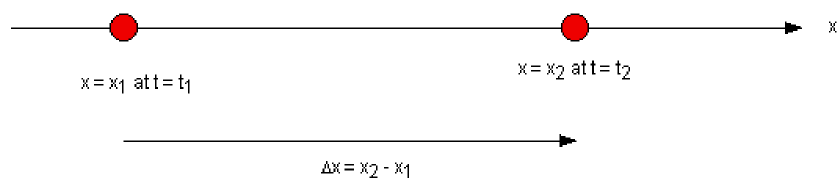
Chapter 2
Motion in One dimension

1. Displacement

The position of an object (particle) moving along the x axis, is described by its x coordinate. The change in the particle's position is its displacement Δx. If the particle is at x_1 at t_1 and at x_2 at t_2, then the displacement is given by

$$\Delta x = x_2 - x_1$$



The displacement is a vector. Δx is positive for $x_2>x_1$ and negative for $x_2<x_1$.

2. Average velocity and instantaneous velocity

The average velocity is defined by

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1} = \frac{\Delta x}{\Delta t}$$

in terms of the diplacement Δx.

The instantaneous velocity v is defined by a derivative of x with respect to t,

$$v = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

or

$$v = \dot{x} \qquad \text{(simpler notation)}$$

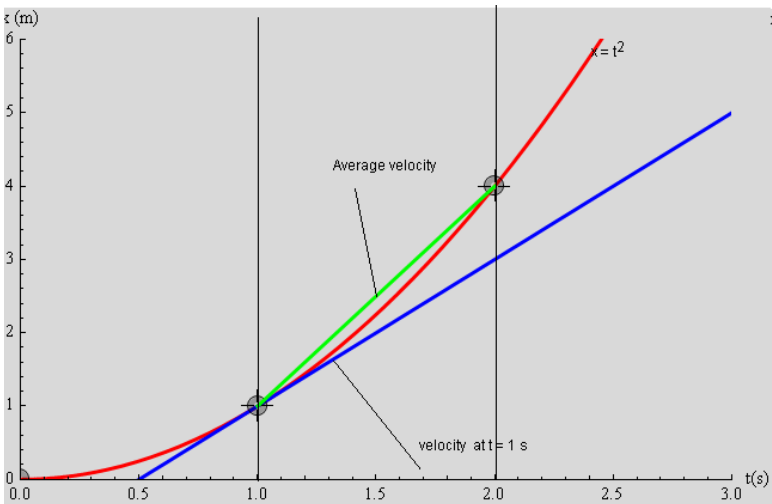


Fig. The definition of the average velocity and the instantaneous velocity at t = 1 s. We assume that x is described by $x = t^2$ in this example.

The velocity is a vector and has a unit of m/s. The magnitude of the velocity is called speed. The speed is always positive.

((Note))
The average speed is defined by

Average speed = (total distance)/(total time)

The definition of the average speed is rather different from that of the average velocity.

3. Average acceleration and instantaneous acceleration

$$\bar{a} = \frac{v_2 - v_1}{t_2 - t_1} = \frac{\Delta v}{\Delta t}$$

$$a = \frac{dv}{dt} = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

The acceleration is the second derivative of displacement,

$$a = \frac{dv}{dt} = \frac{d}{dt}\left(\frac{dx}{dt}\right) = \frac{d^2 x}{dt^2} = \ddot{x}$$

The acceleration a has a unit of m/s^2.

4. Velocity and position

4.1 General case

(a) Position $x(t)$:

The position $x(t)$ can be derived from

$$x(t) = \int_{t_0}^{t} v(t')dt' + x_0$$

where

$$v = \frac{dx}{dt} ,$$

and the initial condition is given by

$$x = x_0 \qquad \text{at } t = t_0.$$

(b) Velocity $v(t)$:

The velocity $v(t)$ can be derived from

$$v(t) = \int_{t_0}^{t} a(t')dt' + v_0$$

where

$$a = \frac{dv}{dt}$$

and

$$v = v_0 \qquad \text{at } t = t_0.$$

4.2 Example

Suppose that $x(t)$ is given by

$$x(t) = t^2 e^{-t}$$

The velocity v and acceleration a are calculated as

$$v = \frac{dx}{dt} = t(2-t)e^{-t}$$

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2} = (t^2 - 4t + 2)e^{-t}$$

Using Mathematica, we make a plot of *x*, *v*, and *a* as a function of *t*.

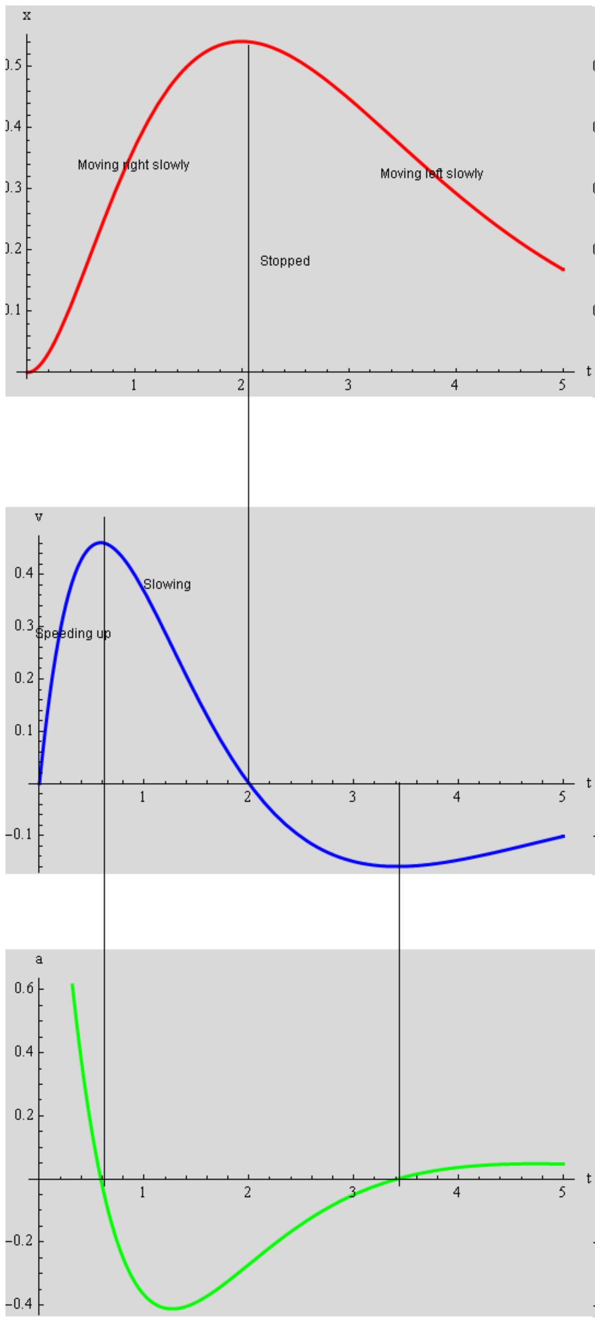


Fig. Plot of *x*, *v*, and *a* as a function of *t*. $x(t) = t^2 e^{-t}$.

5. Constant acceleration

5.1 Formulation

We consider the motion of particle with constant acceleration (a = constant).

$$\frac{dv}{dt} = \frac{d^2x}{dt^2} = a$$

Then we have

$$v = \int a dt = at + c_1$$

where c_1 is constant. The constant c_1 is determined as

$$c_1 = v_0$$

from the initial condition that

$v = v_0$ and $x = x_0$ at $t = 0$.

