Lecture Note (Chapter 4)

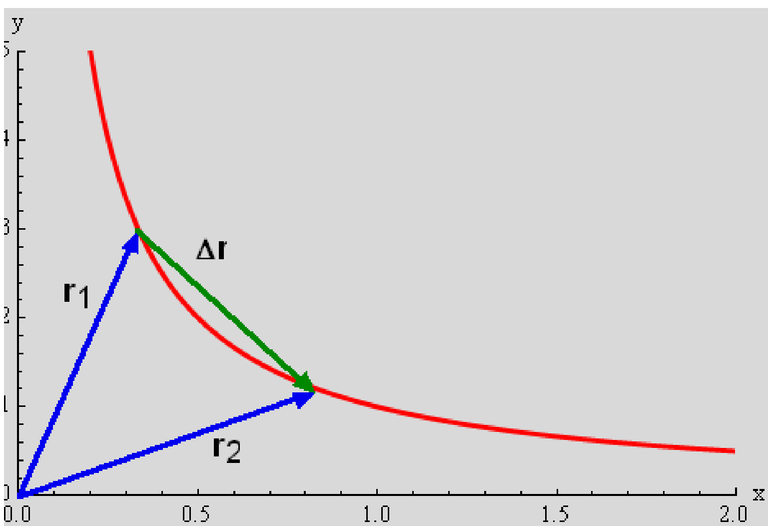
1. Position, velocity, and acceleration for a particle moving in the *xy* plane

The position vector $\boldsymbol{r}$ is defined by

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} = (x, y)$$

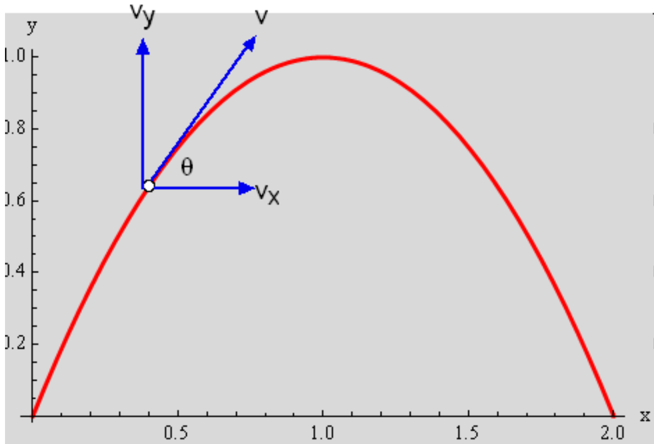
The average velocity is defined by

$$\boldsymbol{v}_{avg} = \frac{\Delta \boldsymbol{r}}{\Delta t} = (\frac{\Delta x}{\Delta t}\boldsymbol{i} + \frac{\Delta y}{\Delta t}\boldsymbol{j}) = (\frac{\Delta x}{\Delta t}, \frac{\Delta y}{\Delta t})$$



((**Note**))

The average velocity is the ratio of the displacement to the time interval for the displacement. The direction of the average velocity is the direction of the displacement vector.



The (instantaneous) velocity vector is defined by

$$\boldsymbol{v} = \frac{d\boldsymbol{r}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \boldsymbol{r}}{\Delta t} = \frac{dx}{dt}\boldsymbol{i} + \frac{dy}{dt}\boldsymbol{j} = v_x \boldsymbol{i} + v_y \boldsymbol{j} = (v_x, v_y) .$$

The average acceleration vector is defined by

$$\boldsymbol{a}_{avg} = \frac{\Delta v}{\Delta t} = \left(\frac{\Delta v_x}{\Delta t}\boldsymbol{i} + \frac{\Delta v_y}{\Delta t}\boldsymbol{j}\right) = \left(\frac{\Delta v_x}{\Delta t}, \frac{\Delta v_y}{\Delta t}\right) .$$

The (instantaneous) acceleration vector:

$$\boldsymbol{a} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2 x}{dt^2}\boldsymbol{i} + \frac{d^2 y}{dt^2}\boldsymbol{j} = a_x \boldsymbol{i} + a_y \boldsymbol{j} = (a_x, a_y) .$$

The magnitudes of these vectors are

$$r = |\boldsymbol{r}| = \sqrt{x^2 + y^2}$$

$$v = |\boldsymbol{v}| = \sqrt{{v_x}^2 + {v_y}^2}$$

$$a = |\boldsymbol{a}| = \sqrt{{a_x}^2 + {a_y}^2}$$

((**Note**))
The slope of the tangential line is equal to v_y/v_x since

$$\tan\theta = \frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{v_y}{v_x} ,$$

where

$$v_x = v\cos\theta$$
$$v_y = v\sin\theta$$

2. Constant acceleration

We now consider a simple case that $\boldsymbol{a}$ = constant. Then we have the vector form

$$\frac{d\mathbf{v}}{dt} = \boldsymbol{a}$$
$$\frac{d\mathbf{r}}{dt} = \boldsymbol{v}$$

or

$$\boldsymbol{a} = const$$
$$\boldsymbol{v} = \boldsymbol{a}t + \boldsymbol{v}_0$$
$$\boldsymbol{r} = \boldsymbol{r}_0 + \boldsymbol{v}_0 t + \frac{1}{2}\boldsymbol{a}t^2$$

Using x and y components, we have

$$v_x = v_{0x} + a_x t$$
$$x = x_0 + v_{0x}t + \frac{1}{2}a_x t^2$$
$$v_x{}^2 - v_{0x}{}^2 = 2a_x(x - x_0)$$

$$v_y = v_{0y} + a_y t$$
$$y = y_0 + v_{0y}t + \frac{1}{2}a_y t^2$$
$$v_y{}^2 - v_{0y}{}^2 = 2a_y(y - y_0)$$

3. Projectiles

3.1 Formulation

Old Faithful Geyser in Yellowstone National Park (Wyoming)

Old Faithful Geyser in Yellowstone National Park, Wyoming (July, 2007, picture taken by the author). Old Faithful was named by the first official expedition to Yellowstone, the Washburn Expedition of 1870. They were impressed by its size and frequency. Old Faithful erupts 90 minutes for 1 1/2 to 5 minutes. Its maximum height is up to 184 feet (≈ 56 m).

We now consider a projectile motion

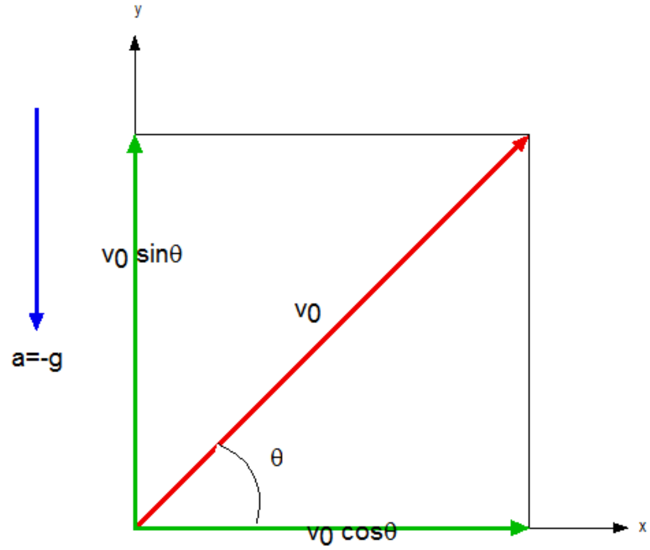
$$\begin{aligned} a_x &= 0 \\ a_y &= -g \end{aligned}, \qquad \begin{aligned} v_{0x} &= v_0 \cos\theta \\ v_{0y} &= v_0 \sin\theta \end{aligned}$$

Then we have

$$\begin{aligned} v_x &= v_0 \cos\theta \\ x &= x_0 + v_0 t \cos\theta \end{aligned}$$

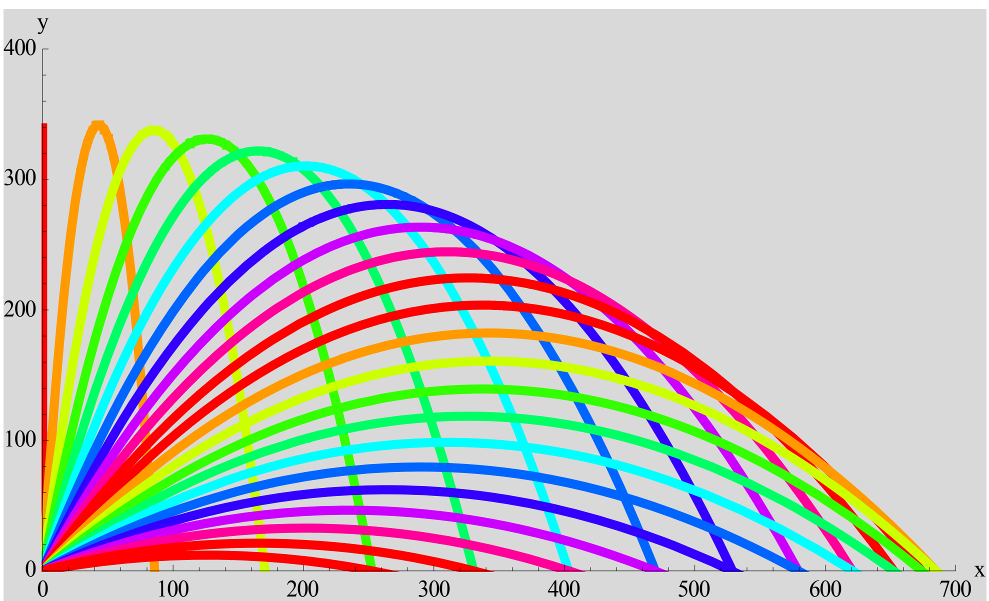
$$\begin{aligned} v_y &= v_0 \sin\theta - gt \\ y &= y_0 + v_0 t \sin\theta - \frac{1}{2} g t^2 \\ v_y{}^2 - v_0{}^2 \sin^2\theta &= -2g(y - y_0) \end{aligned}$$

A key feature of projectile motion is that the horizontal motion is independent of the vertical motion.



