Lecture Note
Chapter 3 Vectors

1. Definition of vectors

Vectors are usually indicated by boldface letters, such as A, and we will follow this most common convention. Alternative notation is a small arrow over the letters such as $\vec{A}$. The magnitude of a vector is also often expressed by $A = |\boldsymbol{A}|$. The displacement vector serves as a prototype for all other vectors. Any quantity that has magnitude and direction and that behaves mathematically like he displacement vector is a **vector**.

((Example))

velocity, acceleration, force,
linear momentum, angular momentum, torque
electric field, magnetic field, current density, magnetization, polarization
electric dipole moment, magnetic moment

By contrast, any quantity that has a magnitude but no direction is called a **scalar**.

((Example))

length, time, mass, area, volume, density, temperature, energy

A unit vector is a vector of unit length; a unit vector in the direction of ***A*** is written with a caret as $\hat{\boldsymbol{A}}$, which we read as "A hat."

$$\boldsymbol{A} = A\hat{\boldsymbol{A}} = \hat{\boldsymbol{A}}A$$

(a) A vector *r*

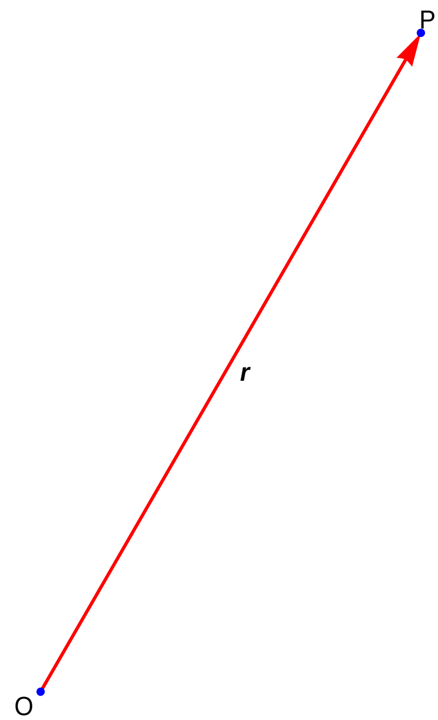


Fig. The vector ***r*** represents the position of a point P relative to another point O as origin.

(b) Negative vector: - ***r***

The negative of a given vector ***r*** is a vector of the same magnitude, but opposite direction.

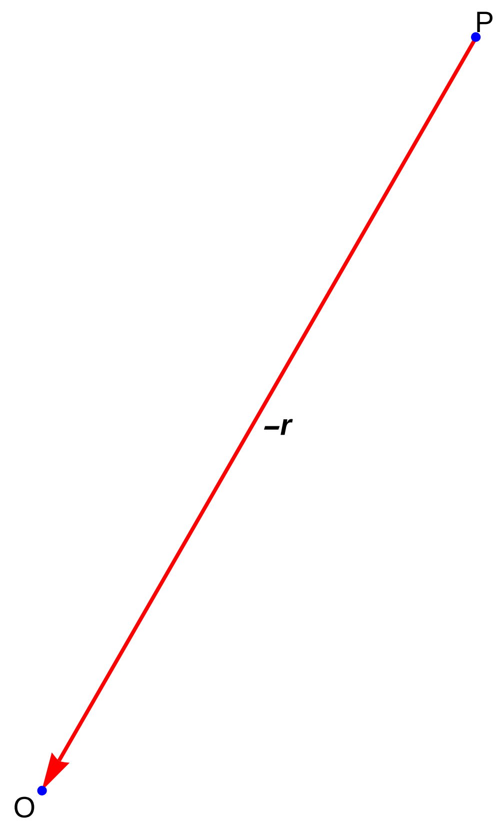


Fig. The vector $-\boldsymbol{r}$ is equal in magnitude but opposite in direction to $\boldsymbol{r}$.

(c) The multiplication of the vector by a scalar

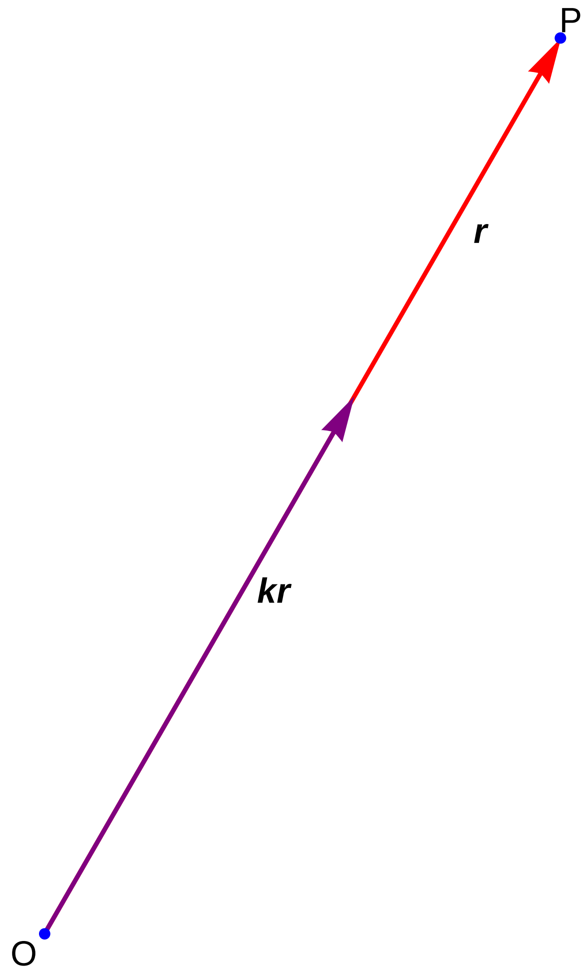


Fig. The vector $k\mathbf{r}$ is in the direction of $\mathbf{r}$ and is of magnitude kr, where $k = 0.6$.

(d) A unit vector

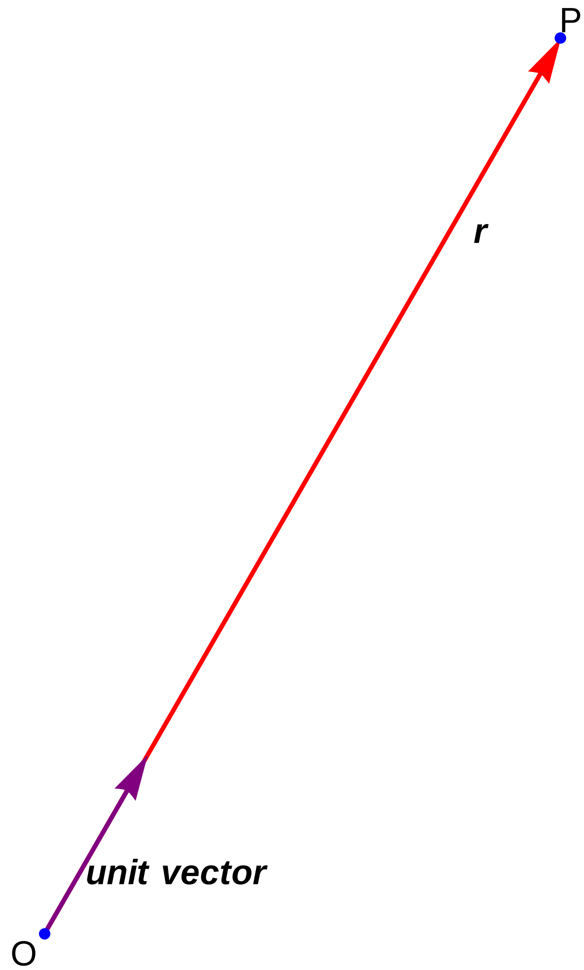
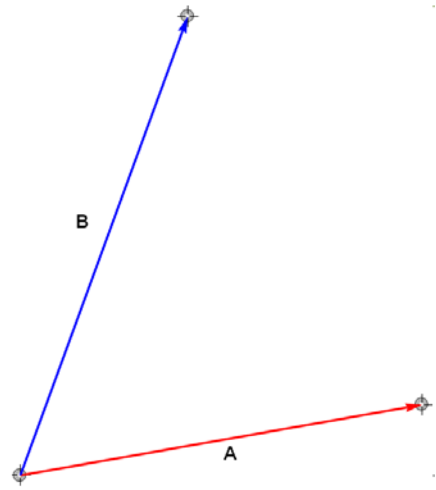