The displacement x can be derived from a first-order differential equation.

$$v = \frac{dx}{dt} = at + v_0 \qquad (1)$$

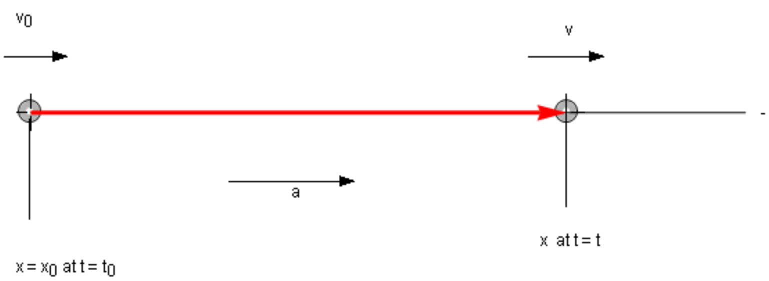
or

$$x = \int_0^t (at + v_0)dt + x_0 = \frac{1}{2}at^2 + v_0 t + x_0 \qquad (2)$$

(i) The formula where the missing quantity is t.

Substitution of $t = \frac{v - v_0}{a}$ into Eq.(2) yields

$$v^2 - {v_0}^2 = 2a(x - x_0) \qquad (3)$$



((Another method)) Derivation of Eq.(3) where a = constant.

$$\frac{dv}{dt} = a = const$$

$$v\frac{dv}{dt} = av = a\frac{dx}{dt}$$

$$\frac{d}{dt}(\frac{1}{2}v^2) = \frac{d}{dt}(ax)$$

$$\frac{d}{dt}(\frac{1}{2}v^2 - ax) = 0$$

$$\frac{1}{2}v^2 - ax = \frac{1}{2}{v_0}^2 - ax_0$$

$$v^2 - {v_0}^2 = 2a(x - x_0)$$

(ii) The formula where the missing quantity is a.

Substitution of $a = \frac{v - v_0}{t}$ into Eq.(2) yields

$$x - x_0 = \frac{1}{2}t(v + v_0)$$

(iii) The formula where the missing quantity is v_0.

Substitution of $v_0 = v - at$ into Eq.(2) yields

$$x - x_0 = -\frac{1}{2}at^2 + vt$$

5.2 Mathematica

```
Clear["Global`*"]; eq1 = x == x0 + v0 t + 1/2 a t^2;
eq2 = v == v0 + a t;
```

Formula-1 Missing quantity; t

```
eq3 = Solve[eq2, t]
```

$$\left\{\left\{t \to \frac{v - v0}{a}\right\}\right\}$$

```
eq11 = eq1 /. eq3[[1]] // Simplify
```

$$x == \frac{v^2 - v0^2 + 2\,a\,x0}{2\,a}$$

Formula - 2 Missing quantity;a

```
eq21 = Solve[eq2, a]
```

$$\left\{\left\{a \to \frac{v - v0}{t}\right\}\right\}$$

```
eq12 = eq1 /. eq21[[1]] // Simplify
```

$$t\,(v + v0) + 2\,x0 == 2\,x$$

Formula - 1 Missing quantity; v0

```
eq31 = Solve[eq2, v0]
```

$$\{\{v0 \to -a\,t + v\}\}$$

```
eq13 = eq1 /. eq31[[1]] // Simplify
```

$$x == -\frac{a\,t^2}{2} + t\,v + x0$$

6. Special case $a = 0$

$v = v_0$ (constant) (1)

and

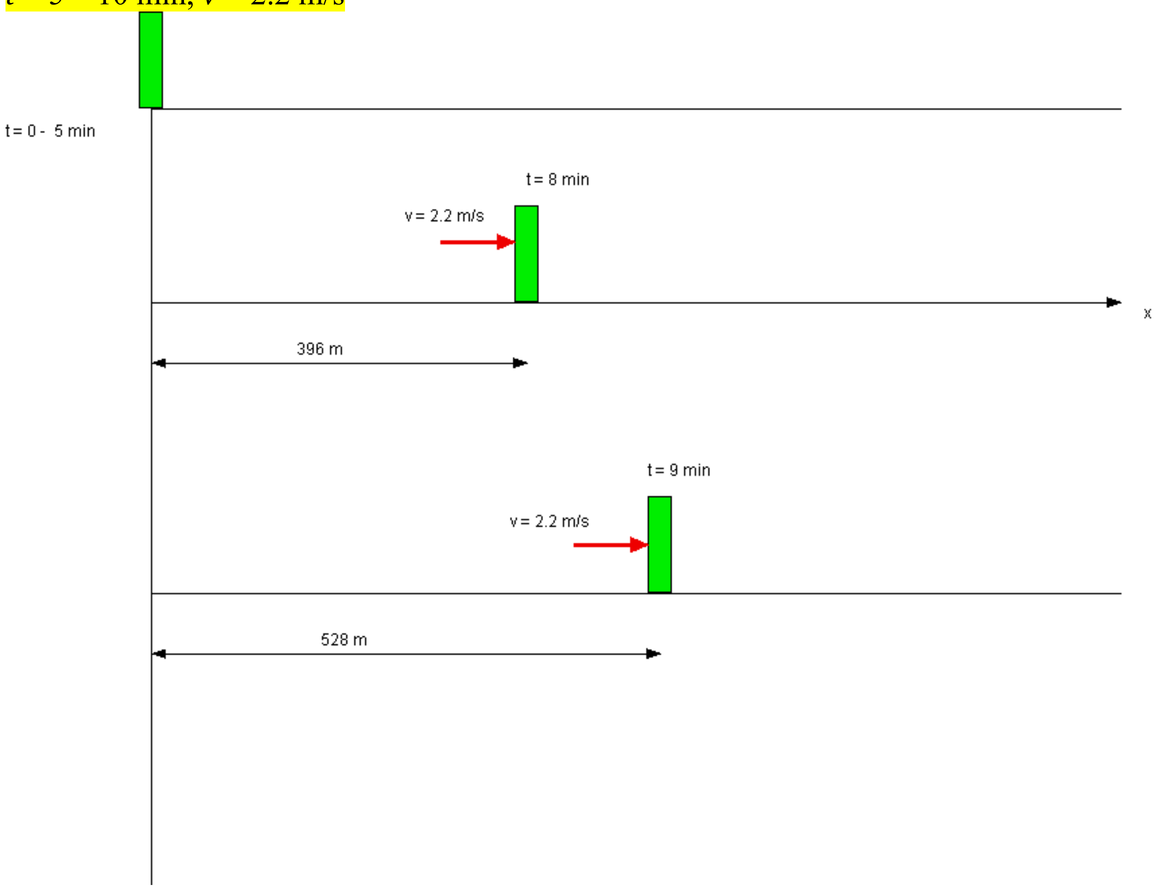
$x = v_0 t + x_0$ (2)

7. Typical problems

Problem 2-22 (8-th edition)
Problem 2-21 (9-th, 10-th edition)

From $t = 0$ to $t = 5.00$ min, a man stands still, and from $t = 5.00$ min to $t = 10.0$ min, he walks briskly in a straight line at a constant speed of 2.20 m/s. What are (a) his average velocity v_{avg} and (b) his average acceleration a_{avg} in the time interval 2.00 min to 8.00 min? What are (c) v_{avg} and (d) a_{avg} in the time interval 3.00 min to 9.00 min? (e) Sketch x versus t and v versus t, and indicate how the answers to (c) and (d) can be obtained from the graphs.