Initial velocity in projectile motic

3.2 The equation of the path



Here we assume that $x_0 = 0$ and $y_0 = 0$.

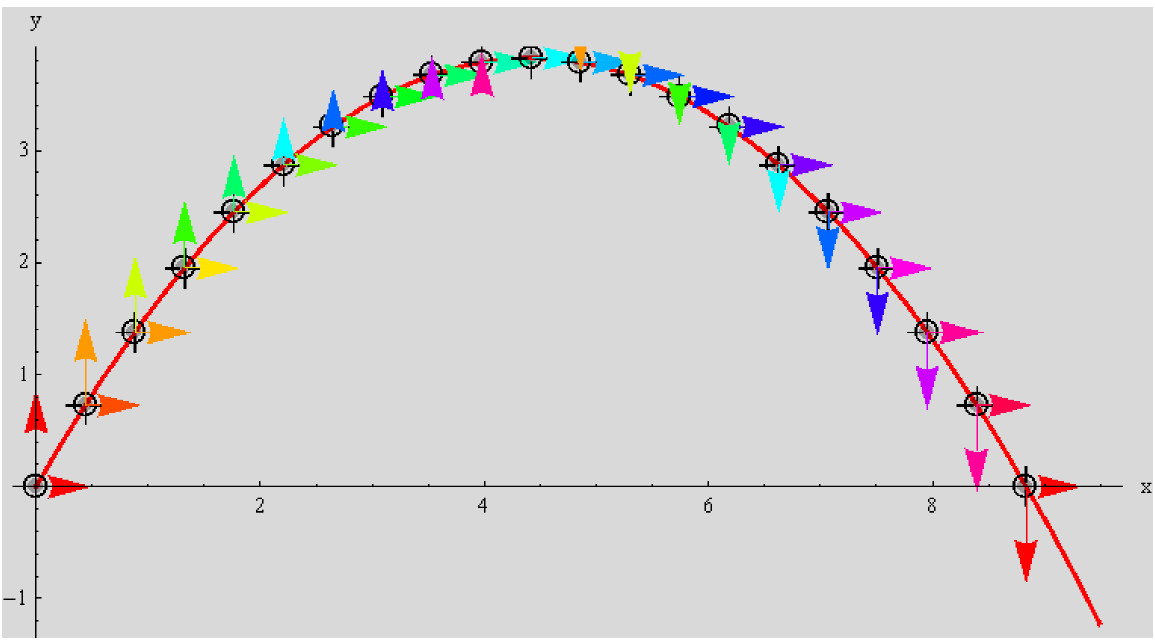
$$x = v_0 \cos\theta t$$

$$y = v_0 \sin\theta t - \frac{1}{2} g t^2$$

Since $t = \dfrac{x}{v_0 \cos\theta}$, we have

$$y = v_0 \sin\theta\left(\frac{x}{v_0 \cos\theta}\right) - \frac{1}{2} g\left(\frac{x}{v_0 \cos\theta}\right)^2 = v_0 \tan\theta x - \frac{g}{2{v_0}^2 \cos^2\theta} x^2$$

This figure shows the path of a projectile that starts at (or path through) the origin at time t = 0. The position and velocity components are shown at equal time intervals. The x component of acceleration is zero, so v_x is constant. The y component of the acceleration is constant an not zero, so v_y changes by equal amounts in equal times, just the same as if the projectile were launched vertically with the same initial velocity. At the highest point in the trajectory, $v_y = 0$.



3.3 The maximum height

We start with the two equations,

$${v_y}^2 - {v_0}^2 \sin^2\theta = -2gy$$
$$v_y = v_0 \sin\theta - gt$$

At the highest point, we have $v_y = 0$ and $y = H$ at $t = t_{\text{max-height}}$.

$$-{v_0}^2 \sin^2\theta = -2gH$$
$$0 = v_0 \sin\theta - gt_{\max-height}$$

or

$$H = \frac{v_0{}^2 \sin^2\theta}{2g}$$

$$t_{\max-hight} = \frac{v_0 \sin\theta}{g}$$

3.4 The horizontal range

We start with the two equations

$$x = v_0 t \cos\theta$$

$$y = v_0 t \sin\theta - \frac{1}{2} g t^2$$

When $y = 0$, $x = R$ and $t = t_{\text{flight}}$. Then we get

$$R = v_0 (\cos\theta) t_{flight}$$

$$v_0 \sin\theta - \frac{1}{2} g t_{flight} = 0$$

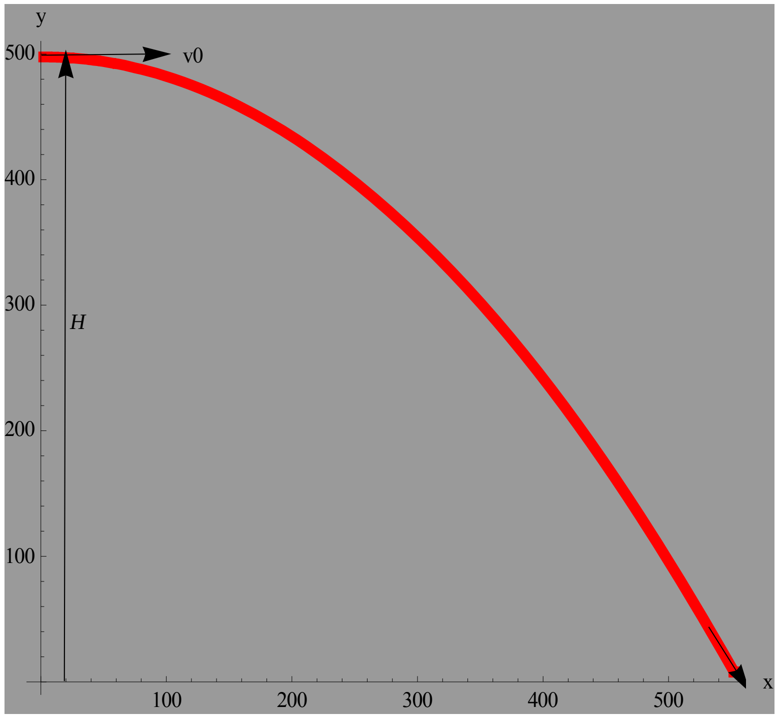
or

$$t_{flight} = \frac{2v_0 \sin\theta}{g}$$

$$R = v_0 (\cos\theta) t_{flight} = v_0 \cos\theta \frac{2v_0 \sin\theta}{g} = \frac{2v_0{}^2}{g} \sin\theta \cos\theta = \frac{v_0{}^2}{g} \sin(2\theta)$$

4. Horizontal flight

We consider the case when the initial velocity is directed along the positive x direction.



We set up the following equations

$$x = v_0 t$$
$$y = H - \frac{1}{2} g t^2$$

$$v_x = v_0$$
$$v_y = -gt$$

When $y = 0$, we have $t = \sqrt{\frac{2H}{g}}$. Then the flight distance along the *x* direction, *d*, is

$$d = v_0 \sqrt{\frac{2H}{g}}$$

The velocity at $t = \sqrt{\frac{2H}{g}}$ is given by

$$v_x = v_0$$
$$v_y = -g\sqrt{\frac{2H}{g}} = -\sqrt{2gH}$$

5. Uniform circular motion

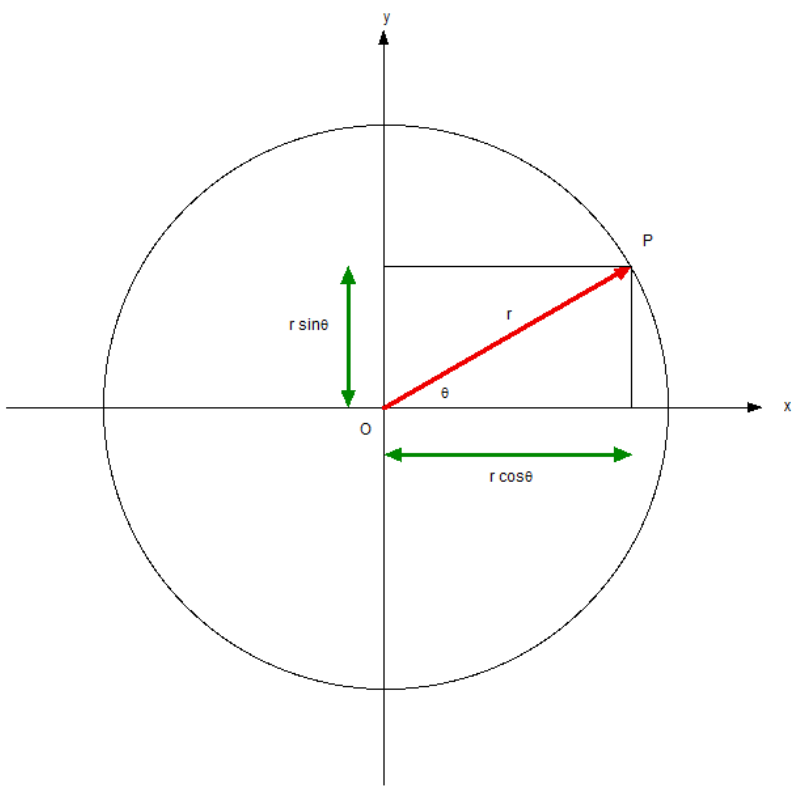
5.1 Polar coordinates

We define the polar coordinate in the *x*-*y* plane;

