Midterm 1 : Key
Econ 616, Spring 2002

1.

(a)
$$\begin{pmatrix}\hat\beta_2\\ \hat\beta_3\end{pmatrix} = \begin{bmatrix}\Sigma x_2^2 & \Sigma x_2x_3\\ \Sigma x_2x_3 & \Sigma x_3^2\end{bmatrix}^{-1}\begin{pmatrix}\Sigma x_2y\\ \Sigma x_3y\end{pmatrix}$$
$$= \begin{bmatrix}\Sigma x_2^2 & 0\\ 0 & \Sigma x_3^2\end{bmatrix}^{-1}\begin{pmatrix}\Sigma x_2y\\ \Sigma x_3y\end{pmatrix} = \begin{pmatrix}\Sigma x_2y/\Sigma x_2^2\\ \Sigma x_3y/\Sigma x_3^2\end{pmatrix} = \begin{pmatrix}\hat\gamma_2\\ \hat\gamma_3\end{pmatrix}$$

$$\left.\begin{aligned}&\text{From (2)}, \ \hat\gamma_2 = \Sigma x_2y/\Sigma x_2^2\\ &\text{and from (3)}, \ \hat\gamma_3 = \Sigma x_3y/\Sigma x_3^2\end{aligned}\right\}$$

(b)
$$V\begin{pmatrix}\hat\beta_2\\ \hat\beta_3\end{pmatrix} = \sigma^2\begin{bmatrix}\Sigma x_2^2 & 0\\ 0 & \Sigma x_3^2\end{bmatrix}^{-1} = \begin{bmatrix}\sigma^2/\Sigma x_2^2 & 0\\ 0 & \sigma^2/\Sigma x_3^2\end{bmatrix} = \begin{bmatrix}V(\hat\gamma_2) & 0\\ 0 & V(\hat\gamma_3)\end{bmatrix}$$

$\Rightarrow \ V(\hat\beta_2) = V(\hat\gamma_2)$ and $V(\hat\beta_3) = V(\hat\gamma_3)$.

From (2), $V(\hat\gamma_2) = \sigma^2/\Sigma x_2^2$ and from (3) $V(\hat\gamma_3) = \sigma^2/\Sigma x_3^2$.

(c)
$$R^2 = \frac{\hat\beta_2\Sigma x_2y + \hat\beta_3\Sigma x_3y}{\Sigma y^2} = \frac{\hat\gamma_2\Sigma x_2y}{\Sigma y^2} + \frac{\hat\gamma_3(\Sigma x_3y)}{\Sigma y^2} = R_1^2 + R_2^2.$$

(d) If the X variables are uncorrelated there is no need to run a multiple regression.

(e) No. If $Y = \beta_2X_2 + \beta_3X_3 + u$, then BLUE 4 β_2 and β_3 is
$$\begin{pmatrix}\hat\beta_2\\ \hat\beta_3\end{pmatrix} = \begin{bmatrix}\Sigma X_2^2 & \Sigma X_2X_3\\ \Sigma X_2X_3 & \Sigma X_3^2\end{bmatrix}^{-1}\begin{bmatrix}\Sigma X_2Y\\ \Sigma X_3Y\end{bmatrix} \neq \begin{pmatrix}\Sigma X_2Y/\Sigma X_2^2\\ \Sigma X_3Y/\Sigma X_3^2\end{pmatrix} = \begin{pmatrix}\hat\gamma_2\\ \hat\gamma_3\end{pmatrix}$$

Note: X_2 and X_2 uncorrelated $\Rightarrow \Sigma x_2x_3 = 0$ but $\Sigma X_2X_3 \neq 0$ (unless $\bar X_2 = \bar X_3 = 0$).

2. (a) $b_* = b + (X'X)^{-1} R' [R (X'X)^{-1} R']^{-1} (r - Rb)$

$R = (1 \ 1 \ 1)$ and $r = 0$.

$$b_* = \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} + s^2 (X'X)^{-1} R' [\underbrace{R s^2 (X'X)^{-1} R'}_{13}]^{-1} \begin{bmatrix} -3 \end{bmatrix}$$

$$= \begin{bmatrix} 5 \\ -4 \\ 2 \end{bmatrix} - \frac{3}{13} \begin{bmatrix} 5 \\ 4 \\ 4 \end{bmatrix} = \begin{bmatrix} 50/13 \\ -64/13 \\ 14/13 \end{bmatrix}$$

(b) Can use either a t or a F test

t test : $\dfrac{b_1 + 2b_2 + b_3}{SE(b_1 + 2b_2 + b_3)} \sim t_{50}$.

$$SE(\cdot) = \{ V(b_1) + 4V(b_2) + V(b_3) + 4 \operatorname{Cov}(b_1, b_2) + 2 \operatorname{Cov}(b_1, b_3) + 4 \operatorname{Cov}(b_2, b_3) \}^{1/2}$$

$$= \{ 3 + 4(2) + 2 + 4(1) + 2(1) + 4(1) \}^{1/2} = \sqrt{23}$$

$$\Rightarrow \quad t = \frac{-1}{\sqrt{23}} = -.2085$$

(c) Changed to predict $E(Y_0)$

$\hat{Y}_0 = X_0' b = 3$

C.I. for $E(Y_0)$: $\hat{Y}_0 \pm \underbrace{t_{.025, 50}}_{2.009} \sqrt{\underbrace{s^2 X_0' (X'X)^{-1} X_0}_{51}}$

$3 \pm (2.009)(7.141) = 3 \pm 14.347$

(d) $\widehat{V\begin{pmatrix} b_2 \\ b_3 \end{pmatrix}} = s^2 (x_2' x_2)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

(e) $b_3 \pm \underbrace{t_{.025, 50}}_{2.009} \underbrace{s\sqrt{a_{33}}}_{\sqrt{2}} = 2 \pm 2.8411$

(f) $t = \dfrac{b_2 - 3}{SE(b_2)} = \dfrac{-4-3}{\sqrt{2}} = -4.9497$

(g) (i) $\frac{ESS}{s^2} = \frac{b_2' (x_2' x_2) b_2}{s^2}$

From (d) $s^2 (x_2' x_2)^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \Rightarrow \frac{1}{s^2}(x_2' x_2) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^{-1} = \frac{1}{3}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$

$\therefore \frac{ESS}{s^2} = \frac{1}{3}(-4 \quad 2)\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -4 \\ 2 \end{pmatrix} = \frac{56}{3}$

(ii) $s^2 = \frac{RSS}{n-K} \Rightarrow \frac{RSS}{s^2} = n-K = 50$

(iii) $R^2 = \frac{ESS/s^2}{ESS/s^2 + RSS/s^2} = \frac{56/3}{56/3 + 50} = .2718$

(h) $F = \frac{R^2/q}{1-R^2/n-K} = \frac{.2718}{1-.2718} \cdot \frac{56}{2} = 9.3318$

Compare it with $F_{.05;\,2,50}$.