((**Solution**))
$t = 0 – 5$ min, $x = 0$, $v = 0$
$t = 5 – 10$ min, $v = 2.2$ m/s



(a) $2 \le t \le 8$ min

At $t = 8$ min, $x = (8-5) \times 60 \times 2.2 = 396$ m

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{396m}{(8-2)\times 60s} = 1.1 m/s$$

(b) $2 \le t \le 8$ min

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{2.2m/s}{(8-2)\times 60s} = 6.11\times 10^{-3} m/s^2$$

(c) $3 \le t \le 9$ min

At t = 9 min, x = (9-5) x 60 x 2.2 = 528 m

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{528m}{(9-3)\times 60s} = 1.467 m/s$$

(d) $3 \le t \le 9$ min

$$a_{av} = \frac{\Delta v}{\Delta t} = \frac{2.2m/s}{(9-3)\times 60s} = 6.11\times 10^{-3} m/s^2$$

(e)

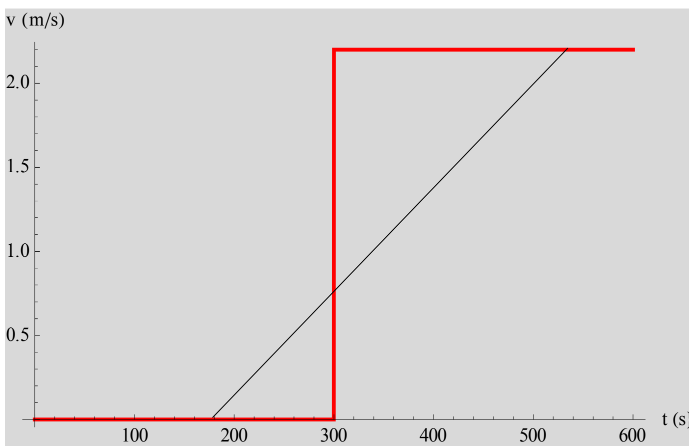


Fig. The slope of the straight line for v vs t gives the average acceleration for $3 \le t \le 9$ min.

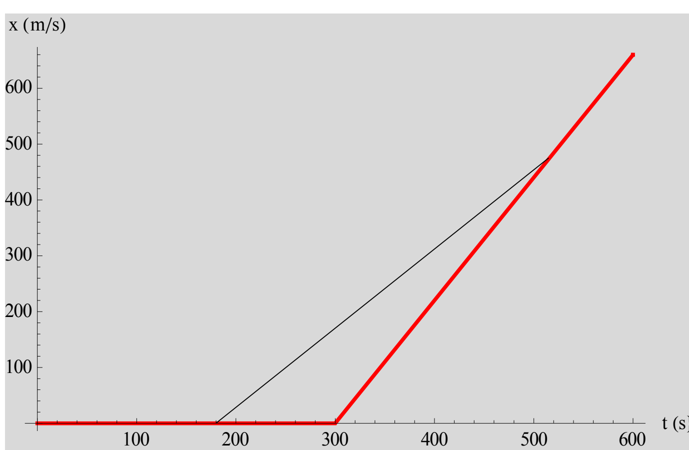
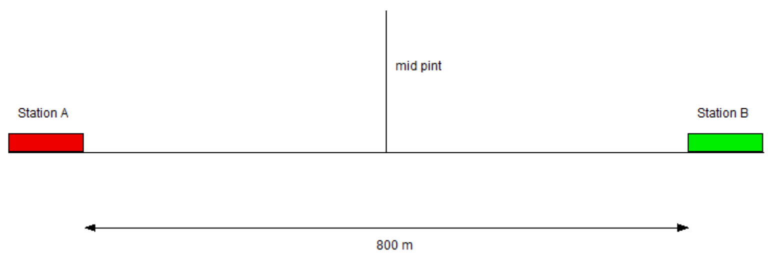


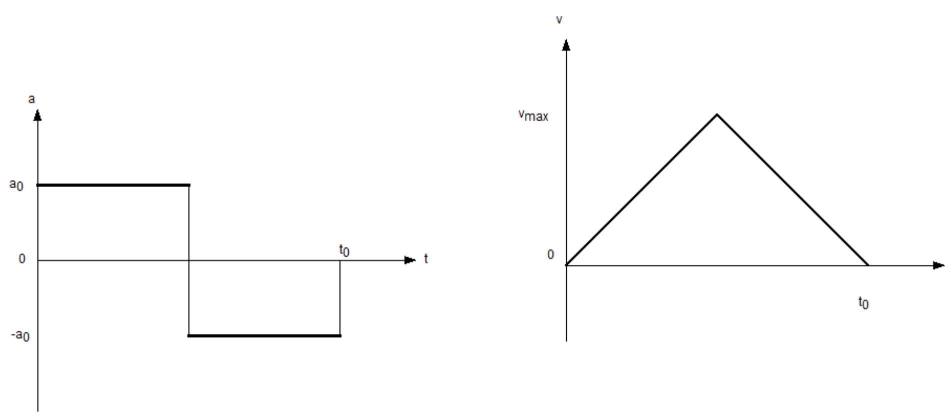
Fig. The slope of the straight line for x vs t gives the average velocity $3 \le t \le 9$ min.

Problem 2-36 (8-th edition)
Problem 2-38 (9-th, 10-th edition)

(a) If the maximum acceleration that is tolerable for passengers in a subway train is 1.34 m/s^2 and subway stations are located 806 m apart, what is the maximum speed a subway train can attain between stations? (b) What is the travel time between stations? (c) If a subway train stops for 20 s at each station, what is the maximum average speed of the train, from one start-up to the next? (d) Graph x, v, and a versus t for the interval from one-start-up to the next.

((**Hint**))
$a \le 1.34$ m/s^2
We assume that the velocity as a function of t is symmetric with respect to the axis $t = t_0/2$.





The curve of v vs t is obtained from the curve of a vs t.

$$v(t) = v(t=0) + \int_0^t a(t')dt' = 0 + a_0 t$$

for $0 \le t \le t_0/2$ and

$$v(t) = v(t = t_0/2) + \int_{t_0/2}^{t} a(t')dt'$$
$$= a_0(t_0/2) - \int_{t_0/2}^{t} a_0 dt'$$
$$= a_0(t_0/2) - a_0(t - t_0/2)$$
$$= -a_0(t - t_0)$$

for $t_0/2 \le t \le t_0$.