$$x = r\cos\theta$$
$$y = r\sin\theta$$

The position vector $\overrightarrow{OP}$ is expressed by

$$\overrightarrow{OP} = \mathbf{r} = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$$

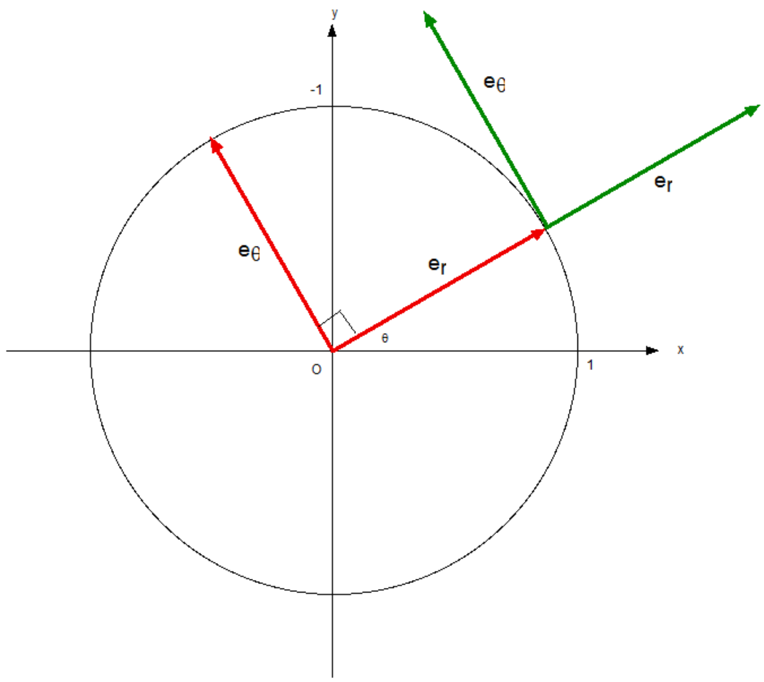


The two unit vectors (radial component and tangential component) are given by

$$\mathbf{e}_r = (\cos\theta, \sin\theta)$$

$$\mathbf{e}_\theta = (\cos(\theta + \frac{\pi}{2}), \sin(\theta + \frac{\pi}{2})) = (-\sin\theta, \cos\theta)$$

$$\mathbf{e}_r \cdot \mathbf{e}_\theta = 0$$



5.2 Velocity and acceleration for the uniform circular motion

We consider a case when a particle rotates around a center at the constant velocity v (**uniform circular motion**)

We define the angular velocity given by

$$\omega = \frac{d\theta}{dt} = \frac{2\pi}{T}$$

The period T is given by

$$T = \frac{2\pi r}{v}$$

where r is the radius of the circle and v is the velocity around the circumference of the circle,

$$v = \omega r$$

The angle θ is expressed by

$$\theta = \omega t$$

Then we can calculate the velocity and acceleration in terms of the polar coordinates,

$$
\begin{aligned}
\mathbf{r} &= r\mathbf{e}_r \\
&= r[\cos(\theta)\mathbf{i} + \sin(\theta)\mathbf{j}] \\
&= r\cos(\omega t)\mathbf{i} + r\sin(\omega t)\mathbf{j}
\end{aligned}
$$

$$
\begin{aligned}
\mathbf{v} &= \frac{d\mathbf{r}}{dt} = -r\omega\sin(\omega t)\mathbf{i} + r\omega\cos(\omega t)\mathbf{j} \\
&= r\omega[-\sin(\omega t)\mathbf{i} + \cos(\omega t)\mathbf{j}] \\
&= r\omega\mathbf{e}_\theta
\end{aligned}
$$

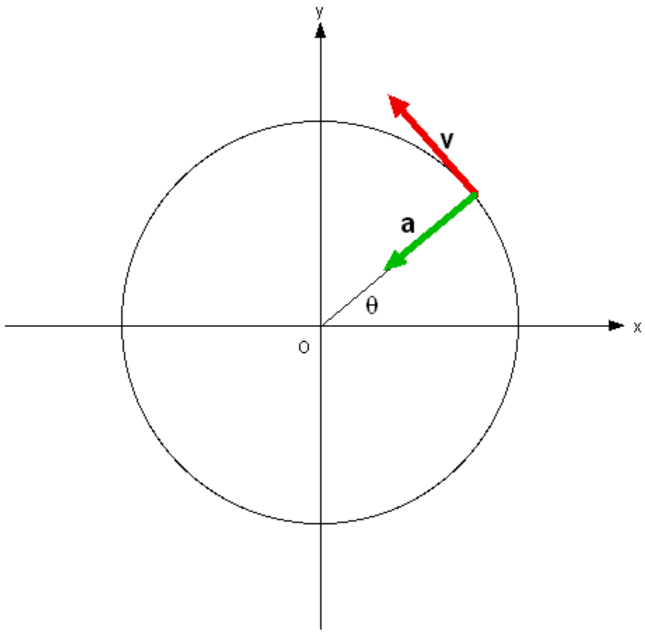
$$
\begin{aligned}
\mathbf{a} &= \frac{d^2\mathbf{r}}{dt^2} = -r\omega^2\cos(\omega t)\mathbf{i} - r\omega^2\sin(\omega t)\mathbf{j} \\
&= -r\omega^2[\cos(\omega t)\mathbf{i} + \sin(\omega t)\mathbf{j}] = -r\omega^2\mathbf{e}_r
\end{aligned}
$$

Note that

$$
\begin{aligned}
&(\sin\theta)' = \cos\theta \\
&(\cos\theta)' = -\sin\theta
\end{aligned}
$$

The acceleration ***a*** is called a centripetal acceleration. The direction of ***a*** is directed toward the center of the circle: the ($-\boldsymbol{e}_r$) direction.

$$
\boxed{
\begin{aligned}
&a = \omega^2 r = \frac{v^2}{r} \\
&v = \omega r \\
&\omega = \frac{2\pi}{T}
\end{aligned}
}
$$



Uniform circular motion

5.3 Geometrical consideration

5.3.1 Tangential velocity

We now consider the uniform circular motion from a view point of geometry.

OP = OQ = r.

$\angle POQ = \Delta\theta$, $\qquad \angle QPR = \angle PQR = \Delta\theta/2$

Arc(PQ) = $\Delta s = r\Delta\theta$

From the definition,

$$v_\theta = v = \frac{ds}{dt} = \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0} r\frac{\Delta\theta}{\Delta t} = r\omega$$

((**Note**))

$$\overrightarrow{OP} = (r, 0)$$

$$\overrightarrow{OQ} = (r\cos\Delta\theta, r\sin\Delta\theta)$$

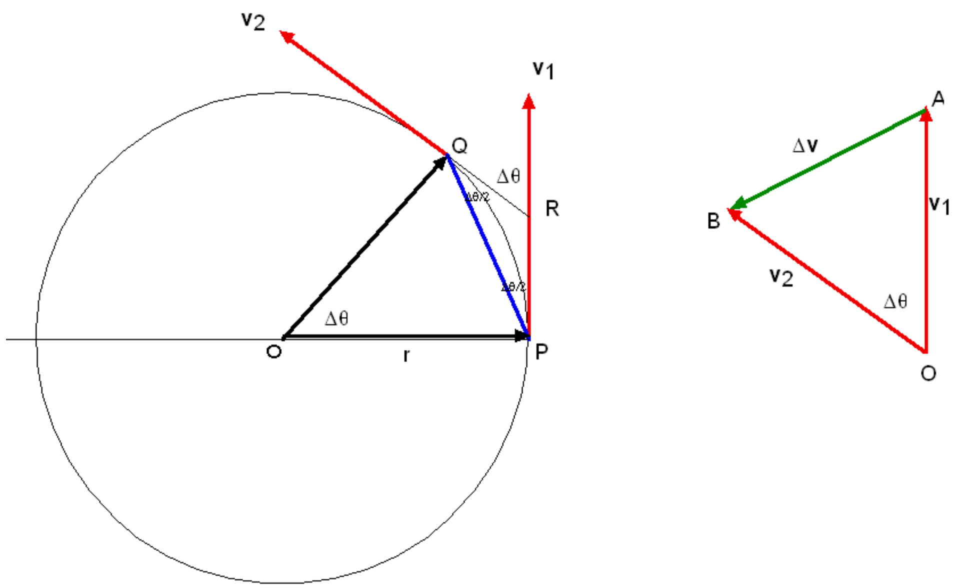
$$\Delta\mathbf{r} = \overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OR} = (r\cos\Delta\theta - r, r\sin\Delta\theta)$$

$$\approx (-\frac{r}{2}(\Delta\theta)^2, r\Delta\theta)$$

Then we have

$$\frac{\Delta\mathbf{r}}{\Delta t} = (-\frac{r}{2}\Delta\theta\frac{\Delta\theta}{\Delta t}, r\frac{\Delta\theta}{\Delta t})$$

$$\mathbf{v} = \frac{d\mathbf{r}}{dt} = \lim_{\Delta\theta\to 0}\frac{\Delta\mathbf{r}}{\Delta t} = (0, r\omega)$$



