Economics 616-Exam I
Spring 2003

1. Based on 29 observations a researcher obtains the following X'X matrix:

$$X'X = \begin{bmatrix} 29 & 0 & 0 \\ 0 & 50 & 10 \\ 0 & 10 & 80 \end{bmatrix}$$

a) Find $(X'X)^{-1}$

Noticing that the matrix is block diagonal we can partition it and use the inverse formula so that:

$$(X'X)^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix} = \begin{bmatrix} 1/29 & 0 & 0 \\ 0 & 4/195 & -1/390 \\ 0 & -1/390 & 1/78 \end{bmatrix}$$

$$A = [29] \qquad B = \begin{bmatrix} 50 & 10 \\ 10 & 80 \end{bmatrix}$$

b) if $b = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 4 \\ .4 \\ .9 \end{bmatrix}$, show that $X'Y = \begin{bmatrix} 116 \\ 29 \\ 76 \end{bmatrix}$ and ESS is equal to 80.

From OLS we know that $b = (X'X)^{-1}X'Y$ so $(X'X)\ b = X'Y$. Therefore

$$\begin{bmatrix} 29 & 0 & 0 \\ 0 & 50 & 10 \\ 0 & 10 & 80 \end{bmatrix} \begin{bmatrix} 4 \\ .4 \\ .9 \end{bmatrix} = \begin{bmatrix} 116 \\ 29 \\ 76 \end{bmatrix}$$

$$ESS = B_2'X_2'y = b_2 \sum x_2 y + b_3 \sum x_3 y = .4 * 29 + .9 * 76 = 80$$

c) If RSS = 520 find R^2 and the estimated variance covariance matrix for $\begin{bmatrix} b_2 \\ b_3 \end{bmatrix}$.

$$TSS = RSS + ESS = 520 + 80 = 600, \quad R^2 = {ESS}/{TSS} = 80/600 = .133\overline{3}$$

$$Var\begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = s^2 (X_2 ' X_2)^{-1}, \quad s^2 = \frac{RSS}{n-k} = \frac{520}{26} = 20$$

$$Var\begin{pmatrix} b_2 \\ b_3 \end{pmatrix} = 20\begin{bmatrix} 4/195 & -1/390 \\ -1/390 & 1/78 \end{bmatrix}$$

d) Using the above information, obtain the restricted least squares estimator b^* when $\beta_2 + \beta_3 = 1$.

$$b^* = b + (X'X)^{-1} R'\left[R(X'X)^{-1} R'\right]^{-1}(r - Rb)$$

$$\begin{bmatrix} b_1^* \\ b_2^* \\ b_3^* \end{bmatrix} = \begin{bmatrix} 4 \\ .4 \\ .9 \end{bmatrix} + \begin{bmatrix} 0 \\ .017949 \\ .010256 \end{bmatrix}[.028205]^{-1}[-.3] = \begin{bmatrix} 4 \\ .4 \\ .9 \end{bmatrix} + \begin{bmatrix} 0 \\ .017949 \\ .010256 \end{bmatrix}[-10.6364] = \begin{bmatrix} 4 \\ .4 \\ .9 \end{bmatrix} + \begin{bmatrix} 0 \\ -.190912 \\ -.109087 \end{bmatrix} = \begin{bmatrix} 4 \\ .209088 \\ .790913 \end{bmatrix}$$

e) Using the information above, test the hypothesis that $2\beta_2 + \beta_3 = 2$ at the 5% level of significance.

$$\frac{(Rb - r)'\left[R(X'X)^{-1} R'\right]^{-1}(Rb - r)/q}{e'e/(n-k)} \sim F(q, n-k)$$

$$\frac{(2b_2 + b_3 - 2)\left[\begin{bmatrix} 0 & 2 & 1 \end{bmatrix}\begin{bmatrix} 1/29 & 0 & 0 \\ 0 & 4/195 & -1/390 \\ 0 & -1/390 & 1/78 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}\right]^{-1}(2b_2 + b_3 - 2)/1}{520/26} = \frac{(-.3)[.084615]^{-1}(-.3)/1}{20}$$

$\frac{1.06364}{20} = .050879 \sim F(2{,}26) = 3.35$ so we fail to reject the null hypothesis.

f) Given $X_0' = \begin{bmatrix} 1 & -1 & 1 \end{bmatrix}$ predict E(Y) and obtain a 95% confidence interval for Y.

$$\hat{Y}_f \pm t_{0.025} s\sqrt{R(X'X)^{-1} R'} \Rightarrow 4.5 \pm 2.052\sqrt{20}\sqrt{.072944} \Rightarrow (5.70784, 3.29216)$$

2. Consider the simple regression model $Y = \beta_0 + \beta_1 X + u$ which satisfies all the assumptions of the classical linear regression model.

a) Show that the t test of the hypothesis that there is no regression can be expressed as $t = \sqrt{(n-2)R^2 / (1 - R^2)}$.