The distance x vs t is obtained from that of a vs t.

$$x(t) = x(t = 0) + \int_0^t v(t')dt' = \int_0^t a_0 t' dt' = a_0 \frac{t^2}{2}$$

for $0 \le t \le t_0/2$ and

$$x(t) = x(t = t_0/2) + \int_{t_0/2}^{t} v(t')dt'$$
$$= a_0 \frac{1}{2}\left(\frac{t_0}{2}\right)^2 - \int_{t_0/2}^{t} a_0(t' - t_0)dt'$$
$$= a_0 \frac{t_0^2}{4} - \frac{1}{2} a_0 (t - t_0)^2$$

((**Solution**))

(a) The total distance d is the area enclosed by v vesus t and $v = 0$ for $0 \le t \le t_0$.

$$d = \frac{t_0 v_{\max}}{2} = 806m$$
$$v_{\max} = a\frac{t_0}{2}$$
$$a = \frac{2v_{\max}}{t_0} = 1.34 m/s^2$$

From these equations, we have

$$v_{\max} = 32.864 m/s$$
$$t_0 = 49.05 s$$

(c) Stop time at each station is 20 s.

$$v_{av} = \frac{\Delta x}{\Delta t} = \frac{806m}{(49.05+20)s} = 11.67 m/s$$

(d)

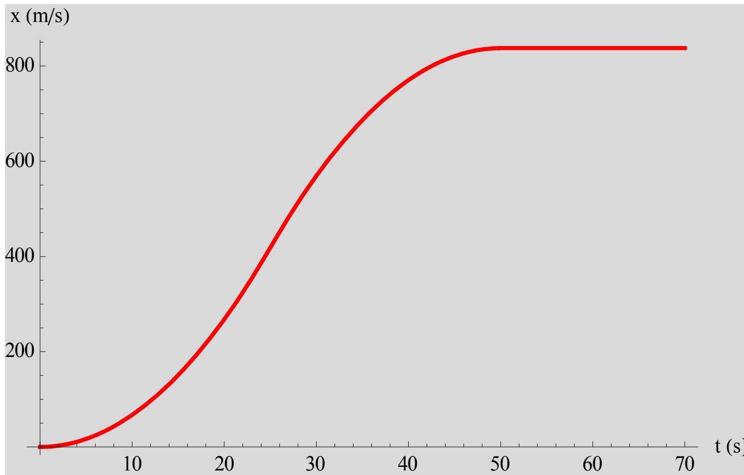


Fig. the distance vs *t*

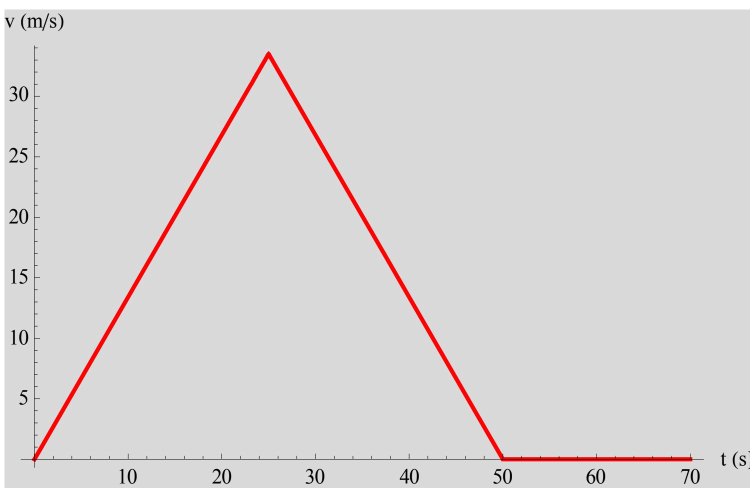


Fig.   Velocity vs time

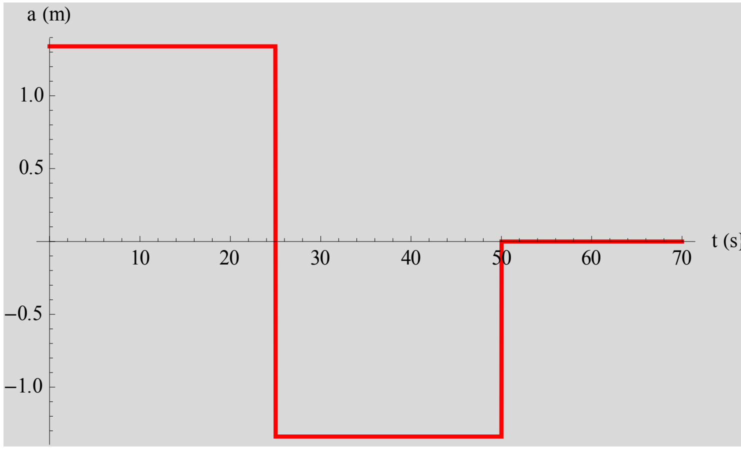
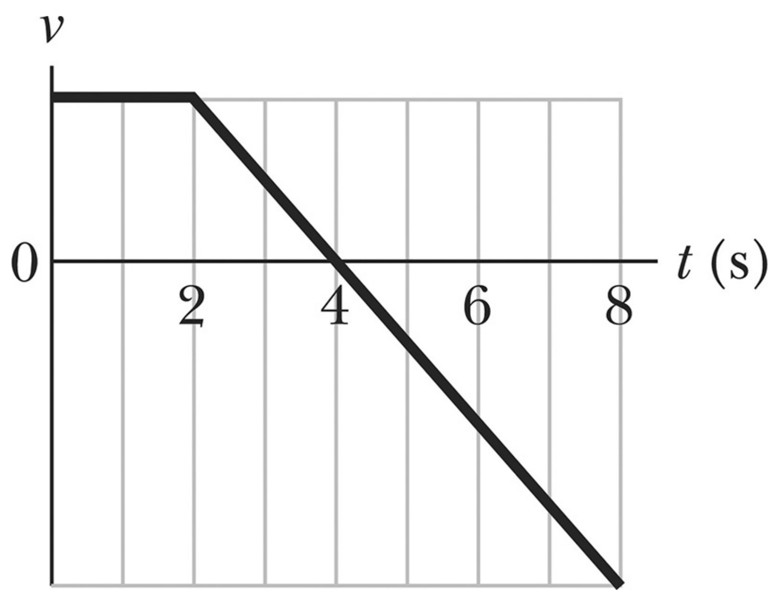


Fig. Acceleration *a* vs *t*

Problem 2-53 HW-02 (8-th edition)**
Problem 2-51HW-02 (9-th, 10-th edition)**

As a runaway scientific balloon ascends at 19.6 m/s, one of its instrument packages breaks free of a harness and free-falls. Figure gives the vertical velocity of the package versus time, from before it breaks free to when it reaches the ground. (a) What maximum height above the break-free point does it rise? (b) How high is the break-free point above the ground?



((**Hint**))
$v_0 = 19.6$ m/s
$t_0 = 2$ s
$a_0 = -19.6/2 = -9.8$ m/s^2 ($t \geq t_0 = 2$ s)

$$v = \frac{dy}{dt}$$

The distance y vs t is obtained from that of v vs t.

$$y(t) = y_0 + \int_0^t v(t')dt'$$

(i) For $0 \le t \le 2$

$$v = v_0.$$

and

$$y(t) = y_0 + \int_0^t v_0 dt' = y_0 + v_0 t$$

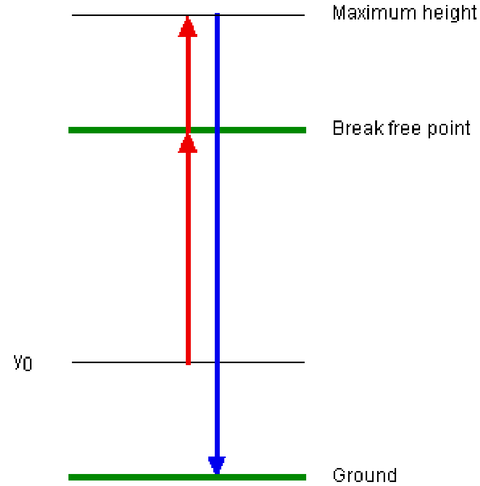
(ii) For $t \ge 2$

$$v = v_0 - \frac{v_0}{2}(t-2)$$
$$= -\frac{v_0}{2}(t-4)$$

and

$$y(t) = y(t=2) + \int_2^t [-\frac{v_0}{2}(t'-4)]dt'$$
$$= y_0 + 3v_0 - \frac{v_0}{4}(t-4)^2$$

where H is the height of the break-free point from the ground. $H = y_0 + v_0 t_0$. Δt (= t-2) is the time measured after the balloon breaks free.