((Note)) In the limit of small θ, we have

$$\sin\theta \approx \theta$$

$$\cos\theta \approx 1 - \frac{\theta^2}{2}$$

5.3.2 Centripetal acceleration

In the limit of $\Delta\theta \to 0$, the difference vector $\Delta\mathbf{v}_r$ is directed toward the center of the circle.

$$\Delta\mathbf{v}_r = \mathbf{v}_2 - \mathbf{v}_1$$

The magnitude of $\Delta\mathbf{v}_r$ is given by

$$\Delta v_r = 2v\sin\left(\frac{\Delta\theta}{2}\right) \approx 2v\frac{\Delta\theta}{2} = v\Delta\theta\,.$$

The centripetal acceleration $\boldsymbol{a}_r$ is defined by

$$\mathbf{a}_r = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}_r}{\Delta t}$$

The vector $\boldsymbol{a}_r$ is directed toward the center of the circle.

$$a_r = \lim_{\Delta t \to 0} \frac{\Delta v_r}{\Delta t} = \lim_{\Delta t \to 0} v \frac{\Delta \theta}{\Delta t} = \omega v = v \frac{v}{r} = \frac{v^2}{r}$$

((**Note**))

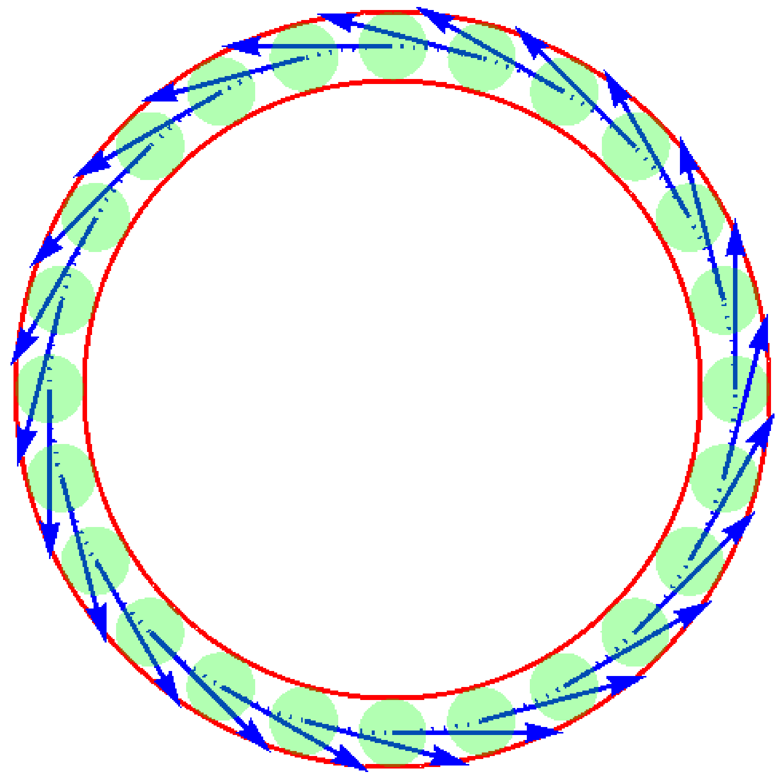
$$\mathbf{v}_1 = (0, v)$$
$$\mathbf{v}_2 = (-v \sin \Delta\theta, v \cos \Delta\theta)$$
$$\Delta \mathbf{v} = \mathbf{v}_2 - \mathbf{v}_1 = -v(\sin \Delta\theta, 1 - \cos \Delta\theta)$$
$$\approx -v(\Delta\theta, \frac{(\Delta\theta)^2}{2})$$

Then we have

$$\frac{\Delta \mathbf{v}}{\Delta t} = -v(\Delta\theta, \frac{(\Delta\theta)^2}{2})$$
$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}}{\Delta t} = -v(\omega, 0) = (-v\omega, 0)$$

5.3.3 The small angle approximation

Suppose that car rotates uniformly on the circular track. As shown in the Fig below, the direction of the velocity of car is clearly changing as a function of time, even though the magnitude of the velocity remains constant. This means that the acceleration of the car is non-zero. The velocity vector changes in time because its direction changes in time.



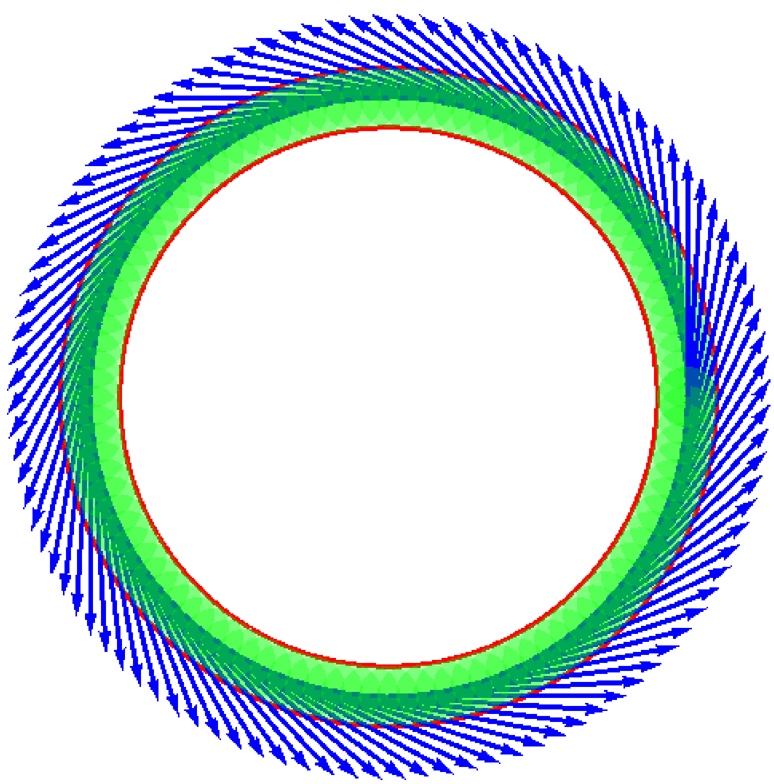


Fig. A car (here denoted by small green circle) moves with constant velocity around a circular track. Since the direction of the velocity vector is constantly changing, the acceleration of the car is not zero. In fact, the acceleration's magnitude is constant and its direction is always toward the center of the circle.

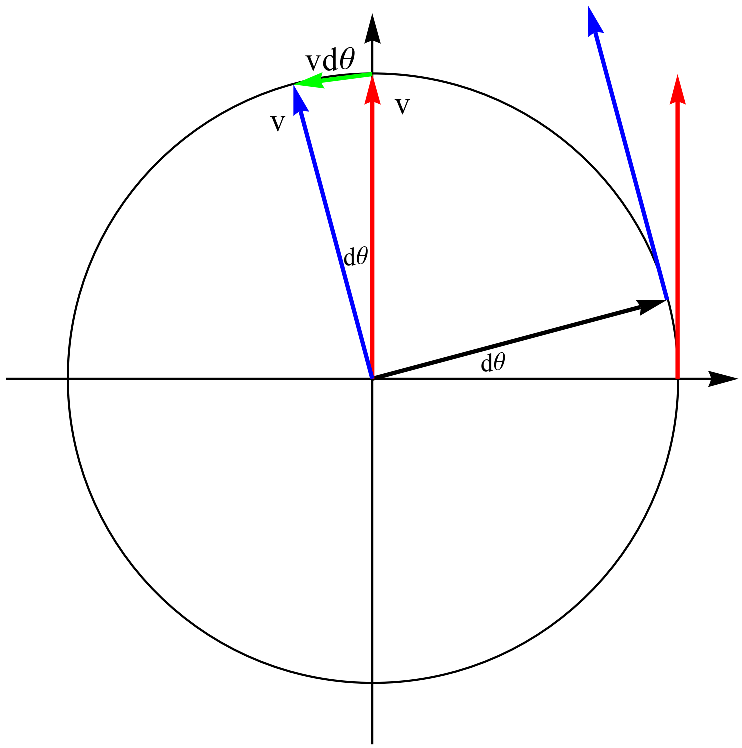


Fig. We use the small angle approximation to determine the magnitude of the centripetal acceleration.