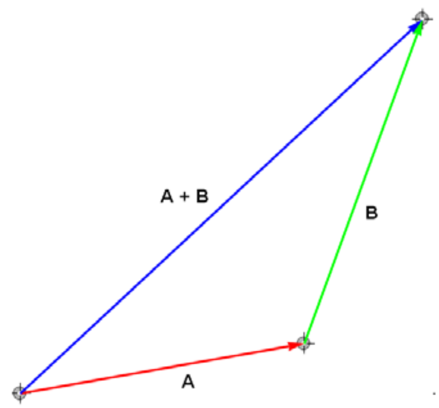
Fig. The vector $\hat{\boldsymbol{r}}$ is the unit vector in the direction of $\boldsymbol{r}$. Note that $\boldsymbol{r} = r\hat{\boldsymbol{r}}$. $\hat{\boldsymbol{r}} = \boldsymbol{e}_r$

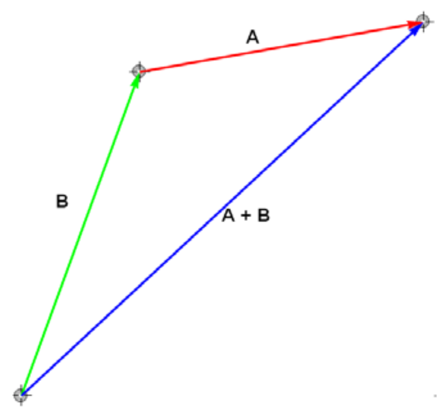
2. Vector addition

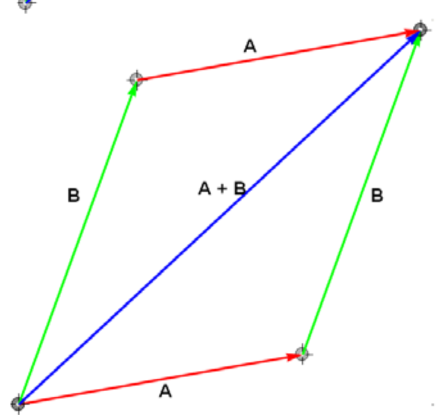
$\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B} = \boldsymbol{B} + \boldsymbol{A}$ (commutative)

The sum of two vectors is defined by the geometrical construction shown below. This construction is often called the parallelogram of addition of vectors.


B
A


A + B
B
A

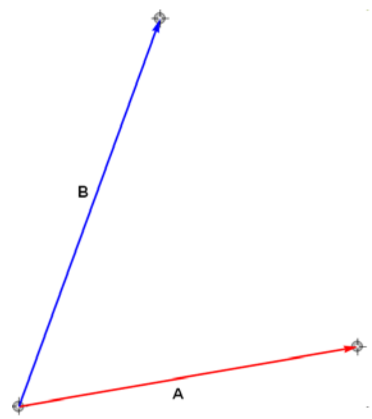

A
B
A + B

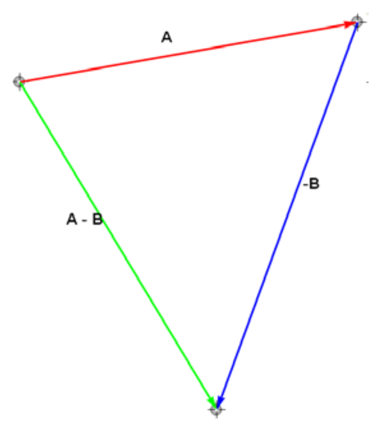

A
B
A + B
B
A

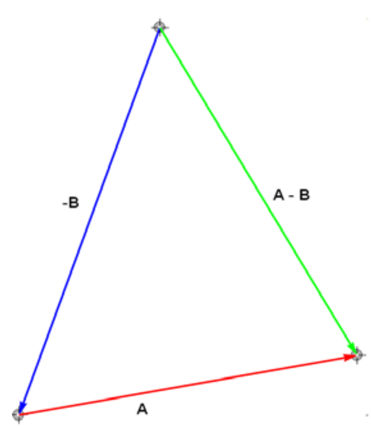
3. Vector subtraction

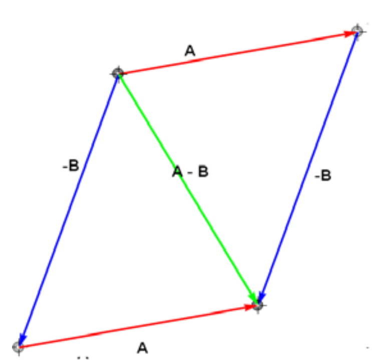
$$\boldsymbol{C} = \boldsymbol{A} - \boldsymbol{B}$$

The subtraction of two vectors is also defined by the geometrical construction shown below.

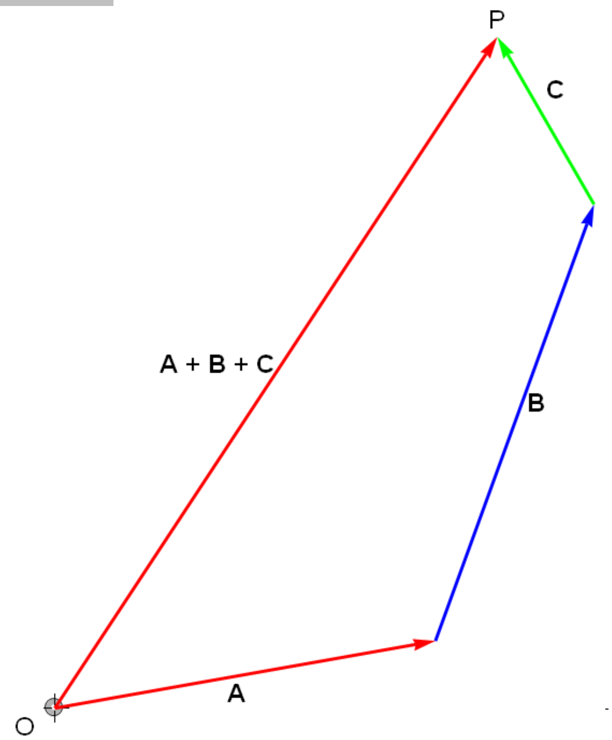

B
A


A
A - B
-B

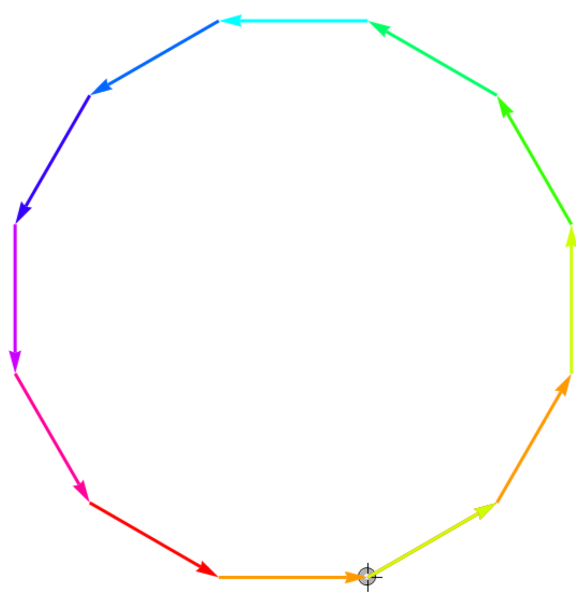

-B
A - B
A

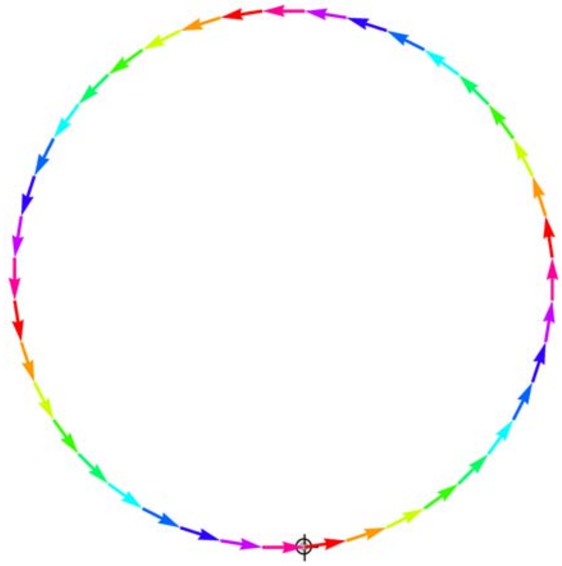

A
-B
A - B
-B
A

4. Sum of three vectors



5. Sum of many vectors





6. Imporrtant theorem for the geometry

6.1 Theorem

When the point P is between the point Q and P on the line connecting the two points P and Q, the vector $\overrightarrow{OP}$ is expressed in terms of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ by

$$\overrightarrow{OP} = \alpha\boldsymbol{A} + \beta\boldsymbol{B}$$

where $\alpha + \beta = 1$ ($\alpha > 0$ and $\beta > 0$).