((Note))

This problem can be more easily solved by using the fact that the area enclosed by the curve $v(t)$ vs t and v= 0 is a distance.

Problem 2-109 (8-th edition)
This problem was removed in 9-th edition

The speed of a bullet is measured to be 640 m/s as the bullet emerges from a barrel of length 1.20 m. Assuming constant acceleration, find the time that the bullet spends in the barrel after it is fired.

((**Solution**))

v_0 = 640 m/s

$$0^2 - v_0{}^2 = -2a(d-0)$$

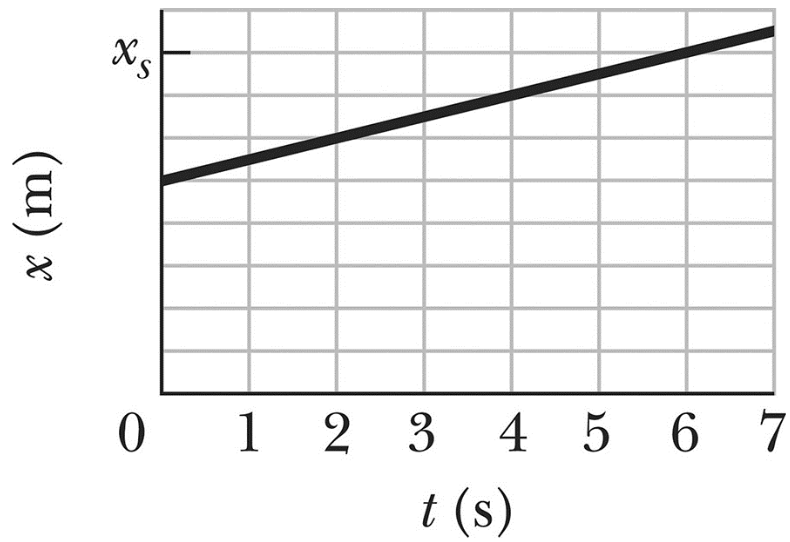
$$a = \frac{v_0{}^2}{2d} = 1.707\times 10^5 m/s^2$$

$$0 = v_0 - at$$

$$t = \frac{v_0}{a} = 3.75ms$$

Problem 2-37 (8-th edition)
Problem 2-39 (9-th, 10-th edition)

Cars A and B move in the same direction in adjacent lanes. The position x of car A is given in Fig, from time t = 0 to t = 7.0 s. The figure's vertical scaling is set by x_s = 32.0 m. At t = 0, car B is at x = 0, with a velocity of 12 m/s and a negative constant acceleration a_B. (a) What must a_B be such that the cars are (momentarily) side by side (momentarily at the same value of x) at t = 4.0 s? (b) For that value of a_B, how many times are the cars side by side? (c) Sketch the position x of car B versus time t on Fig. How many times will the cars be side by side if the magnitude of acceleration a_B is (d) more than and (e) less than the answer to part (a)?



((**Solution**))
For $0 \le t \le 7$s,

$$x_A(t) = 20 + 2t$$
$$x_B(t) = 12t + \frac{1}{2}a_B t^2$$

(a)

$$x_B(4) = 48 + 8a_B$$
$$x_A(4) = 20 + 8 = 28$$

Since $x_A(4) = x_B(4)$, we have

$$a_B = -2.5 \text{ m/s}^2$$

$$x_B(t) = 12t - \frac{5}{4}t^2$$
$$v_B(t) = \frac{dx_B(t)}{dt} = 12 - \frac{5}{2}t$$

$$v_B = 0 \text{ at } t = \frac{24}{5} = 4.8s$$

Then $x_B(t)$ has a local maximum at $t = 4.8$s.

(b) We now consider the solution of

$$20 + 2t = 12t + \frac{1}{2}a_B t^2$$

$$\frac{1}{2}a_B t^2 + 10t - 20 = 0$$

$$D = 100 - 4\times\frac{1}{2}a_B(-20) = 100 + 4a_B$$

If a_B>-2.5, two positive solutions
If a_B=-2.5, double solutions.
If a_B<-2.5, no solutions.

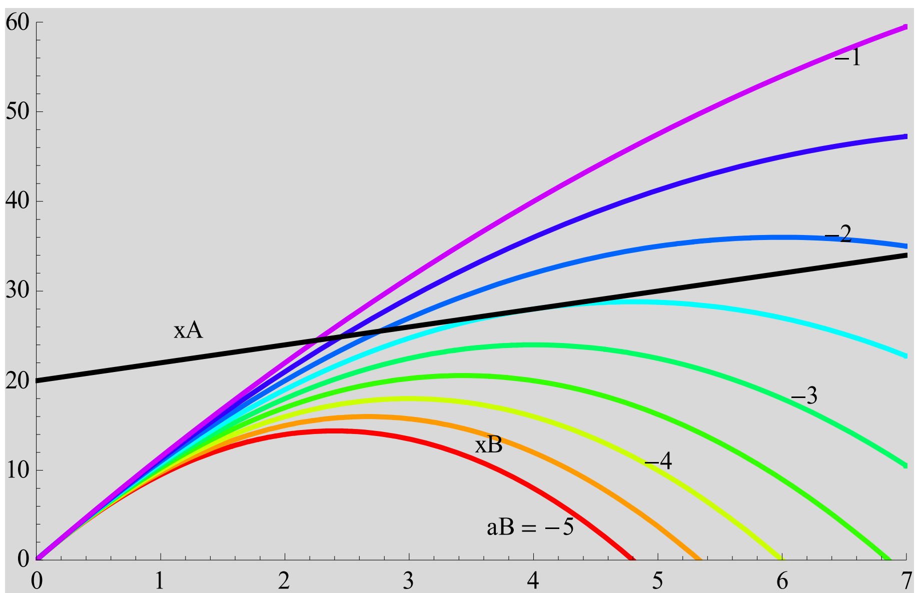
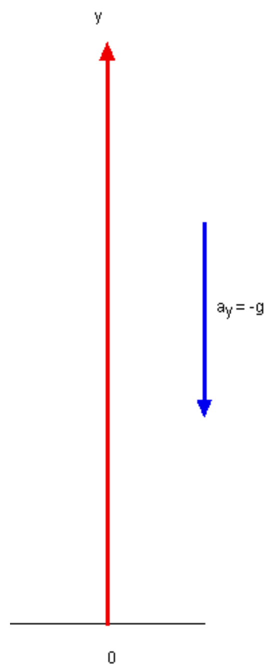


Figure Plot of $x_A(t)$ and $x_B(t)$ as a function of t, where a_B is changed as a parameter. a_B = -5, -4.5, -4, -3.5, -3, -2.5, -2, -1.5, -1.

8. Freely falling bodies

8.1 Formulation

We consider an object moving upward or downward along a vertical axis y. We neglect any air effects and consider only the influence of gravity on such an object. It is found that all objects, large and small, experience the same acceleration due to the gravity. The acceleration is always directed downward, since it is caused by the downward force of gravity.



$$a_y = -g$$
$$v_y = v_0 - gt$$
$$y = y_0 + v_0 t - \frac{1}{2} g t^2$$

where g is acceleration due to the gravity,

((Initial condition))

$y = y_0$ and $v = v_0$ at $t = 0$.