In the above figure, we define the angle $d\theta$ between the two velocity vectors during the infinitesimal time *dt* to be equal to $d\theta$. We can use the small angle approximation to obtain the magnitude of *dv*, the change in velocity vector during the time *dt*.

$$dv = vd\theta .$$

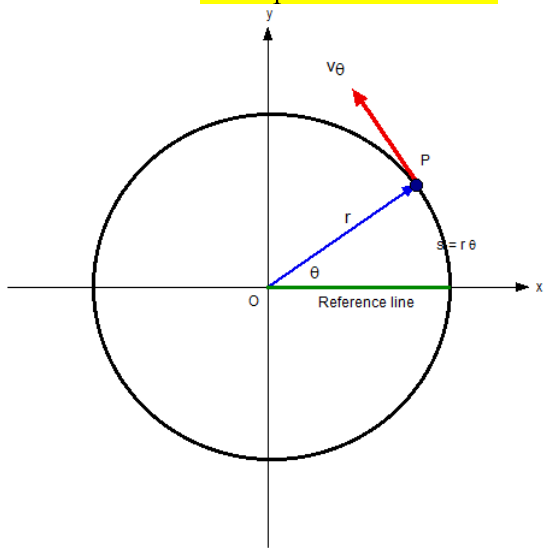
Then the magnitude of the acceleration is

$$a = \frac{dv}{dt} = v\frac{d\theta}{dt} = v\omega .$$

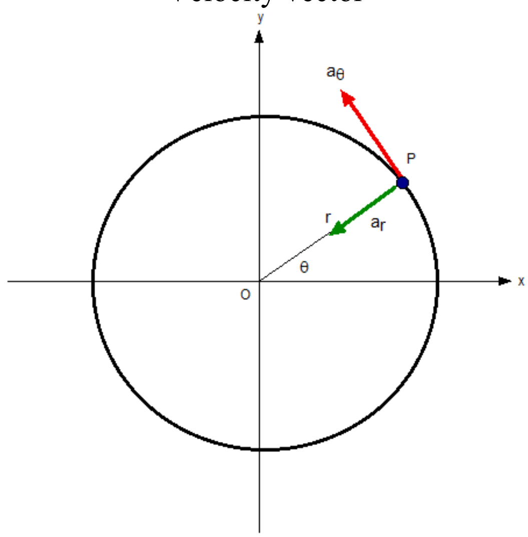
The direction of the acceleration vector is perpendicular to the velocity vector on each point of the circle.

6. Non-uniform circular motion

We assume that the particle rotates around the center of circle with time dependent velocity. This problem will be discussed later (Chapter 10). Here we just show that result. The acceleration vector consists of the centripetal acceleration and tangential acceleration.



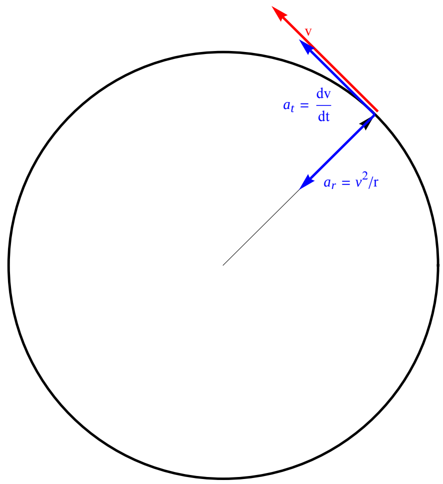
Velocity vector



Acceleration vector

$$v_r = 0$$
$$v_\theta = v,$$

$$a_r = \frac{v^2}{r}$$
$$a_\theta = \frac{dv}{dt}$$



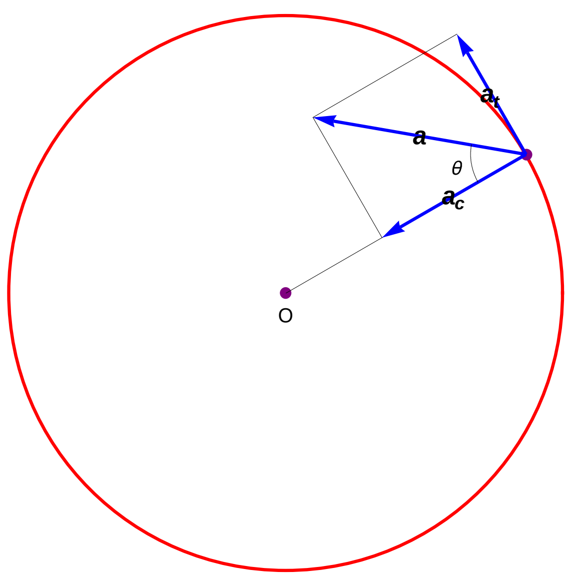
Non-uniform circular motion

We consider a more general motion. A particle moves to the right along a circle, Its velocity changes both in direction and in magnitude. The total acceleration is the sum of the tangential component and the centripetal acceleration (radial acceleration).

$$\boldsymbol{a} = \boldsymbol{a}_t + \boldsymbol{a}_r \, .$$

where

$$|\boldsymbol{a}_t| = \frac{dv}{dt}, \qquad a_r = -a_c = -\frac{v^2}{r} \, .$$



((Example-1)) **Serway 3-29**

A train slows down as it rounds a sharp horizontal turn, going from 90.0 km/h to 50.0 km/h in the 15.0 s it takes to round the bend. The radius of the curve is 150 m. Compute the acceleration at the moment the train speed reaches 50.0 km/h. Assume the train continues to slow down at this time at the same rate.

((**Solution**))

$r = 150$ m. $\Delta t = 15.0$ s.
$\Delta v = (90 – 50)$ km/h = 11.11 m/s.
$v = 50$ km/h= 13.88 m/s.

The tangential acceleration:

$$a_t = \frac{\Delta v}{\Delta t} = \frac{11.11}{15.0} = 0.74 \text{ m/s}^2.$$

The centripetal acceleration:

$$a_c = \frac{v^2}{r} = \frac{13.88^2}{150} = 1.284 \text{ m/s}^2.$$

Then the magnitude of the acceleration is

$$|\boldsymbol{a}| = \sqrt{a_t^{\ 2} + a_c^{\ 2}} = 1.48 \text{ m/s}^2.$$

The angle θ is obtained as

$$\theta = \arctan(\frac{a_t}{a_c}) = 29.9°.$$

((**Example-2**)) **Serway 3-31**

The same Figure (above described) represents the total acceleration of a particle moving clockwise in a circle of radius r = 2.50 m at a certain instant of time. For that instant, find
(a) the radial acceleration of the particle,
(b) the speed of the particle,
and
(c) its tangential acceleration.

((**Solution**))

$r = 2.50$ m, $\theta = 30.0°$, $a = 15.0$ m/s^2.

(a) radial (centripetal) acceleration:

$$a_c = \frac{v^2}{r} = a\cos\theta = 15.0\ \cos 30° = 13.0\ \text{m/s}^2.$$

(b)

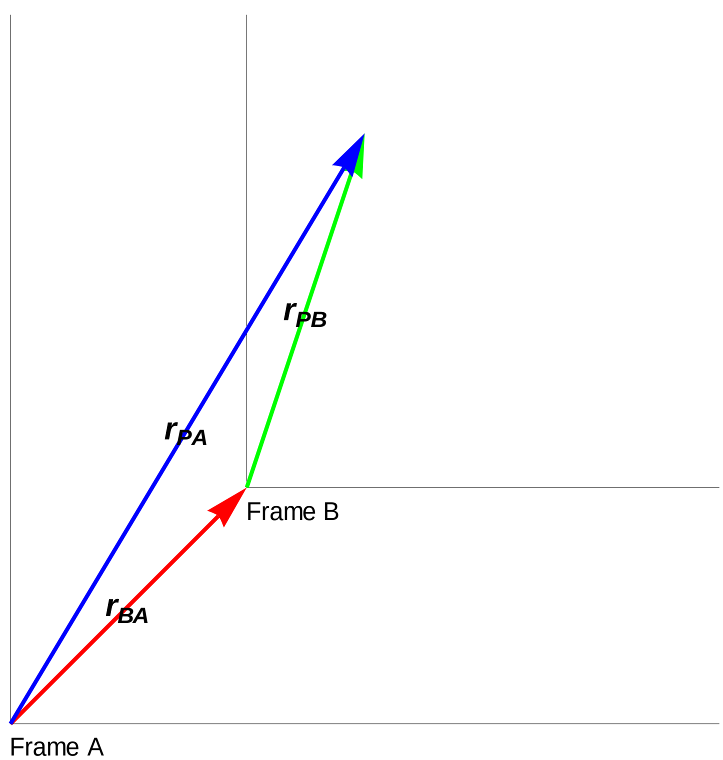
$$v = \sqrt{ra_c} = 5.70\ \text{m/s}.$$

(c) Tangential acceleration:

$$a_t = \frac{dv}{dt} = a\sin\theta = 15.0\ \sin 30° = 7.50\ \text{m/s}^2$$

6. Relative motion in two dimension

When two frames of reference A and B are moving relative to each other at constant velocity, the velocity of a particle as measured by an observer in the frame A ($\boldsymbol{v}_{PA}$) usually differs from that measured from the frame B ($\boldsymbol{v}_{PB}$).