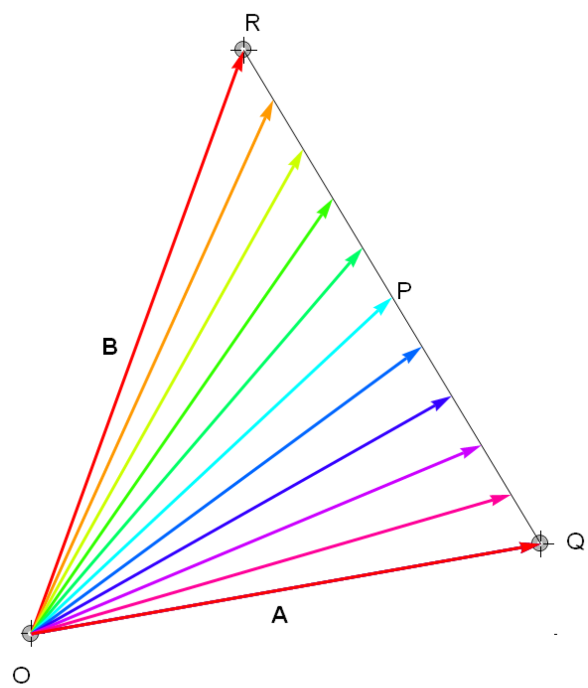
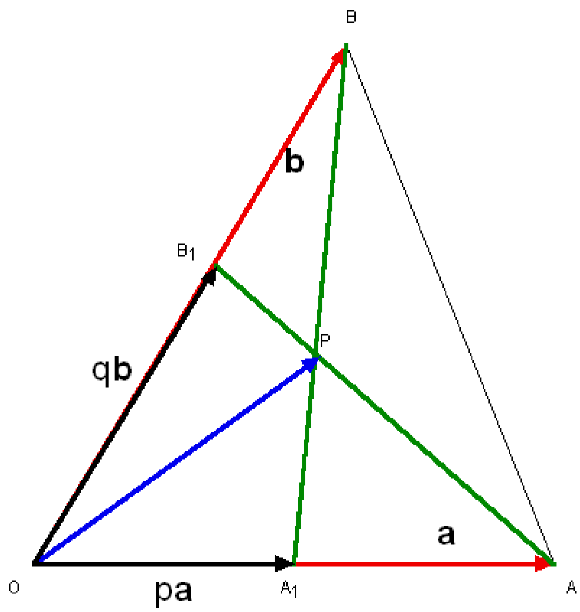


Fig. $\overrightarrow{OP} = \alpha \boldsymbol{A} + \beta \boldsymbol{B}$ where $\alpha + \beta = 1$. α is changed between $\alpha =$ 0.1 and 0.9 with $\Delta\alpha = 0.1$.

6.2 Example

We now consider the following case.



$$\overrightarrow{OA} = \boldsymbol{a}$$

$$\overrightarrow{OB} = \boldsymbol{b}$$

$$\overrightarrow{OA_1} = p\boldsymbol{a}$$

$$\overrightarrow{OB_1} = q\boldsymbol{b}$$

where p and q are between 0 and 1. From the above theorem, the vector $\overrightarrow{OP}$ is expressed by

$$\overrightarrow{OP} = \alpha(\boldsymbol{a}) + \beta(q\boldsymbol{b}) = \frac{\alpha}{p}(p\boldsymbol{a}) + \beta q(\boldsymbol{b})$$

where

$$\alpha + \beta = 1$$

$$\frac{\alpha}{p} + \beta q = 1$$

From these Eqs., we have

$$\alpha = \frac{p(1-p)}{1-pq}$$

$$\beta = \frac{1-p}{1-pq}$$

Then $\overrightarrow{OP}$ is expressed by

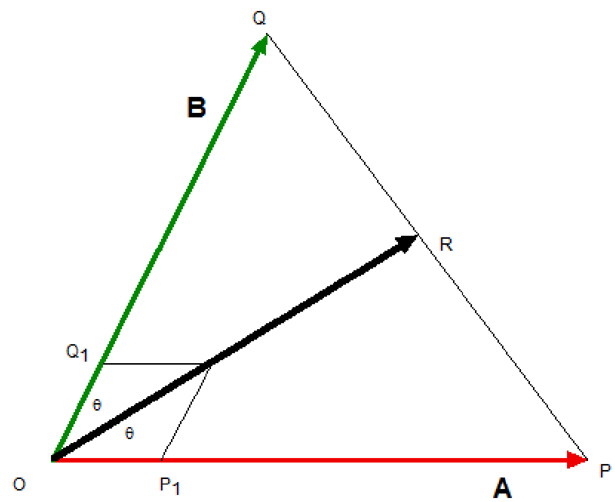
$$\overrightarrow{OP} = \frac{p(1-p)}{1-pq}\boldsymbol{a} + \frac{q(1-p)}{1-pq}\boldsymbol{b}$$

6.3 Example: bisecting vector

In a triangle of this figure, the angle POR is equal to the angle QOR. The point R is on the line PQ. What is the expression of $\overrightarrow{OR}$ in terms of the vectors ***A*** and ***B***?. Since R is on the line AB, $\overrightarrow{OR}$ is described by

$$\overrightarrow{OR} = \alpha\boldsymbol{A} + \beta\boldsymbol{B} \tag{1}$$

where $\alpha + \beta = 1$ (α>0 and β>0).



The vector $\overrightarrow{OR}$ is also described by

$$\overrightarrow{OR} = k(\hat{\boldsymbol{A}} + \hat{\boldsymbol{B}}) = k(\frac{\boldsymbol{A}}{A} + \frac{\boldsymbol{B}}{B}) \tag{2}$$

where A and B are the magnitudes of ***A*** and ***B***., $\hat{\boldsymbol{A}}$ and $\hat{\boldsymbol{B}}$ are the unit vectors for ***A*** and ***B***. From Eqs.(1) and (2), we have

$$\alpha = \frac{k}{A}$$
$$\beta = \frac{k}{B}$$

or

$$\beta = \frac{A}{B}\alpha \qquad (3)$$

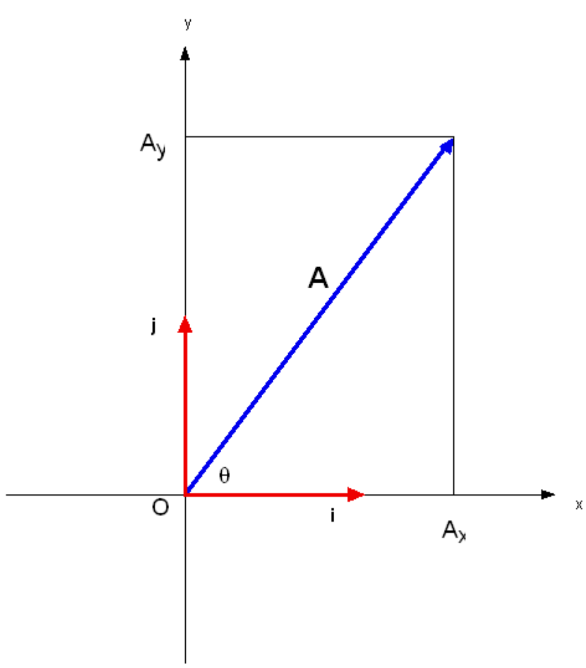
Then we get

$$\alpha = \frac{B}{A+B}$$
$$\beta = \frac{A}{A+B}$$

7. Cartesian components of vectors

7.1 2D system

Let ***I*** and ***j***, and ***k*** denote a set of mutually perpendicular unit vectors. Let ***i*** and ***j*** drawn from a common origin O, give the positive directions along the system of rectangular axes O*xy*.