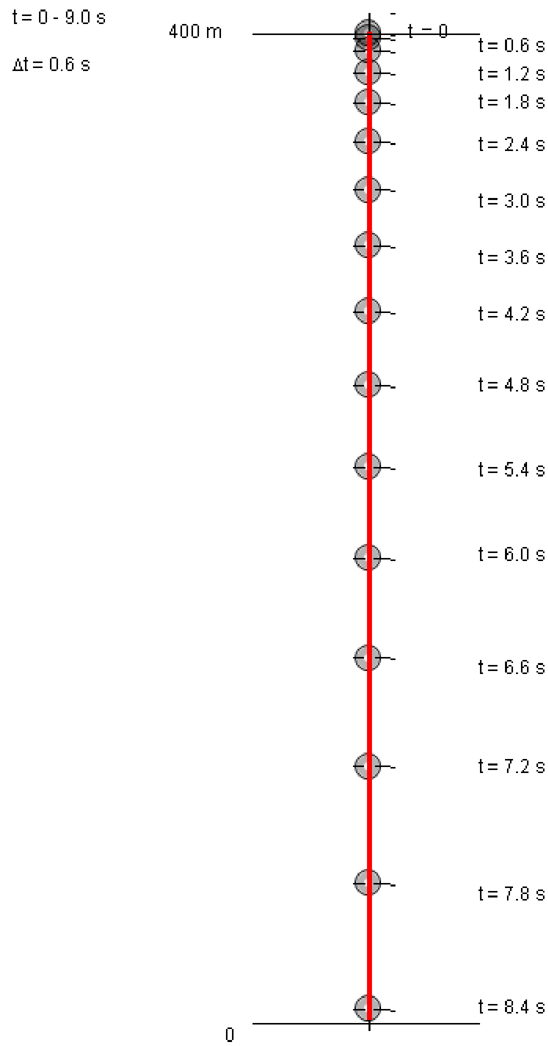
((Example)) Free falling

$y_0 = h = 400$ m
$v_0 = 0$
g = 9.8 m/s^2

Then we have

$$y(t) = h - \frac{1}{2} g t^2$$

((**Mathematica**)) Strobe pictures



8.2 The gravity in the Earth and Moon

Suppose that a stone is dropped from rest from a cliff in the Earth (Moon)? How far will it fall after a time *t* (s)? How fast will it move?

g = 9.8 m/s^2 (Earth)
g = 1.6 m/s^2 (Moon)
g = 274.0 m/s^2 (Sun)

Suppose that $v_0 = 0$ and $y_0 = H$. Then we have

$$a = -g$$
$$v = -gt$$
$$y = H - \frac{1}{2}gt^2$$

or

$$\Delta y = y - H = -\frac{1}{2}gt^2$$

Note that the expression of Δy depends only on g.

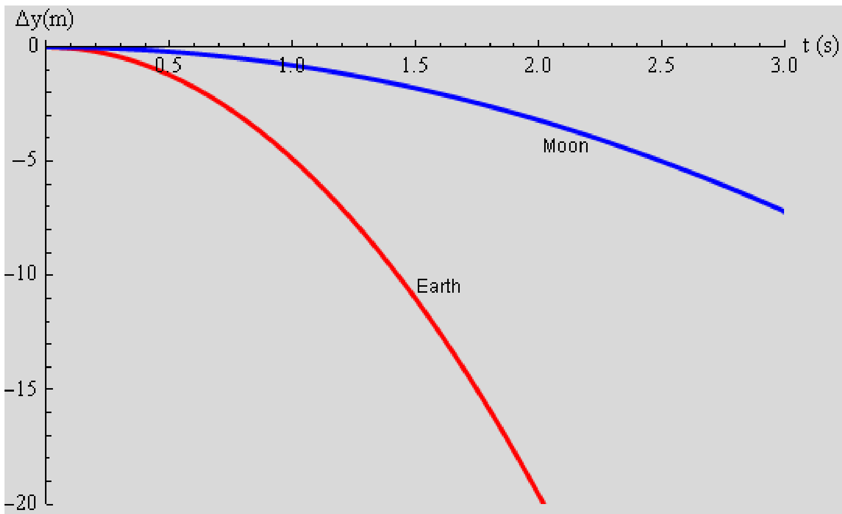


Fig. Δy vs t for the Earth and Moon

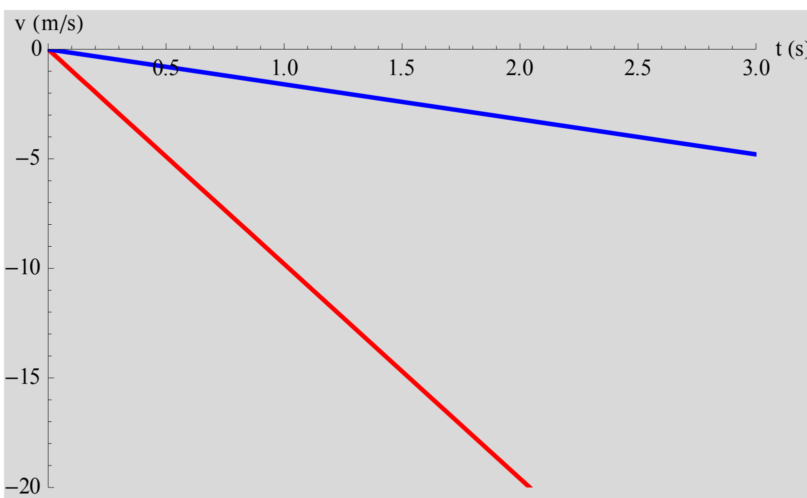


Fig. v (m/s) vs t for the Earth and Moon. The slope of the straight line is –g.

8.3 Tossing up of a ball with an initial velocity

Suppose that a ball is tossed up along the y axis with an initial velocity v_0?

$y = y_0 = 0$ at $t = 0$.

We discuss the time dependence of y and v.

$$a = -g \tag{1}$$

$$v = v_0 - gt \tag{2}$$

$$\begin{aligned} y &= v_0 t - \frac{1}{2} g t^2 = v_0 t - \frac{1}{2} g t^2 = -\frac{1}{2} g (t^2 - \frac{2v_0}{g} t) \\ &= -\frac{1}{2} g (t^2 - \frac{2v_0}{g} t + \frac{v_0^{\ 2}}{g^2}) + \frac{v_0^{\ 2}}{2g} \\ &= -\frac{1}{2} g (t - \frac{v_0}{g})^2 + \frac{v_0^{\ 2}}{2g} \end{aligned} \tag{3}$$

$$v^2 - v_0^{\ 2} = 2(-g)(y - 0) = -2gy \tag{4}$$

(a)
The peak height is given by

$$y_{peak} = \frac{v_0^{\ 2}}{2g}.$$

since $v^2 - v_0^{\ 2} = -2gy$ and $v = 0$.