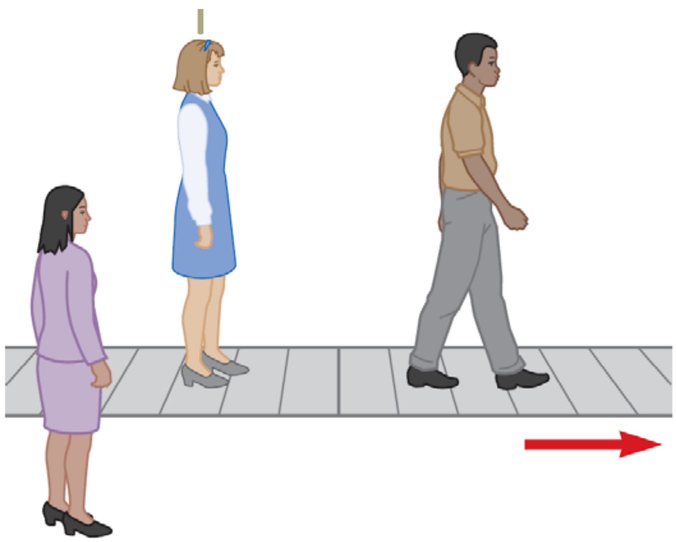
In this figure we have the following relations

$$\mathbf{r}_{PA} = \mathbf{r}_{PB} + \mathbf{r}_{BA}$$
$$\mathbf{v}_{PA} = \mathbf{v}_{PB} + \mathbf{v}_{BA}$$
$$\mathbf{a}_{PA} = \mathbf{a}_{PB}$$

since $\boldsymbol{a}_{BA} = 0$.

((**Note**))
The vector $\boldsymbol{r}_{PA}$ is the position vector of P relative to the origin of the reference frame A.

((**Example**))
We explore this phenomenon by considering two observers watching a man walking on a moving beltway at an airport in Fig. shown below. The woman standing on the moving beltway (in blue dress) sees the man moving at a normal walking speed. The woman observing from the stationary floor (in pink dress) sees the man moving with a higher speed because the beltway speed combines with his walking speed. Both observers look at the same man and arrive at different values for his speed. Both are correct; the difference in their measurements results from the relative velocity of their frames of reference



$$\boldsymbol{v}_{MG} = \boldsymbol{v}_{MB} + v_{BG}$$

where M is the man, B is the beltway, and G is the ground. The velocity of the walking man on the belway, against the ground is the sum of the velocity of the moving beltway and the velocity of the walking man against the moving beltway.

((**Example**))

Problem 4-75 (SP-4) (8-th edition)
Problem 4-77 (SP-4) (9-th, 10-th edition)

Snow is falling vertically at a consant speed of 8.00 m/s. At what angle from the vertical do the snow flakes appear to be falling as viewed by the driver of a car travelling on a straight, level road with a speed of 50 km/h?

((**Solution**))

((Note)) We use the following abbreviation
s: snow, G: ground, and c: car.

50 km/h = 50 x 1000m/3600s = 13.89 m/s

$$\mathbf{v}_{sc} = \mathbf{v}_{sG} + \mathbf{v}_{Gc} = \mathbf{v}_{sG} - \mathbf{v}_{cG}$$

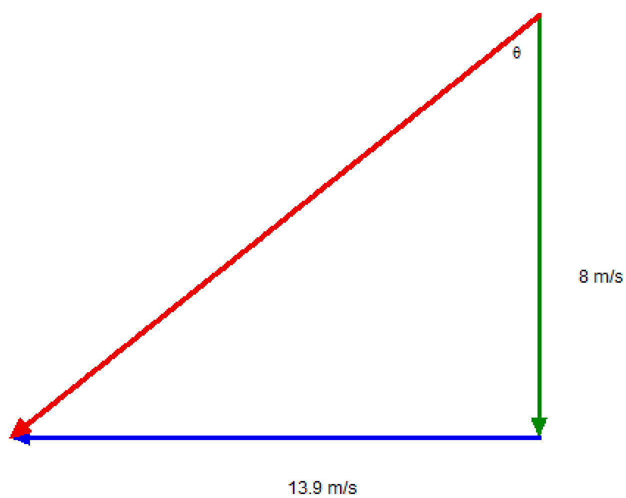
where

$\boldsymbol{v}_{sG} = (0, -8)$ m/s

$\boldsymbol{v}_{cG} = (13.89, 0)$ m/s

Then we have

$$\mathbf{v}_{sc} = \mathbf{v}_{sG} - \mathbf{v}_{cG} = (0,-8) - (13.89,0) = (-13.89,-8) \text{ m/s}$$



$$\tan\theta = \frac{13.9}{8} = 1.738$$

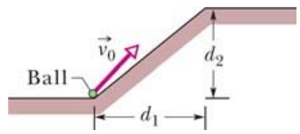
$$\theta = 60°$$

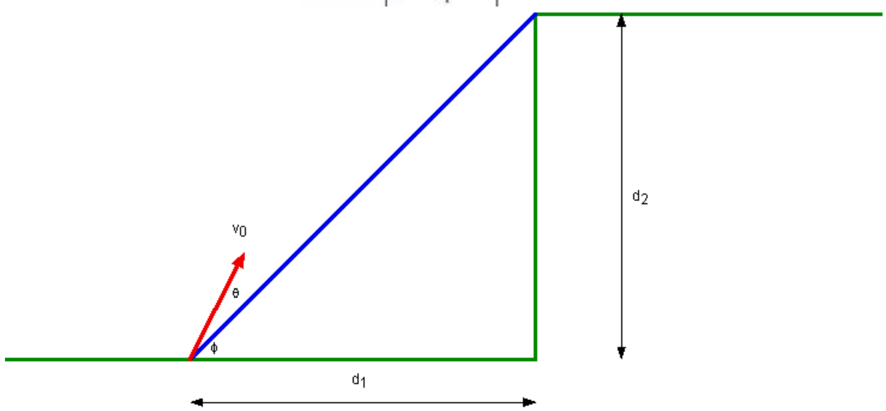
7. Examples

7.1 Projectile

Problem 4-43 (SP-4) (8-th edition)**
Problem 4-45 (SP-4) (9-th, 10-th edition)**

In Fig. a ball is launched with a velocity of magnitude 10.0 m/s, at an angle of 50° to the horizontal. The launch point is at the base of a ramp of horizontal length d_1 = 6.00 m and height d_2 = 3.60 m. A plateau is located at the top of the ramp. (a) Does the ball land on the ramp or the plateau? When it lands, what are the (b) magnitude and (c) angle of its displacement from the launch point?





$d_1 = 6.0$ m. $d_2 = 3.6$ m.
$v_0 = 10.0$ m/s. $\theta = 50^{\circ}$ from the horizontal
$\tan\phi = d_2/d_1 = 0.6$, or $\phi = 30.96\ ^{\circ}$.

The x and y positions of the particle:

$$x = v_0 \cos\theta t$$
$$y = v_0 \sin\theta t - \frac{1}{2} g t^2,$$

or

$$y = \tan\theta x - \frac{g}{2 v_0^2 \cos^2\theta} x^2$$

The ramp is described by

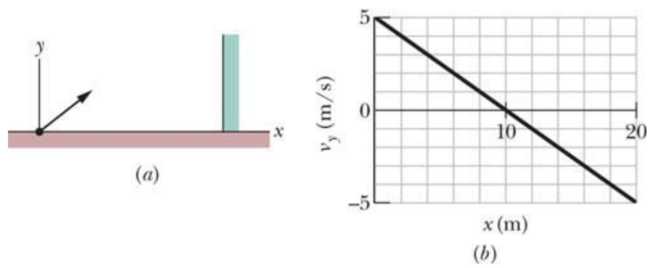
$$y = \frac{d_2}{d_1} x$$

Find the intersection of the two curves.

7.2 Projectile

Problem 4-50 (SP-4)* (8-th edition)**
Problem 4-52 (SP-4)* (9-th, 10-th edition)**

A ball is to be shot from level ground toward a wall at distance x (Fig.a). Figure (b) shows the y component v_y of the ball's velocity just as it would reach the wall, as a function of the distance x. What is the launch angle?



$$x = v_0 \cos\theta t, \qquad \text{or} \qquad t = \frac{x}{v_0 \cos\theta}$$

$$v_y = v_0 \sin\theta - gt, \qquad \text{or} \qquad v_y = v_0 \sin\theta - \frac{gx}{v_0 \cos\theta}$$

v_y is proportional to x. (The curve of v vs x is a straight line).