We consider a vector A lying in the *xy* plane and making an angle θ with the positive *x* axis. The vector ***A*** can be expressed by

$$\boldsymbol{A} = (A_x, A_y) = A_x\boldsymbol{i} + A_y\boldsymbol{j} = A(\cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j})$$

where

$$A = |\boldsymbol{A}| = \sqrt{A_x^{\,2} + A_y^{\,2}} \qquad \text{and} \qquad \tan\theta = \frac{A_y}{A_x}$$

When the vector ***B*** is expressed by

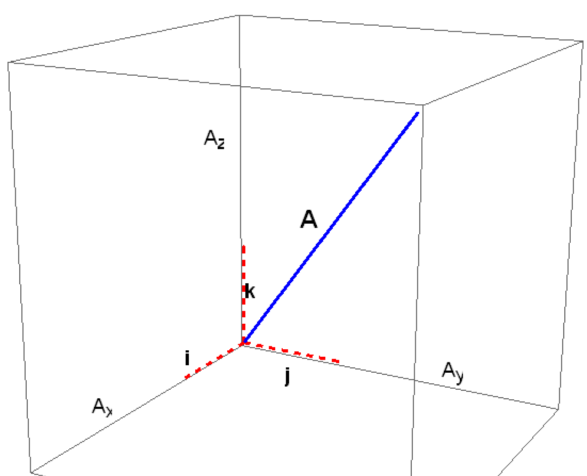
$$\boldsymbol{B} = (B_x, B_{y_z}) = B_x\boldsymbol{i} + B_y\boldsymbol{j}$$

the sum of ***A*** and ***B*** is

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\boldsymbol{i} + (A_y + B_y)\boldsymbol{j}$$

7.2 3D system

Let ***i***, ***j***, and ***k*** denote a set of mutually perpendicular unit vectors. Let ***i***, ***j***, and ***k*** drawn from a common origin O, give the positive directions along the system of rectangular axes O*xyz*.



An arbitrary vector ***A*** can be expressed by

$$\boldsymbol{A} = (A_x, A_y, A_z) = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$

where A_x, A_y, and A_z are called the Cartesian components of the vector $\boldsymbol{A}$. When the vector $\boldsymbol{B}$ is expressed by

$$\boldsymbol{B} = (B_x, B_y, B_z) = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$$

the sum of $\boldsymbol{A}$ and $\boldsymbol{B}$ is

$$\boldsymbol{A} + \boldsymbol{B} = (A_x + B_x)\boldsymbol{i} + (A_y + B_y)\boldsymbol{j} + (A_z + B_z)\boldsymbol{k}$$

8. Scalar product of vectors

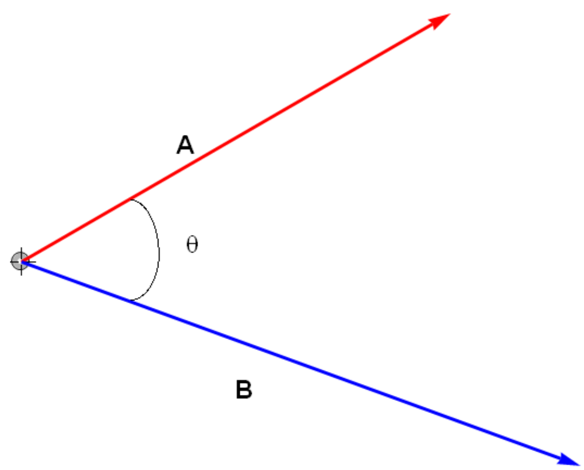
8.1 Definition

The scalar product (or dot product) of the vectors $\boldsymbol{A}$ and $\boldsymbol{B}$ is defined as

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = AB\cos\theta$$

where θ is the angle between $\boldsymbol{A}$ and $\boldsymbol{B}$ and is between 0 and π. The scalar product is a scalar and is commutative,

$$\boldsymbol{A} \cdot \boldsymbol{B} = \boldsymbol{B} \cdot \boldsymbol{A}$$



(i) **Magnitude:**

When $\boldsymbol{B} = \boldsymbol{A}$, we have

$$\boldsymbol{A} \cdot \boldsymbol{A} = |\boldsymbol{A}|^2 = A^2$$

since $\theta = 0$.

(ii) **Orthogonal ($\boldsymbol{A} \perp \boldsymbol{B}$):**

If

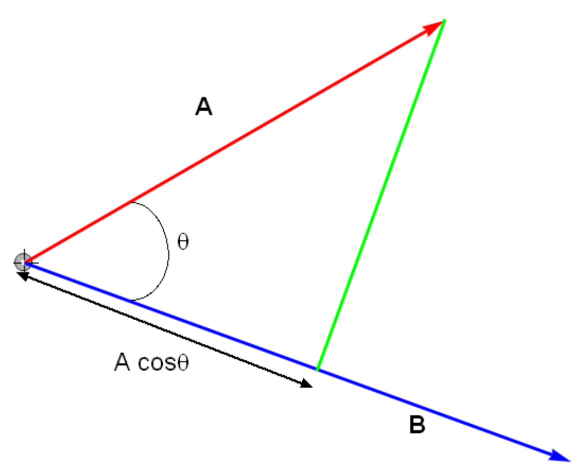
$$\boldsymbol{A} \cdot \boldsymbol{B} = 0 \;\; (A \neq 0 \text{ and } B \neq 0),$$

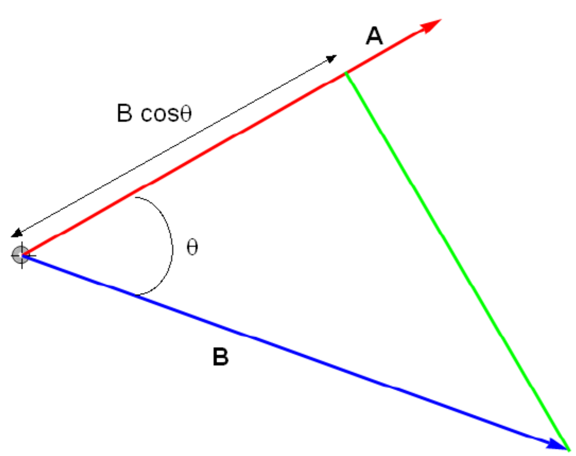
we say that $\boldsymbol{A}$ is orthogonal to B or perpendicular to $\boldsymbol{B}$.

(iii) **Projection:**

The magnitude of the projection of $\boldsymbol{A}$ on $\boldsymbol{B}$ is $A\cos\theta$. So $\mathbf{A} \cdot \mathbf{B}$ is the product of the projection of $\boldsymbol{A}$ on $\boldsymbol{B}$ with the magnitude of $\boldsymbol{A}$. We also consider that the magnitude of $\boldsymbol{A} \cdot \boldsymbol{B}$ is the product of the projection of on $\boldsymbol{B}$ on $\boldsymbol{A}$ with the magnitude of $\boldsymbol{B}$.

$$\boldsymbol{A} \cdot \boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\cos\theta = B(A\cos\theta) = A(B\cos\theta)$$





8.2 The expression of the scalar product using Cartesian components of vectors Inner product of *A* and *B*

We now consider two vectors given by

$$\boldsymbol{A} = (A_x, A_y, A_z) = A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k}$$
$$\boldsymbol{B} = (B_x, B_y, B_z) = B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}$$

The scalar product of these two vectors ***A*** and ***B*** can be expressed in terms of the components

$$\begin{aligned}\boldsymbol{A}\cdot\boldsymbol{B} &= (A_x\boldsymbol{i} + A_y\boldsymbol{j} + A_z\boldsymbol{k})\cdot(B_x\boldsymbol{i} + B_y\boldsymbol{j} + B_z\boldsymbol{k}) \\ &= (A_xB_x\boldsymbol{i}\cdot\boldsymbol{i} + A_xB_y\boldsymbol{i}\cdot\boldsymbol{j} + A_xB_z\boldsymbol{i}\cdot\boldsymbol{k}) + (A_yB_x\boldsymbol{j}\cdot\boldsymbol{i} + A_yB_y\boldsymbol{j}\cdot\boldsymbol{j} + A_yB_z\boldsymbol{j}\cdot\boldsymbol{k}) \\ &\quad + (A_zB_x\boldsymbol{k}\cdot\boldsymbol{i} + A_zB_y\boldsymbol{k}\cdot\boldsymbol{j} + A_zB_z\boldsymbol{k}\cdot\boldsymbol{k})\end{aligned}$$

or

$$\boldsymbol{A}\cdot\boldsymbol{B} = A_xB_x + A_yB_y + A_zB_z$$

Here we use the above relations for the inner products of the unit vectors.
where

