'X_* \end{bmatrix}^{-1} \begin{pmatrix} i'Y \\ X_*'Y \end{pmatrix}$$

- lets consider $\hat{\beta}_X$ first: $\hat{\beta}_X = (X_*' M_0 X_*)^{-1} X_*' M_0 Y$

where $M_0 = I - i(i'i)^{-1}i'$ is symmetric and idempotent ⇒

$$\hat{\beta}_{X_*} = \left((X_*'M_0')(M_0 X_*)\right)^{-1}(X_*'M_0'M_0 Y) = \left[((M_0 Z)'M_0')(M_0 M_0 Z)\right]^{-1} \times$$

$$\times \left((M_0 Z)'M_0'M_0 Y\right) = [\underbrace{Z'M_0'M_0'M_0 M_0}_{\text{put together}} Z]^{-1} \times (\underbrace{Z'M_0'M_0'M_0}_{\text{put together}} Y) =$$

$$= \left((Z'M_0')(M_0 Z)\right)^{-1}\left((Z'M_0')Y\right) =$$

$$= (X_*'X_*)^{-1}(X_*'Y)$$

- now consider $\hat{\beta}_i$ first of the normal equations (6-24) in Greene gives:

$$\hat{\beta}_i = (i'i)^{-1}i'Y - \underbrace{(i'i)^{-1}i'}_{\text{"an average taker"}} \overbrace{X_*}^{\text{already in deviation form}} \hat{\beta}_2 \quad \dots$$

intuitively taking an average of something already in mean deviation form produces zero

consider the term $i'X_* = i'(I - i(i'i)^{-1}i')Z =$

$= i'IZ - i'i(i'i)^{-1}i'Z = i'Z - i'Z = 0$ matrix

$$\dots = (i'i)^{-1}i'Y - 0 = (i'i)^{-1}i'Y = \bar{Y}$$

problem 5 model $Y = (X_1\ X_2)\begin{pmatrix}\beta_1\\ \beta_2\end{pmatrix} + u$

(a) from $X'X$ follows that X_1 and X_2 are not correlated
apply normal equations (6-24 for example)

(i) $\hat{\beta} = \begin{pmatrix}\hat{\beta}_1\\ \hat{\beta}_2\end{pmatrix} = \begin{bmatrix} X_1'X_1 & 0 \\ 0 & X_2'X_2 \end{bmatrix}^{-1}\begin{pmatrix}X_1'y\\ X_2'y\end{pmatrix} \Rightarrow \hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y - (X_1'X_1)^{-1}X_1'X_2\hat{\beta}_2 =$

$= (X_1'X_1)^{-1}X_1'y - 0 = (X_1'X_1)^{-1}X_1'y$

by analogy $\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'y - (X_2'X_2)^{-1}X_2'X_1\hat{\beta}_1 =$

$= (X_2'X_2)^{-1}X_2'y - 0 = (X_2'X_2)^{-1}X_2y$

(ii) model1 $Y = X_1\beta_1 + \varepsilon \Rightarrow \tilde{\varepsilon} = M_{X_1}y$ where
M_{X_1} is a residual maker $M_{X_1} = (I - X_1(X_1'X_1)^{-1}X_1')$

model2 $\tilde{e} = X_2\beta_2 + v$

$\hat{\beta}_2 = (X_2'X_2)^{-1}X_2'\tilde{e} = (X_2'X_2)^{-1}X_2'(I - X_1(X_1'X_1)^{-1}X_1')y$

open brackets ...

$= (X_2'X_2)^{-1}X_2'y - (X_2'X_2)^{-1}\underbrace{X_2'X_1}_{=0}(X_1'X_1)^{-1}X_1'y = (X_2'X_2)^{-1}X_2'y - 0 =$

$= (X_2'X_2)^{-1}X_2'y$ as before in (i)

also from model1 $\hat{\beta}_1 = (X_1'X_1)^{-1}X_1'y$ as before in (i)

the results are the same because X_1 and X_2 are not correlated, otherwise since $X_1'X_2$ and $X_2'X_1$ will no longer be zero matrices this result doesn't hold.

(b+c) yet again the same result as in (a) part (ii).

(d) model 1 $Y = X\beta_n + \varepsilon$

model 2 $Y = \begin{pmatrix} X \\ x \end{pmatrix} \beta_{n+1} + \varepsilon$

$x \sim$ row of observations

go back to chapter 2 in Greene ...

$Var(\hat{\beta}_n) = \sigma^2 (X'X)^{-1}$

$Var(\hat{\beta}_{n+1}) = \sigma^2 (X'X + xx')^{-1}$

- assume the same variances of error terms
- if $A - B$ is PSD then $A^{-1} - B^{-1}$ is NSD

$\left[Var(\hat{\beta}_{n+1})\right]^{-1} - \left[Var(\hat{\beta}_n)\right]^{-1} = 1/\sigma^2 (X'X + xx') - 1/\sigma^2 (X'X) =$

$= 1/\sigma^2 xx'$ $\quad 1/\sigma^2 > 0$ and xx' is

$(1 \times K)(K \times 1) = (1 \times 1) \Rightarrow$ a number positive

thus $Var(\hat{\beta}_{n+1}) - Var(\hat{\beta}_n)$ is NSD

conclusion additional row of observations reduces variance

problem 6

model $Y = i\beta_i + X\beta + \varepsilon$

$\hat{\beta} = (X' M_0 X)^{-1} X' M_0 Y$

where $M_0 = (I - i(i'i)^{-1} i')$ symmetric/idemp.

$= ((M_0X)'(M_0X))^{-1} (M_0X)' M_0 Y \Rightarrow$ now X and y are in deviation form

$$= (x'x)^{-1} x'y$$

which is OLS estimator from regression of y on x (ie Y and X in mean deviation form)

get yourself a cake if you got that far...

Econ616
Homework #2
Correction to the answer key

Question 5d

Model 1 $y=X\beta_n+\varepsilon$

where X is nxk

$b_n=(X'X)^{-1}X'y$

$Var(b_n)=\sigma^2(X''X)^{-1}$

Model 2 $y=X^*\beta_{n+1}+\varepsilon$

where X* is (n+1)xk

$X^*=\begin{bmatrix} X \\ x \end{bmatrix}$ so that $X*X^*=\begin{bmatrix} X & x \end{bmatrix}\cdot\begin{bmatrix} X \\ x \end{bmatrix}=(X'X+x'x)$

note that x'x is a matrix of the dimension kxk – the same dimension as X'X

$b_{n+1}=(X^{*\prime}X^*)^{-1}X^{*\prime}y$

$Var(b_{n+1})=\sigma^2(X^{*\prime\prime}X^*)^{-1}=\sigma^2(X'X+x'x)^{-1}$

Solution

Assume that the variances of the error terms are the same

$Var(b_{n+1})^{-1}-Var(b_n)^{-1}=1/\sigma^2[(X'X+x'x)-(X'X)]=1/\sigma^2[x'x]$

$1/\sigma^2>0$ and q'x'xq>0 since it is a quadratic form (q is a vector)

thus the difference is PSD

then it must be that $Var(b_{n+1})-Var(b_n)$ is NSD.