(b)
The time taken for the ball to move up from $y = 0$ to y_{peak} is given by

$$t_{peak} = \frac{v_0}{g}$$

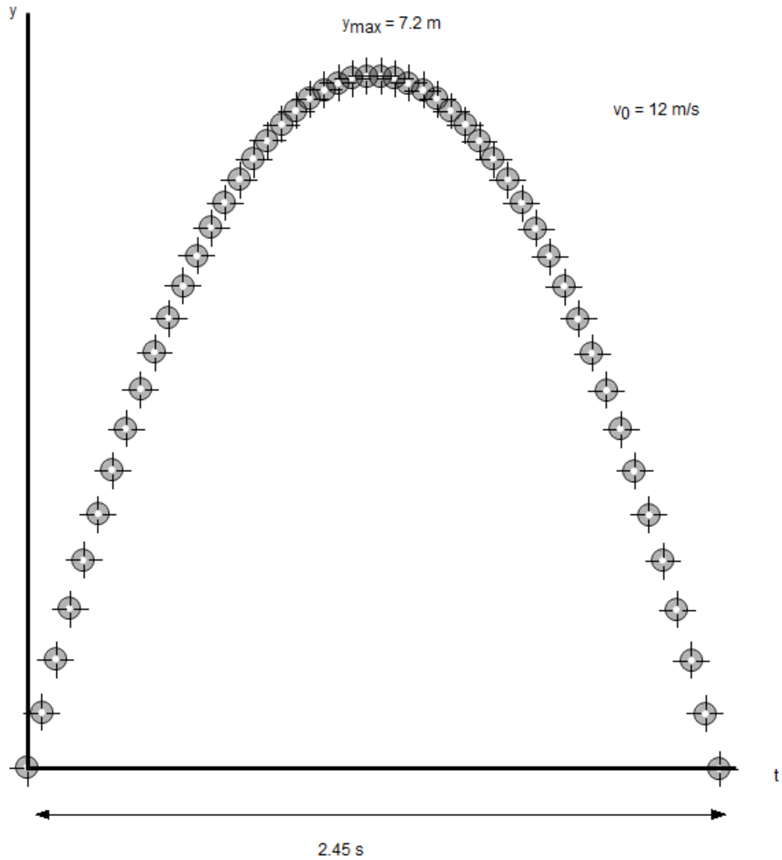
since $v = v_0 - gt = 0$ at $t = t_{\text{peak}}$.

(c)
The total time taken for the ball to move up from $y = 0$ to $y = y_{\text{peak}}$ and then move down from $y = y_{\text{peak}}$ to $y = 0$

$$t_{total} = \frac{2v_0}{g} = 2t_{peak}$$

since $y = v_0 t - \frac{1}{2} g t^2 = 0$.

((Strobe picture))



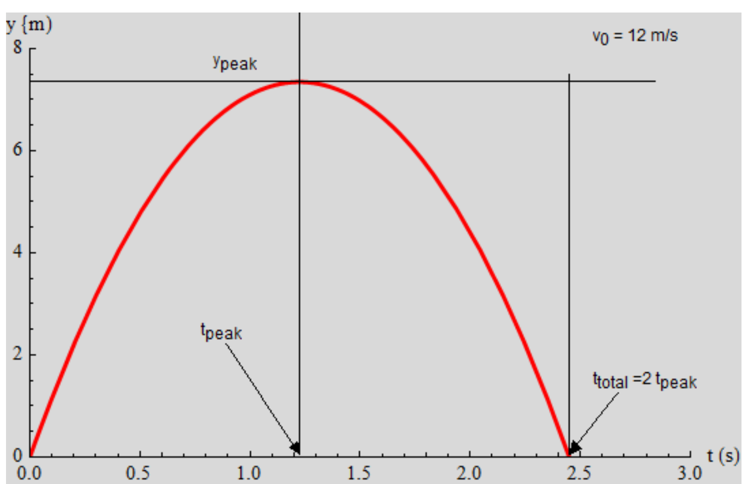


Fig. Plot of *y* vs *t* for v_0 = 12 m/s.

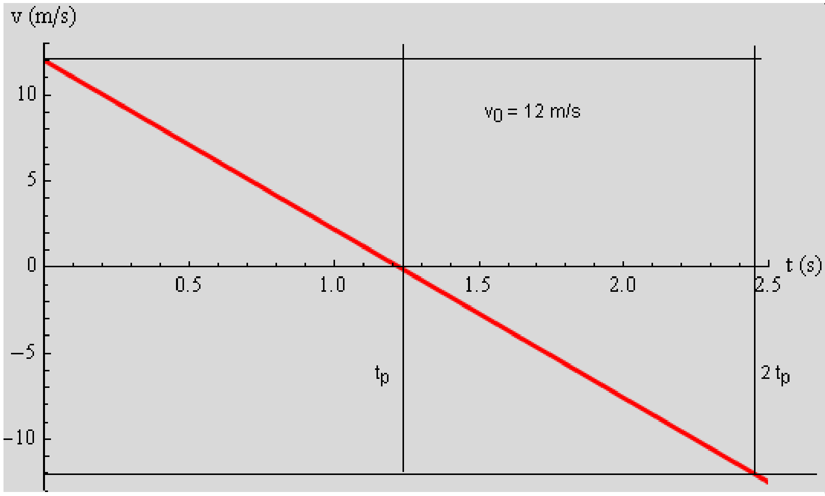


Fig. Plot of v vs t for v_0 = 12 m/s.

Finally we make a plot of y vs t with the initial velocity v_0 changed as a parameter.