$\boldsymbol{i}\cdot\boldsymbol{i} = 1$	$\boldsymbol{j}\cdot\boldsymbol{i} = 0$	$\boldsymbol{k}\cdot\boldsymbol{i} = 0$
$\boldsymbol{i}\cdot\boldsymbol{j} = 0$	$\boldsymbol{j}\cdot\boldsymbol{j} = 1$	$\boldsymbol{k}\cdot\boldsymbol{j} = 0$
$\boldsymbol{i}\cdot\boldsymbol{k} = 0$	$\boldsymbol{j}\cdot\boldsymbol{k} = 0$	$\boldsymbol{k}\cdot\boldsymbol{k} = 1$

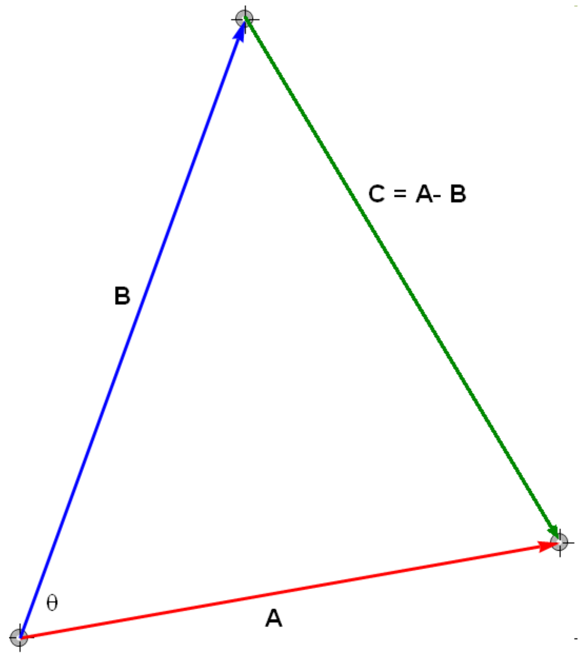
In special cases, the components of ***A*** are given by

$$\boldsymbol{A}\cdot\boldsymbol{i} = A_x\boldsymbol{i}\cdot\boldsymbol{i} + A_y\boldsymbol{j}\cdot\boldsymbol{i} + A_z\boldsymbol{k}\cdot\boldsymbol{i} = A_x$$
$$\boldsymbol{A}\cdot\boldsymbol{j} = A_x\boldsymbol{i}\cdot\boldsymbol{j} + A_y\boldsymbol{j}\cdot\boldsymbol{j} + A_z\boldsymbol{k}\cdot\boldsymbol{j} = A_y$$
$$\boldsymbol{A}\cdot\boldsymbol{i} = A_x\boldsymbol{i}\cdot\boldsymbol{k} + A_y\boldsymbol{j}\cdot\boldsymbol{k} + A_z\boldsymbol{k}\cdot\boldsymbol{k} = A_z$$

The unit vector $\hat{\mathbf{A}}$ of the vector A is expressed by

$$\begin{aligned}\hat{\mathbf{A}} &= \frac{1}{A}(A_x, A_y, A_z) = \frac{A_x}{A}\boldsymbol{i} + \frac{A_y}{A}\boldsymbol{j} + \frac{A_z}{A}\boldsymbol{k} \\ &= \frac{\boldsymbol{A}\cdot\boldsymbol{i}}{A}\boldsymbol{i} + \frac{\boldsymbol{A}\cdot\boldsymbol{j}}{A}\boldsymbol{j} + \frac{\boldsymbol{A}\cdot\boldsymbol{k}}{A}\boldsymbol{k}\end{aligned}$$

8.3 Law of cosine



$$C^2 = \boldsymbol{C}\cdot\boldsymbol{C} = (\boldsymbol{A}-\boldsymbol{B})\cdot(\boldsymbol{A}-\boldsymbol{B}) = A^2 + B^2 - 2(\boldsymbol{A}\cdot\boldsymbol{B})$$
$$= A^2 + B^2 - 2AB\cos\theta$$

This is the famous trigonometric relation (law of cosine).

9. Vector product

9.1 Definition

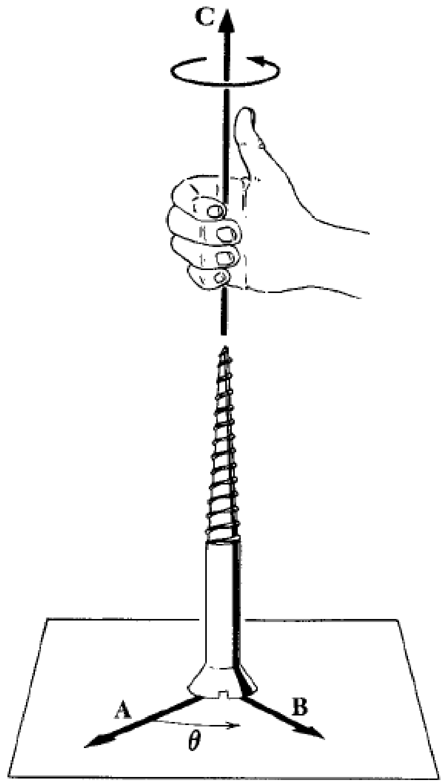
This product is a vector rather than scalar in character, but it is a vector in a somewhat restricted sense. The vector product of ***A*** and ***B*** is defined as

$$\boldsymbol{C} = \boldsymbol{A}\times\boldsymbol{B} = |\boldsymbol{A}||\boldsymbol{B}|\sin\theta\hat{\boldsymbol{n}} = AB\sin\theta\hat{\boldsymbol{n}}$$

where $|\boldsymbol{A}|$ is the magnitude of ***A***. $|\mathbf{B}|$ is the magnitude of ***B***. θ is the angle between ***A*** and ***B***. $\hat{\mathbf{n}}$ is a unit vector, perpendicular to both ***A*** and ***B*** in a sense defined by the right hand thread rule.

We read ***A*** x ***B*** as "***A*** cross ***B***."

The vector ***A*** is rotated by the smallest angle that will bring it into coincidence with the direction of ***B***. The sense of ***C*** is that of the direction of motion of a screw with a right-hand thread when the screw is rotated in the same as was the vector ***A***.



Right-hand-thread rule.

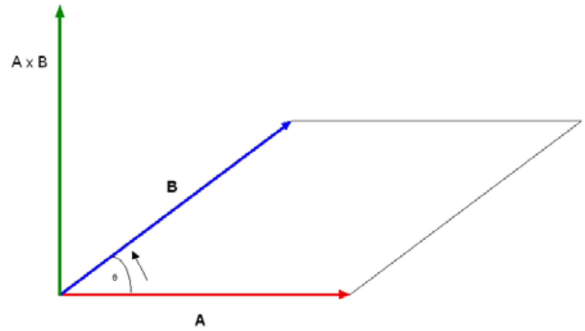
((Note))

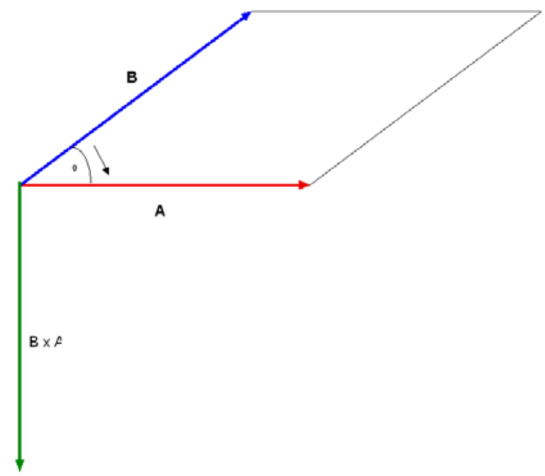
The vector ***C*** is perpendicular to both A and B. Rotate ***A*** into ***B*** through the lesser of the two possible angles – curl the fingers of the right hand in the direction in which ***A*** is rotated, and the thumb will point in the direction of ***C*** = ***A*** x ***B***.