At $x = 0$,

$$v_y = v_0 \sin\theta = 5 m/s$$

At $x = 10$m,

$$v_y = v_0 \sin\theta - \frac{10g}{v_0 \cos\theta} = 5 - \frac{10g}{v_0 \cos\theta} = 0$$

Then we have two equations to determine θ and v_0,

$$v_0 \sin\theta = 5$$
$$v_0 \cos\theta = 2g = 19.6$$

From these we get

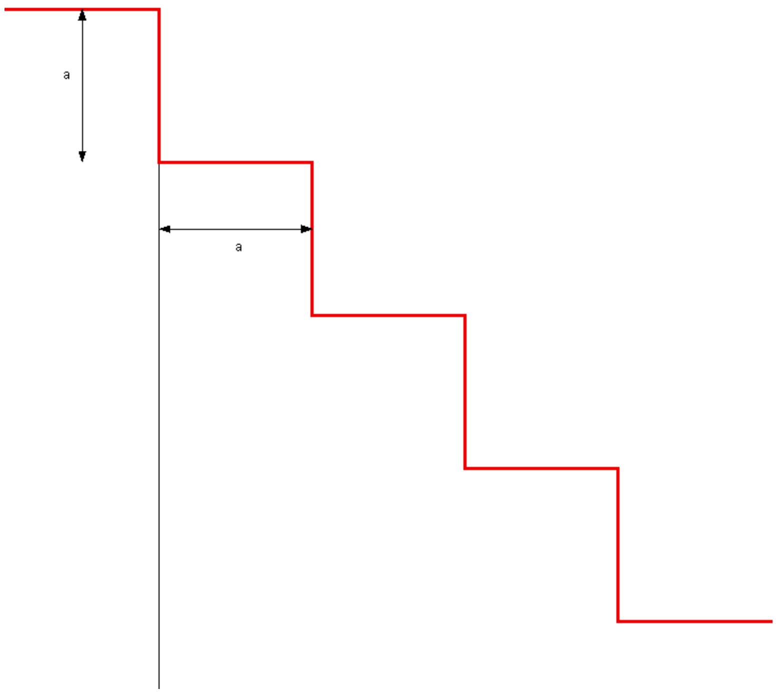
v_0 = 20.2 m/s, θ = 14.3º

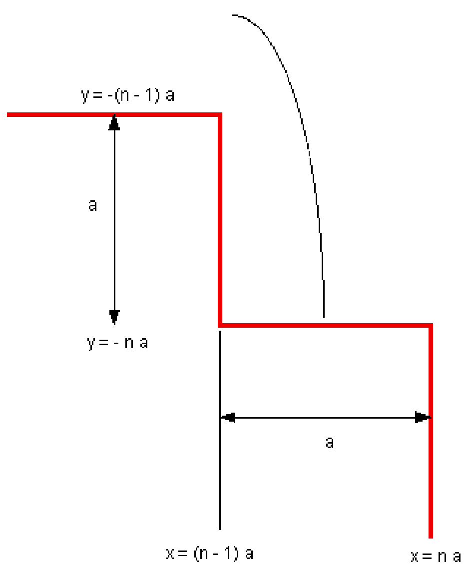
7.3 Projectile

Problem 4-53 (SP-4)* (8-th edition)**
Problem 4-55 (SP-4)* (9-th, 10-th edition)**

A ball rolls horizontally off the top of a stairway with a speed of 1.52 m/s. The steps are 20.3 cm high and 20.3 cm wide. Which step does the ball hit first?

((**Solution**))
a = 0.203 m. v_0 = 1.52 m/s.





$$x = v_0 t$$
$$y = -\frac{1}{2} g t^2,$$

or

$$y = -\frac{g x^2}{2 v_0^{\,2}}$$

The condition that the ball hits the *n*-th step first is as follows.

(1)
For $(n-1)a < x < na$, y takes

$$y(x) = -\frac{g x^2}{2 v_0^{\,2}} = -na$$

or

$$(n-1) < \sqrt{\frac{2 v_0^{\,2} n}{g a}} < n$$

(2) At $x = (n\text{-}1)a$, the value of y should be larger than $-(n\text{-}1)\,a$.

$$y[x = (n-1)a] > -(n-1)a$$

or

$$y[x = (n-1)a] = -\frac{g a^2}{2 v_0^{\,2}}(n-1)^2 > -(n-1)a$$

or

$$n - 1 < \frac{2 v_0^{\,2}}{g a}$$

Then we have two inequalities

$$(n-1)^2 < 2.32 n < n^2$$

$$n < 1 + 2.32$$

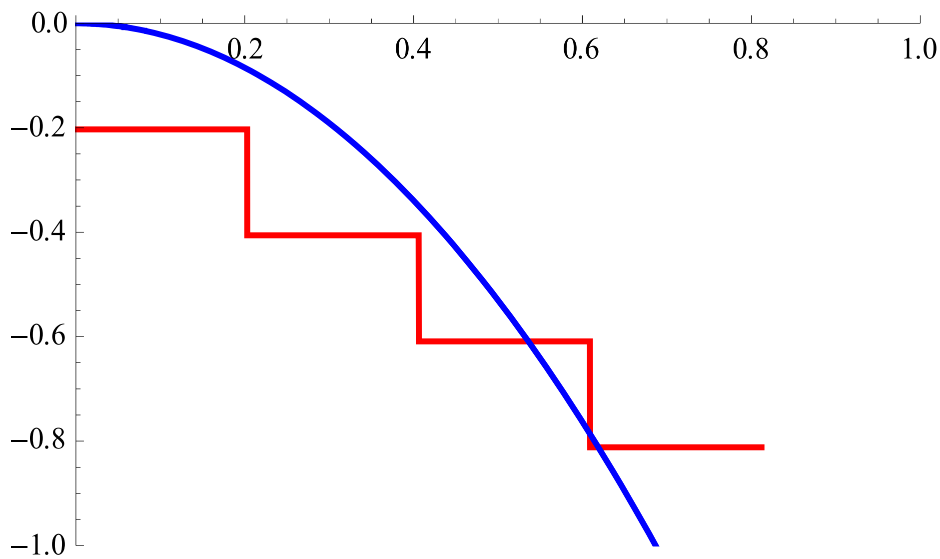
or

$$2.32 < n < 3.32$$

Then we have $n = 3$.

In[1]:=
```
rule1 = {a → 0.203, v0 → 1.52, g → 9.8};
f1 = If[0 < x < a, -a, 0] +
  If[a < x < 2 a, -2 a, 0] +
  If[2 a < x < 3 a, -3 a, 0] +
  If[3 a < x < 4 a, -4 a, 0];
p11 = Plot[f1 /. rule1, {x, 0, 0.812},
  PlotStyle → {Thick, Red},
  PlotRange → {{0, 1}, {0, -1}}];
```
$$f2 = -\frac{g}{2}\frac{x^2}{v0^2};\ f21 = f2\ /.\ rule1;$$
```
p22 = Plot[f21, {x, 0, 0.812},
  PlotRange → {{0, 1}, {0, -1}},
  PlotStyle → {Thick, Blue}];
Show[p11, p22]
```

Out[1]=



7.4 Uniform circular motion

Problem 4-68* (SP-4) (8-th edition)**
Problem 4-68* (SP-4) (9-th, 10-th edition)**

A cat rides a merry-go-round turning with uniform circular motion. At time t_1 = 2.00 s, the cat's velocity is

$$\mathbf{v}_1 = (3.00m/s)\mathbf{i} + (4.00m/s)\mathbf{j},$$

measured on a horizontal *xy* coordinate system. At t_2 = 5.00 s, its velocity is

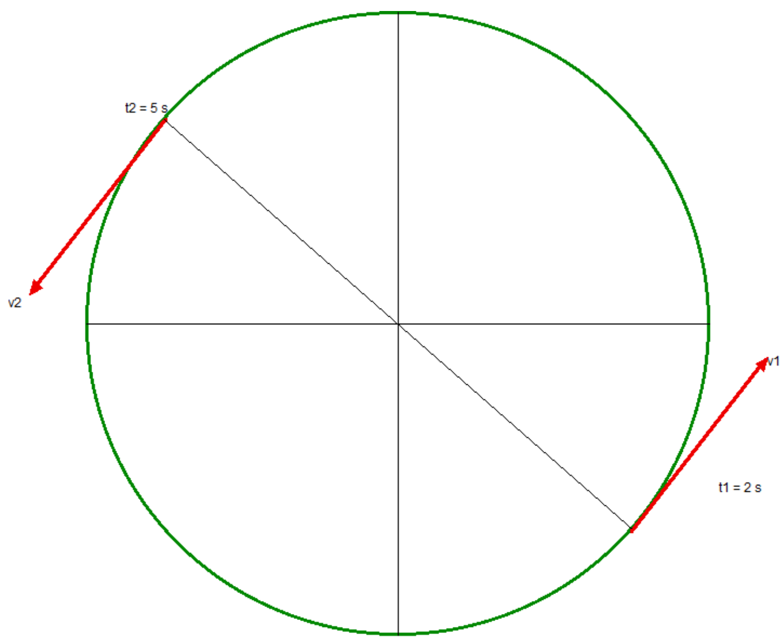
$$\mathbf{v}_2 = (-3.00m/s)\mathbf{i} + (-4.00m/s)\mathbf{j}.$$

What are (a) the magnitude of the cat's centripetal acceleration and (b) the cat's average acceleration during the time interval $|t_2 - t_1|$, which is less than one period?