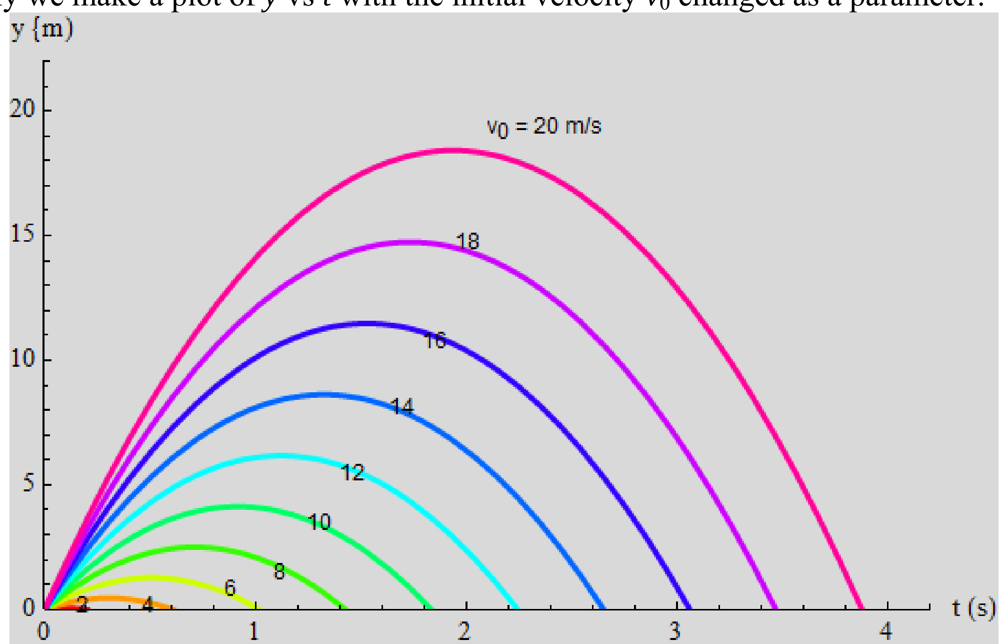


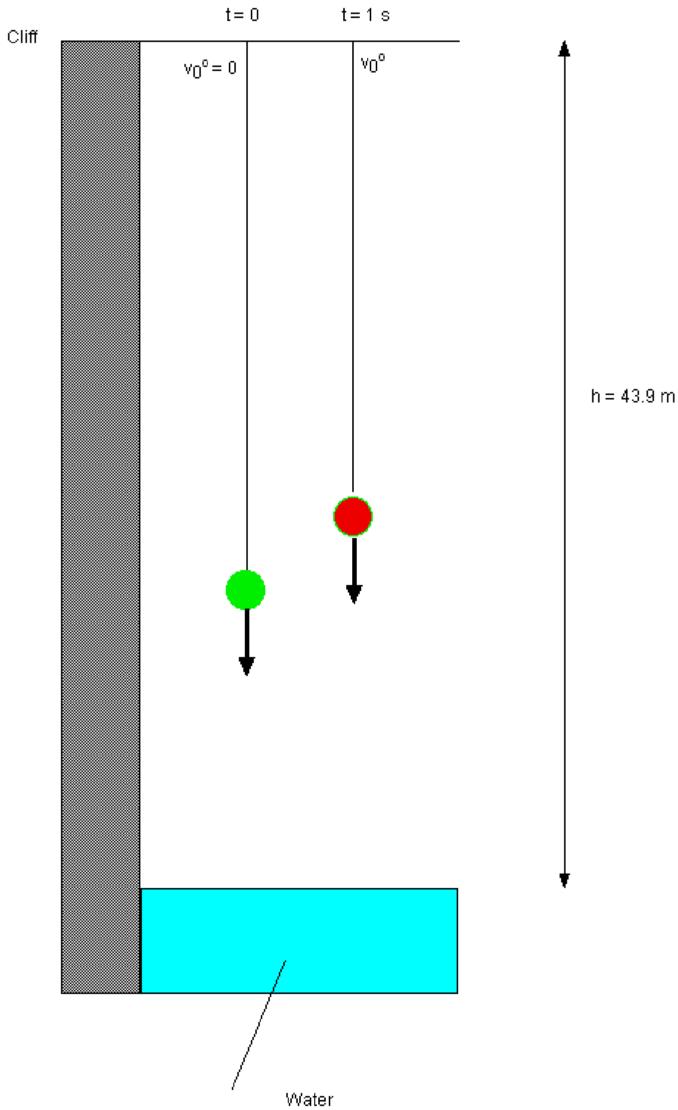
Fig. Plot of y vs t as the initial velocity v_0 is changed as a parameter. v_0 = 2, 4, 6, 8, 10, 12, 14, 16, 18, and 20 m/s.

10. Two Problems

Problem 2-56 8-th edition)**
Problem 2-54 (9-th, 10-th edition)**

A stone is dropped into a river from a bridge 43.9 m above the water. Another stone is thrown vertically down 1.00 s after the first is dropped. The stones strike the water at the same time. (a) What is the initial speed of the second stone? (b) Plot velocity versus time on a graph for each stone, taking zero time as the instant the first stone is released.

((**Solution**))
$H = 43.9$m



(a)

$$y_1 = H - \frac{1}{2}gt^2$$

$$y_2 = H - v_0(t-1) - \frac{1}{2}g(t-1)^2$$

When $y_1 = 0$, $t_0 = \sqrt{\frac{2H}{g}} = 2.9932s$

When $y_2 = 0$, $t = t_0$. Then we have

$$H - v_0(t_0 - 1) - \frac{1}{2}g(t_0 - 1)^2 = 0$$

From this, we get

$v_0 = 12.25$ m/s

(b)

$$v_1 = -gt$$
$$v_2 = -v_0 - g(t-1)$$

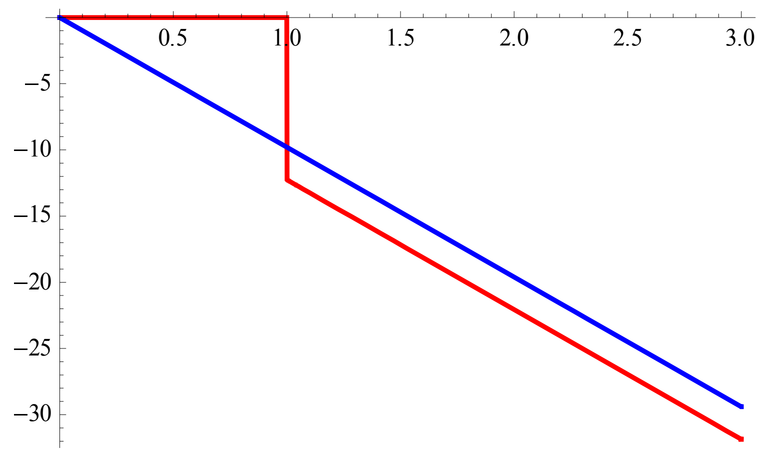
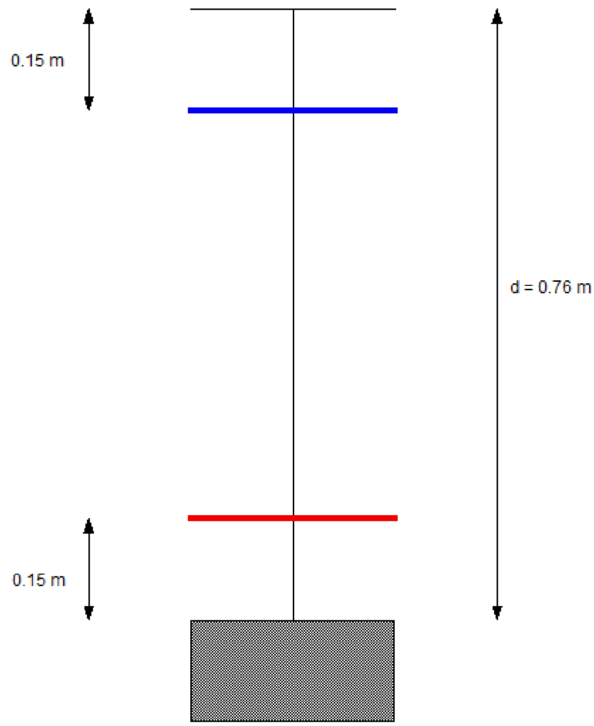


Fig. Plot of v_1 (blue) and v_2 (red) as a function of t.

Problem 2-64* (from SP-02) (8-th edition)**
Problem 2-62* (from SP-02) (9-th, 10-th edition)**

A basketball player grabbing a rebound jumps 76.0 cm vertically. How much total time (ascent and descent) goes the player spend (a) in the top 15.0 cm of this jump and (b) in the bottom 15.0 cm? Do your results explain why such players seem to hang in the air at the top of a jump?

((**Solution**))
$d = 0.76$ m.



$$y = -\frac{1}{2}gt^2 + v_0 t$$

$$0^2 - v_0^{\ 2} = -2gd$$
$$v_0 = \sqrt{2gd} = 3.86 m/s$$

Therefore we have

$$y = -4.9t^2 + 3.86t$$

When $y = 0$, we have

$t = 0$ or $\quad t = \frac{2v_0}{g} = \frac{2 \times 3.86}{9.8} = 0.788s$

(a) The top 15 cm of the jump

$$y = -4.9t^2 + 3.86t = 0.76 - 0.15 = 0.61$$
$$4.9t^2 - 3.86t + 0.61 = 0$$

The solution of this Eq. is

$t = 0.219$ s and 0.5687s

The difference of these times is 0.3497 s.

(b) In the bottom 15 cm

$$y = -4.9t^2 + 3.86t = 0.15$$

$$4.9t^2 - 3.86t + 0.15 = 0$$

The solution of this Eq. is

t= 0.040 s, or t = 0.746 s.

Then we have

0 s	–	0.040 s	the difference time = 0.040 s
0.746 s –		0.788 s	the difference time = 0.040 s

Then the total time is 0.04 x 2 = 0.008 s