(i)

Because of the sign convention, ***B*** x ***A*** is a vector opposite sign to ***A*** x ***B***. In other words, the vector product is not commutative,

$$\boldsymbol{B}\times\boldsymbol{A} = -\boldsymbol{A}\times\boldsymbol{B}\,.$$





(ii)

It follows from the definition of the vector product that

$$\boldsymbol{A} \times \boldsymbol{A} = 0$$

(iii)

The vector product obey the distributive law.

$$\boldsymbol{A} \times (\boldsymbol{B} + \boldsymbol{C}) = \boldsymbol{A} \times \boldsymbol{B} + \boldsymbol{A} \times \boldsymbol{C}$$

9.2 Cartesian components.

The vectors ***A*** and **B** are expressed by

$$\boldsymbol{A} = (A_x, A_y, A_z) = A_x \boldsymbol{i} + A_y \boldsymbol{j} + A_z \boldsymbol{k}$$
$$\boldsymbol{B} = (B_x, B_y, B_z) = B_x \boldsymbol{i} + B_y \boldsymbol{j} + B_z \boldsymbol{k}$$

Then the vector product ***A*** x ***B*** is expressed in terms of the Cartesian components

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} &= (A_x\boldsymbol{i}+A_y\boldsymbol{j}+A_z\boldsymbol{k})\times(B_x\boldsymbol{i}+B_y\boldsymbol{j}+B_z\boldsymbol{k})\\ &= (\boldsymbol{i}\times\boldsymbol{i})A_xB_x+(\boldsymbol{i}\times\boldsymbol{j})A_xB_y+(\boldsymbol{i}\times\boldsymbol{k})A_xB_z\\ &\quad +(\boldsymbol{j}\times\boldsymbol{i})A_yB_x+(\boldsymbol{j}\times\boldsymbol{j})A_yB_y+(\boldsymbol{j}\times\boldsymbol{k})A_yB_z\\ &\quad +(\boldsymbol{k}\times\boldsymbol{i})A_zB_x+(\boldsymbol{k}\times\boldsymbol{j})A_zB_y+(\boldsymbol{k}\times\boldsymbol{k})A_zB_z\end{aligned}$$

Here we use the relations,

$$\begin{array}{lll} \boldsymbol{i}\times\boldsymbol{i}=0 & \boldsymbol{j}\times\boldsymbol{i}=-\boldsymbol{k} & \boldsymbol{k}\times\boldsymbol{i}=\boldsymbol{j}\\ \boldsymbol{i}\times\boldsymbol{j}=\boldsymbol{k} & \boldsymbol{j}\times\boldsymbol{j}=0 & \boldsymbol{k}\times\boldsymbol{j}=-\boldsymbol{i}\\ \boldsymbol{i}\times\boldsymbol{k}=-\boldsymbol{j} & \boldsymbol{j}\times\boldsymbol{k}=\boldsymbol{i} & \boldsymbol{k}\times\boldsymbol{k}=0 \end{array}$$

$$\begin{aligned}\boldsymbol{A}\times\boldsymbol{B} &= \boldsymbol{k}A_xB_y-\boldsymbol{j}A_xB_z-\boldsymbol{k}A_yB_x+\boldsymbol{i}A_yB_z+\boldsymbol{j}A_zB_x-\boldsymbol{i}A_zB_y\\ &= \boldsymbol{i}(A_yB_z-A_zB_y)+\boldsymbol{j}(A_zB_x-A_xB_z)+\boldsymbol{k}(A_xB_y-A_yB_x)\\ &= \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}\end{aligned}$$

It is easier for one to remember if the determinant is used.

Using the cofactor, $\boldsymbol{A}\times\boldsymbol{B}$ can be simplified as

$$\boldsymbol{A}\times\boldsymbol{B}=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}=\boldsymbol{i}\begin{vmatrix}A_y & A_z\\ B_y & B_z\end{vmatrix}-\boldsymbol{j}\begin{vmatrix}A_x & A_z\\ B_x & B_z\end{vmatrix}+\boldsymbol{k}\begin{vmatrix}A_x & A_y\\ B_x & B_y\end{vmatrix}.$$

where a 2x2 determinant is given by $\begin{vmatrix}a & b\\ c & d\end{vmatrix}=ad-bc$.

Note

$$\boldsymbol{C}\cdot(\boldsymbol{A}\times\boldsymbol{B})=\begin{vmatrix}C_x & C_y & C_z\\ A_x & A_y & A_z\\ B_x & B_y & B_z\end{vmatrix}$$

((**Mathematica**))

```
A = {A1, A2, A3}
```

{A1, A2, A3}

```
B = {B1, B2, B3}
```

{B1, B2, B3}

```
CC = {C1, C2, C3}
```

{C1, C2, C3}

```
DD = {D1, D2, D3}
```

{D1, D2, D3}

```
Cross[A, B]
```

{-A3 B2 + A2 B3, A3 B1 - A1 B3, -A2 B1 + A1 B2}

```
A.B
```

A1 B1 + A2 B2 + A3 B3

```
Cross[CC, Cross[A, B]] // Simplify
```

{-A2 B1 C2 + A1 B2 C2 - A3 B1 C3 + A1 B3 C3,
 -B2 (A1 C1 + A3 C3) + A2 (B1 C1 + B3 C3),
 -B3 (A1 C1 + A2 C2) + A3 (B1 C1 + B2 C2)}

```
Cross[Cross[CC, DD], Cross[A, B]] //
 Simplify
```

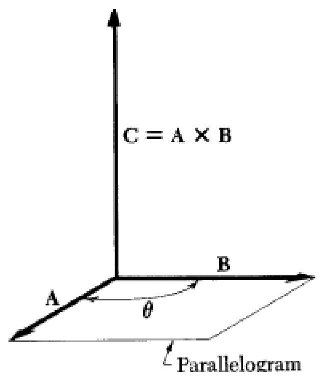
{A3 B1 (C2 D1 - C1 D2) +
  A2 B1 (-C3 D1 + C1 D3) + A1
   (-B3 C2 D1 + B2 C3 D1 + B3 C1 D2 - B2 C1 D3),
 A3 B2 (C2 D1 - C1 D2) +
  A1 B2 (C3 D2 - C2 D3) + A2
   (-B3 C2 D1 + B3 C1 D2 - B1 C3 D2 + B1 C2 D3),
 B3 (-A2 C3 D1 + A1 C3 D2 + A2 C1 D3 - A1 C2 D3) +
  A3 (B2 C3 D1 - B1 C3 D2 - B2 C1 D3 + B1 C2 D3)}

9.3 Application of the vector product

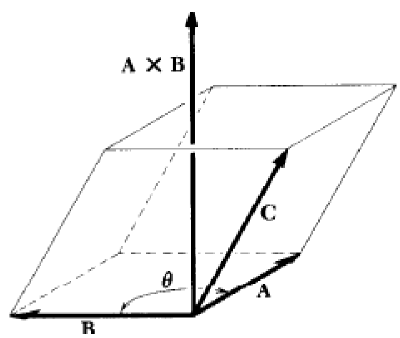
9.3.1 Area of parallelogram

The magnitude of ***A*** x ***B*** is the area of the parallelogram.

$$|\boldsymbol{A} \times \boldsymbol{B}| = AB|\sin\theta|$$



9.3.2 Volume of a parallelepiped

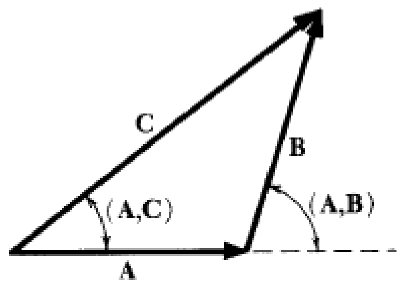


The scalar given by

$$|(\boldsymbol{A} \times \boldsymbol{B}) \cdot \boldsymbol{C}| = V$$

is the volume of parallelepiped

9.3.3 Law of sine



We consider the triangle defined by $\boldsymbol{C} = \boldsymbol{A} + \boldsymbol{B}$, and take the vector product

$$\boldsymbol{A}\times\boldsymbol{C} = \boldsymbol{A}\times(\boldsymbol{A}+\boldsymbol{B}) = \boldsymbol{A}\times\boldsymbol{A} + \boldsymbol{A}\times\boldsymbol{B} = \boldsymbol{A}\times\boldsymbol{B}$$

The magnitude of both sides must be equal so that

$$AC\sin(A,C) = AB\sin(A,B) = AB\sin(\pi-(A,B))$$

or

$$\frac{\sin(A,C)}{B} = \frac{\sin[\pi-(A,B)]}{C} \qquad \text{(Law of sine).}$$

where sin(***A***, ***B***) denotes the sine of the angle between ***A*** and ***B***.