The t statistic for a single variable regression is

$$t = \frac{b-\beta}{s \big/ \sqrt{\sum x^2}} = \sqrt{\frac{(b-\beta)^2}{s^2 \big/ \sum x^2}} = \sqrt{\frac{b^2 \sum x^2}{s^2}} = \sqrt{\frac{ESS(n-2)}{RSS}} = \sqrt{\frac{ESS/TSS(n-2)}{RSS/TSS}} = \sqrt{\frac{(n-2)R^2}{(1-R^2)}}$$

b) What is the F statistic to test the same hypothesis? Do you get the same expression to test the hypothesis of no regression in the K regressor case?

Using the well know identity $F(1,v) = t^2(v)$ so the resulting test statistic is $F(1,n-2) = \frac{(n-2)R^2}{1-R^2}$ and this is not the same expression when we test the hypothesis of no regression in the K regressor case because the numerator is no longer divided by 1, in that case we would need to divide the numerator by the number of regressors in our linear regression model.

c) Researcher A measured both X and Y in kilograms, whereas researcher B measured both of these variables in metric tons (1ton=1000 kgs).

A: $Y = \beta_0 + \beta_1 X + u$

B: $Y^* = \beta_0 + \beta_1 X^* + u$

i) $$b_1^* = \frac{\sum x^* y^*}{\sum (x^*)^2} = \frac{\sum \frac{1}{1000} x \frac{1}{1000} y}{\sum \left(\frac{1}{1000}\right)^2 x^2} = \frac{\left(\frac{1}{1000}\right)^2 \sum xy}{\left(\frac{1}{1000}\right)^2 \sum x^2} = b_1$$

ii) $se(b_1^*) = se(b_1)$ therefore the t - statistics are equivalent

iii) $$R^{*2} = \frac{b_1^* \sum x^* y^*}{\sum (y^*)^2} = \frac{b_1 \sum \left(\frac{1}{1000}\right) x \left(\frac{1}{1000}\right) y}{\sum \left(\frac{1}{1000}\right)^2 y^2} = \frac{b_1 \left(\frac{1}{1000}\right)^2 \sum xy}{\left(\frac{1}{1000}\right)^2 \sum y^2} = R^2$$

d) Researcher A measured both X and Y in kilograms, whereas researcher B measured Y in metric tons and X in kilograms.

A: $Y = \beta_0 + \beta_1 X + u$

B: $Y^* = \beta_0 + \beta_1 X + u$

i) $b_1^* = \dfrac{\sum x\ \ y^*}{\sum x^2} = \dfrac{\sum x \dfrac{1}{1000} y}{\sum\ \ x^2} = \dfrac{\left(\dfrac{1}{1000}\right)\ \sum xy}{\sum x^2} = \left(\dfrac{1}{1000}\right) b_1$

ii) $se(b_1^*) = \sqrt{Var\left(\dfrac{1}{1000} b_1\right)} = \left(\dfrac{1}{1000}\right) se(b_1) \Rightarrow t^* = \dfrac{\left(\dfrac{1}{1000}\right) b_1}{\left(\dfrac{1}{1000}\right) se(b_1)} = t$

iii) $R^{*2} = \dfrac{b_1^* \sum x\ \ y^*}{\sum \left(y^*\right)^2} = \dfrac{\left(\dfrac{1}{1000}\right) b_1 \sum x \left(\dfrac{1}{1000}\right) y}{\sum \left(\dfrac{1}{1000}\right)^2 y^2} = \dfrac{b_1 \left(\dfrac{1}{1000}\right)^2 \sum xy}{\left(\dfrac{1}{1000}\right)^2 \sum y^2} = R^2$