10. Example

10.1
Problem 3-42 (8-th edition)
Problem 3-44 (9-th, 10-th edition)

In the product $\mathbf{F} = q(\mathbf{v}\times\mathbf{B})$, take $q = 2$,

$\mathbf{v} = 2.0\mathbf{i} + 4.0\mathbf{j} + 6.0\mathbf{k}$,and $\mathbf{F} = 4.0\mathbf{i} - 20\mathbf{j} + 12\mathbf{k}$,

What then is ***B*** in unit-vector notation if $B_x = B_y$?

((**Solution**))

$$\mathbf{F} = q(\mathbf{v}\times\mathbf{B}) = 2\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 4 & 6 \\ B_x & B_x & B_z \end{vmatrix} = 2(4B_z - 6B_x, 6B_x - 2B_z, -2B_x) = (4,-20,12)$$

or

$$2B_z - 3B_x = 1$$
$$3B_x - B_z = -5$$
$$B_x = -3$$

or

$$B_z = -4$$
$$B_x = -3$$

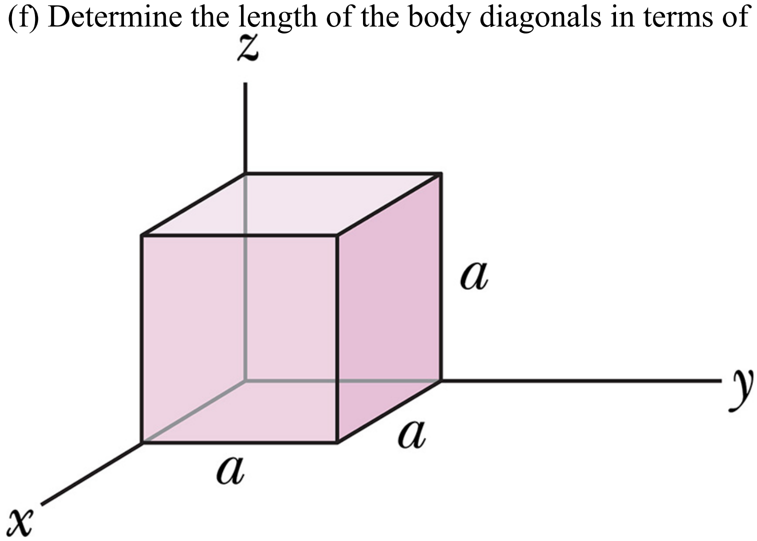
((**Mathematica**))

```
V = {2, 4, 6}

{2, 4, 6}

B = {Bx, Bx, Bz}

{Bx, Bx, Bz}

F = {4, -20, 12}

{4, -20, 12}

eq1 = F - 2 Cross[v, B] // Simplify

{4 + 12 Bx - 8 Bz, -4 (5 + 3 Bx - Bz), 4 (3 + Bx)}

eq11 = eq1[[1]] == 0

4 + 12 Bx - 8 Bz == 0

eq12 = eq1[[2]] == 0

-4 (5 + 3 Bx - Bz) == 0

Solve[{eq11, eq12}, {Bx, Bz}]

{{Bx → -3, Bz → -4}}
```

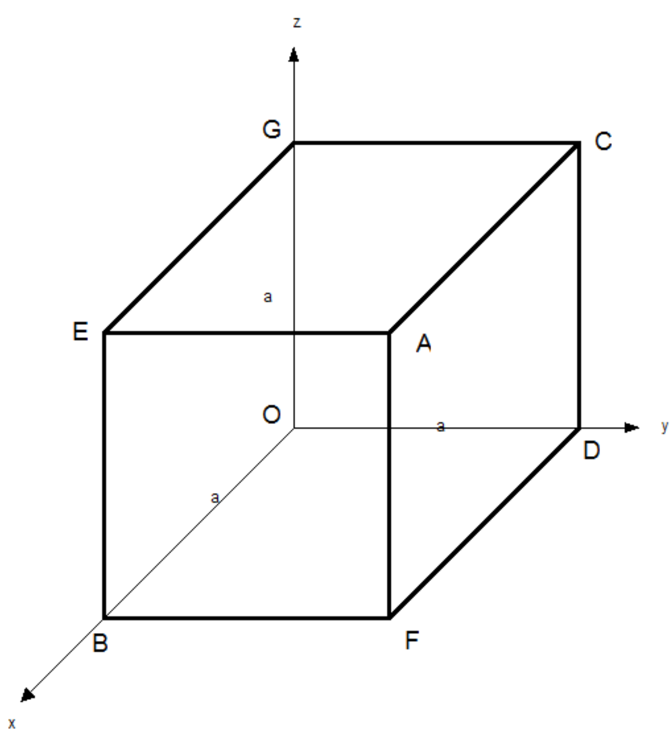
Problem 3-31*(from SP-3)   (8-th edition)**

Problem 3-32* (form SP-3) (9-th, 10-th edition)**

In Fig., a cube of edge length *a* sits with one corner at the origin of an *xyz* coordinate system. A body diagonal is a line that extends from one corner to another through the center. In unit-vector notation, what is the body diagonal that extends from the corner at (a) coordinates (0, 0, 0), (b) coordinates (*a*, 0, 0), (c) coordinates (0, *a*, 0), and (d) coordinates (a, a, 0)? (e) Determine the angles that the body diagonals make with the adjacent edges. (f) Determine the length of the body diagonals in terms of *a*.



((**Solution**))



```
B1 = {a, 0, 0}; C1 = {0, a, a}; E1 = {a, 0, a};
A1 = {a, a, a}; D1 = {0, a, 0}; F1 = {a, a, 0};
G1 = {0, 0, a}
```

{0, 0, a}

```
BC1 = C1 - B1
```

{-a, a, a}

```
DE1 = E1 - D1
```

{a, -a, a}

```
FG = G1 - F1
```

{-a, -a, a}

```
EA1 = A1 - E1
```

{0, a, 0}

```
A1.EA
```

a^2

$\sqrt{EA1.EA1}$ // Simplify[#, a > 0] &

a

$\sqrt{A1.A1}$ // Simplify[#, a > 0] &

$\sqrt{3}\ a$

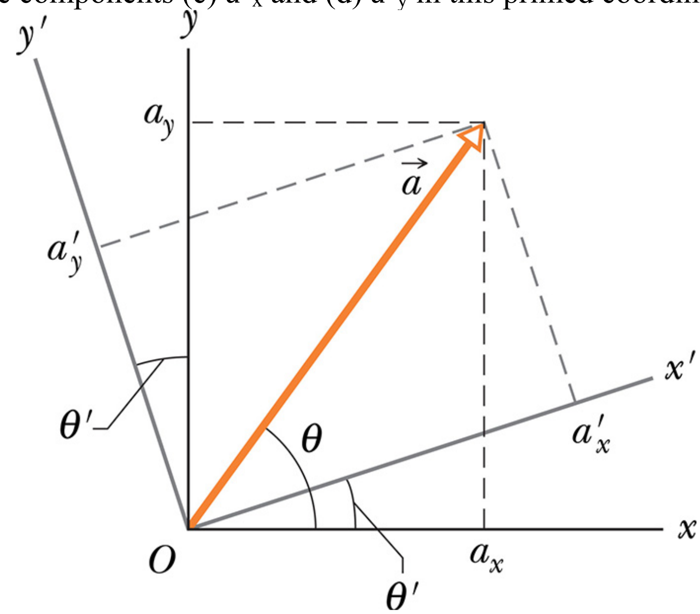
$\frac{180}{\pi}\ \text{ArcCos}\left[\frac{EA1.A1}{\sqrt{A1.A1}\ \sqrt{EA1.EA1}}\right]$ // N

54.7356

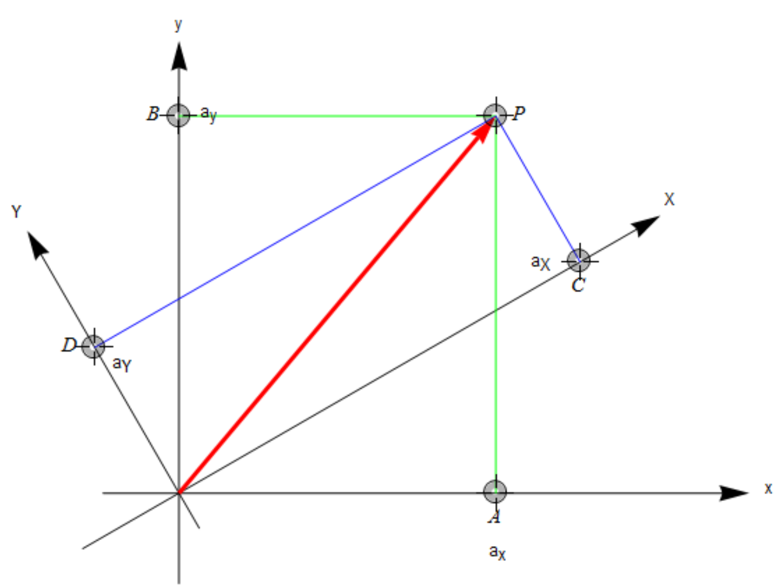
10.3

Problem 3-32 (from SP-3) (8-th edition)
Problem 3-31 (from SP-3) (9-th, 10-th edition)

In Fig., a vector ***a*** with a magnitude of 17.0 m is directed at angle θ = 56.0° counterclockwise from the $+x$ axis. What are the components (a) a_x and (b) a_y of the vector? A second coordinate system is inclined by angle θ' = 18.0° with respect to the first. What are the components (c) a'_x and (d) a'_y in this primed coordinate system?



Instead of solving this problem, here we consider the more general case shown below.



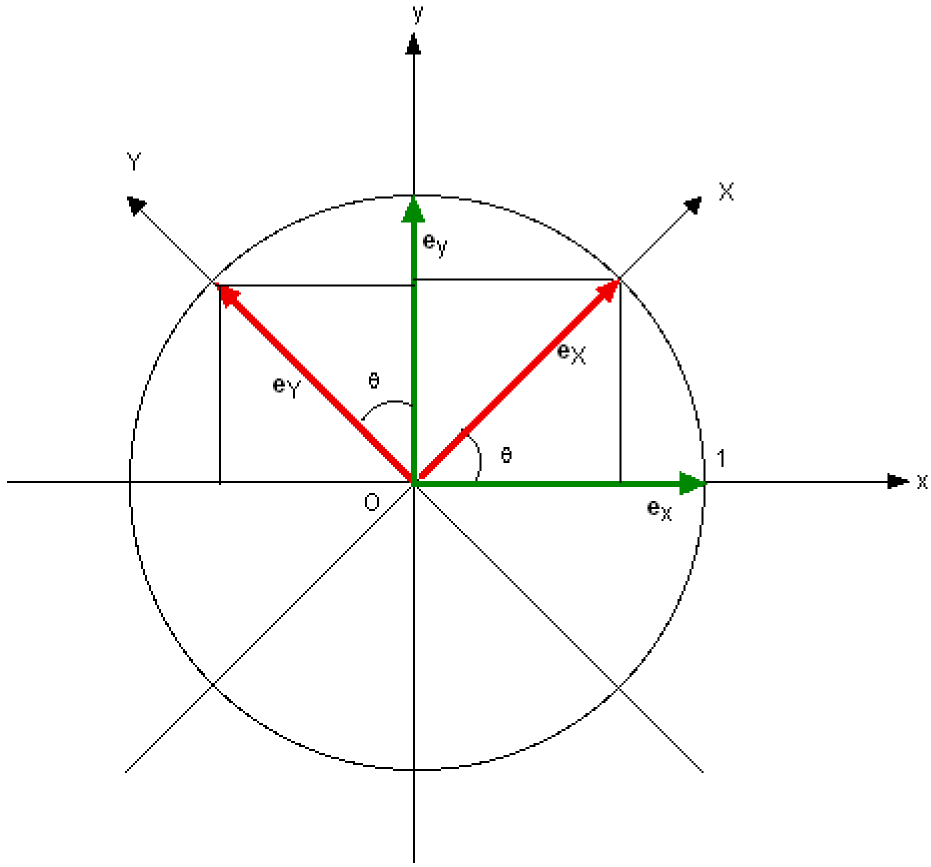
We assume that

$$\mathbf{a} = a_x\hat{e}_x + a_y\hat{e}_y = a_X\hat{e}_X + a_Y\hat{e}_Y$$

where the unit vectors are related through a relation

$$\hat{e}_X = (\cos\theta)\hat{e}_x + (\sin\theta)\hat{e}_y$$
$$\hat{e}_Y = (-\sin\theta)\hat{e}_x + (\cos\theta)\hat{e}_y$$

where θ is the angle between the x axis and X axis.



or

$$\hat{e}_x = (\cos\theta)\hat{e}_X + (-\sin\theta)\hat{e}_Y$$
$$\hat{e}_y = (\sin\theta)\hat{e}_X + (\cos\theta)\hat{e}_Y$$

Then we have

$$a_X \hat{e}_X + a_Y \hat{e}_Y = a_x \hat{e}_x + a_y \hat{e}_y$$
$$= a_x (\cos\theta \hat{e}_X - \sin\theta \hat{e}_Y) + a_y (\sin\theta \hat{e}_X + \cos\theta \hat{e}_Y)$$

or

$$a_X \hat{e}_X + a_Y \hat{e}_Y =$$
$$= (a_x \cos\theta + a_y \sin\theta)\hat{e}_X + (-a_x \sin\theta + a_y \cos\theta)\hat{e}_Y$$

or

$$a_X = a_x \cos\theta + a_y \sin\theta$$
$$a_Y = -a_x \sin\theta + a_y \cos\theta$$

or

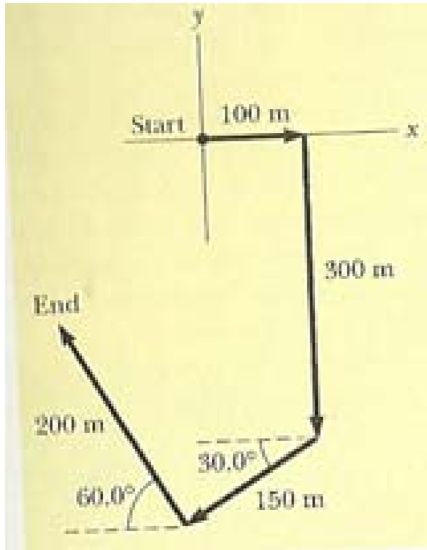
$$\begin{pmatrix} a_X \\ a_Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

or

$$\begin{pmatrix} a_x \\ a_y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_X \\ a_Y \end{pmatrix}$$

11. Problem
3-59 (Serway and Jewett)

A person going for a walk, follows the path shown in Fig. The total trip consists of four straight-line paths. At the end of the walk, what is the person's resultant displacement measured from the starting point.




```
A1 = {100, 0};

B1 = {0, -300};

C1 = {150 Cos[-150 °], 150 Sin[-150 °]};

D1 = {200 Cos[120 °], 200 Sin[120 °]};

E1 = A1 + B1 + C1 + D1 // N
{-129.904, -201.795}
```

$\sqrt{E1.E1}$ // N

239.992

```
eq1 = ArcTan[E1[[2]] / E1[[1]]] // N
0.998833

eq1 (180 / π) // N
57.2289
```

(Angle = 180 + 57.2 = 237.2°

```
Graphics[{Red, Thick, Arrow[{{0, 0}, A1}], Arrow[{A1, A1 + B1}],
  Arrow[{A1 + B1, A1 + B1 + C1}], Arrow[{A1 + B1 + C1, A1 + B1 + C1 + D1}],
  Black, Thin, Arrow[{{0, -500}, {0, 150}}] ,
  Arrow[{{-200, 0}, {200, 0}}], Green, Thick, Arrow[{{0, 0}, E1}],
  Black, Text[Style["x", 12], {210, 0}],
  Text[Style["y", 12], {0, 160}], Blue, Thick,
  Text[Style["A", Bold, 12], {50, 10}],
  Text[Style["B", Bold, 12], {110, -150}],
  Text[Style["C", Bold, 12], {50, -350}],
  Text[Style["D", Bold, 12], {-80, -330}]}]
```