((**Solution**))

Uniform circular motion

t_1 = 2 s,	$\boldsymbol{v}_1$ = (3, 4) m/s.	v_1 = 5 m/s
t_2 = 5 s,	$\boldsymbol{v}_2$ = (-3, -4) m/s.	v_2 = 5 m/s



The circular motion is counterclockwise.

$\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ are antiparallel.

$$v = \sqrt{3^2 + 4^2} = 5 \text{ m/s}$$

The period *T*

$$t_2 - t_1 = \frac{T}{2} = 3$$
$$T = 6s$$

The radius *r*,

$$T = \frac{2\pi r}{v}$$

$$r = \frac{vT}{2\pi} = \frac{30}{2\pi} = \frac{15}{\pi} = 4.77m$$

(a) The acceleration *a*,

$$a = \frac{v^2}{r} = \frac{25}{4.77} = 5.24 m/s^2$$

(b) The average acceleration

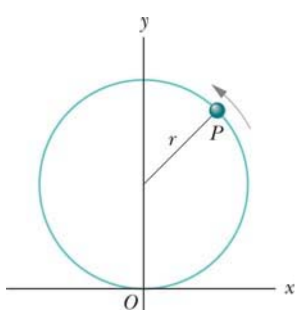
$$\mathbf{a}_{av} = \frac{\Delta \mathbf{v}}{\Delta t} = \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} = \frac{(-6,-8)}{3} = (-2,-\frac{8}{3})$$

$$a_{av} = 3.3 m/s^2$$

7.5 Uniform circular motion

Problem 4-95 (HW Hint) **(8-th edition)**
Problem 4-107 **(9-th, 10-th edition)**

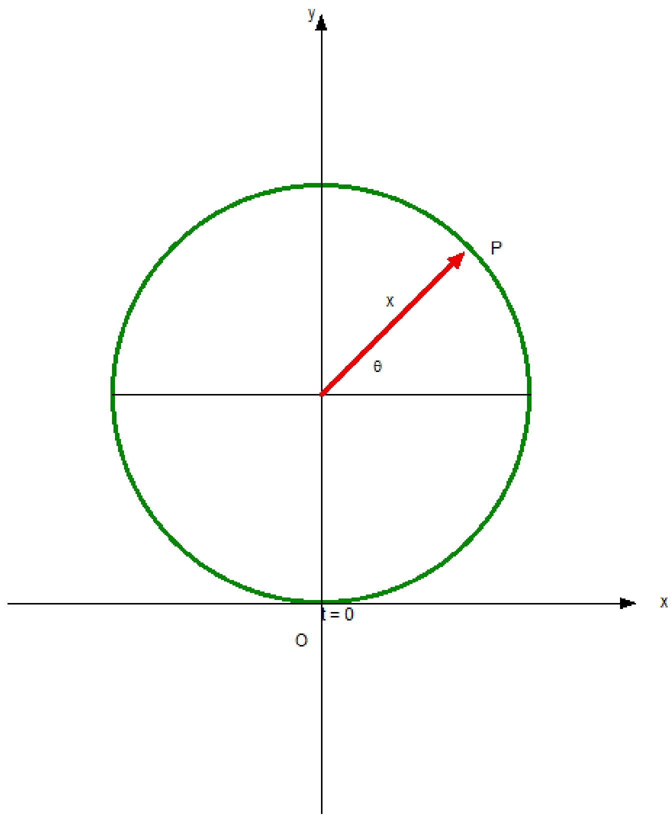
A particle P travels with constant speed on a circle of radius r = 3.00 m (Fig.) and completes one revolution in 20.0 s. The partricle passes through O at time t = 0. State the following vectors in magnitude-angle notation (angle relative to the positive direction of x). With respect to O, find the particle's position vector at the times t of (a) 5.00 s, (b) 7.50 s, and (c) 10.0 s. (d) 10.0 s. (d) For the 5.00 s interval from the end of the fifth second to the end of the tenth second, find the particle's displacement. For that interval, find (e) its average velocity and its velocity at the (f) beginning and (g) end. Next, find the acceleration at the (h) beginning and (i) end of that interval.



((**Solution**))
Uniform circular motion
r = 3.00 m
T = 20 s

The center of the circle is O_1. The position vector of P is $\boldsymbol{r}$. The position vector of O_1 is (0, r). The angle between PO_1 and the positive x axis is θ. At $t = 0$, the point P is at O.

$$\overrightarrow{OP} = \mathbf{r} = \overrightarrow{OO_1} + \overrightarrow{O_1P} = (0, r) + (r\cos(\theta), r\sin(\theta) = (r\sin(\omega t), r - r\cos(\omega t))$$



The angular velocity ω is given by

$$\omega = \frac{2\pi}{T}$$

The angle θ vs t

$$\theta = \omega t - \frac{\pi}{2}$$

The x and y coordinates for the position of the particle

$$\mathbf{r} = (x, y)$$

with

$$x = r\sin(\omega t)$$
$$y = r - r\cos(\omega t)$$

The velocity of the particle is

$$v = (\frac{dx}{dt}, \frac{dy}{dt}) = (r\omega\cos(\omega t), r\omega\sin(\omega t))$$

The acceleration vector of the particle;

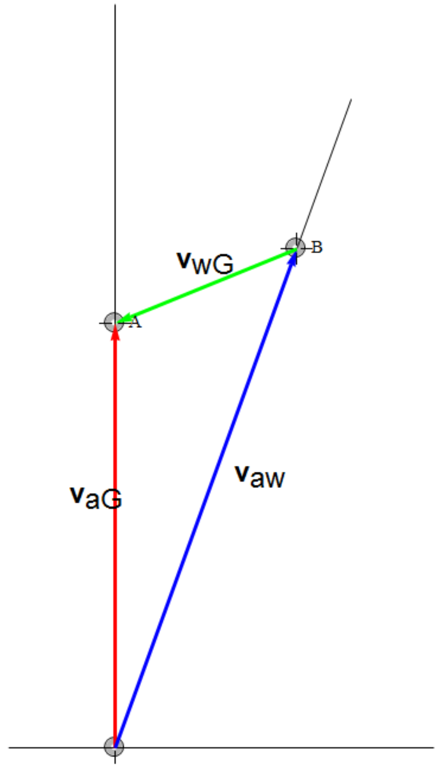
$$a = (\frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}) = (-r\omega^2\sin(\omega t), r\omega^2\cos(\omega t))$$

7.6 Relative motion

Problem 4-74 (SP-4) (8-th edition)**
Problem 4-76 (SP-4) (9-th, 10-th edition)**

A light plane attains an airspeed of 500 km/h. The pilot sets out for a destination 800 km due north but discovers that the plane must be headed 20.0° east of due north to fly there directly. The plane arrives in 2.00 h. What were the (a) magnitude and (b) direction of the wind velocity?

((**Solution**))



((Note)) We use the following abbreviation
a: airplane, G: ground, and w: wind.

$t = 2$ hours
Airspeed of the plane: 500 km/h

$$\mathbf{v}_{aG} = \mathbf{v}_{aw} + \mathbf{v}_{wG}$$

where

$$\mathbf{v}_{aG} = (0,400) km/h$$
$$\mathbf{v}_{aw} = (500\sin(20°), 500\cos(20°)) = (171, 469.8) km/h$$

Then we have

$$\mathbf{v}_{wG} = \mathbf{v}_{wa} + \mathbf{v}_{aG} = -\mathbf{v}_{aw} + \mathbf{v}_{aG} = (-171, -69.8) \text{ km/h}$$

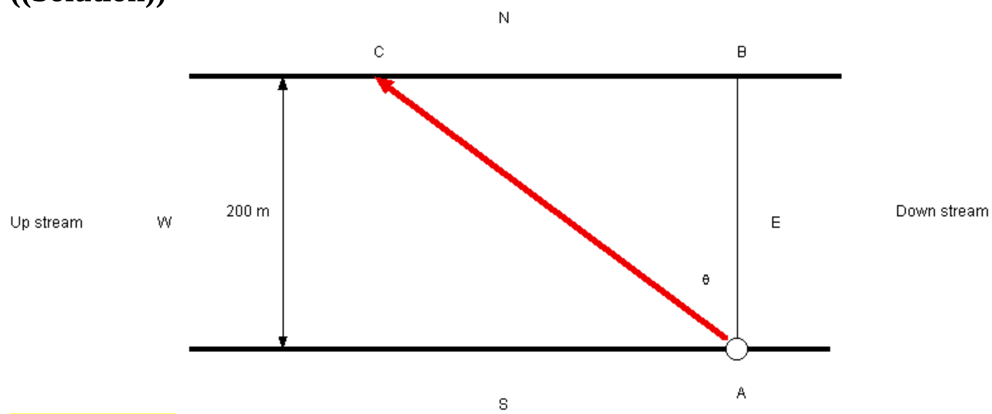
$$\tan\theta = \frac{69.8}{171} = 0.408$$
$$\theta = 22.2°$$
$$v_{wG} = 185 km/h$$

7.7 Relative motion

Problem 4-82* (8-th edition)**
Problem 4-82* (9-th, 10-th edition)**

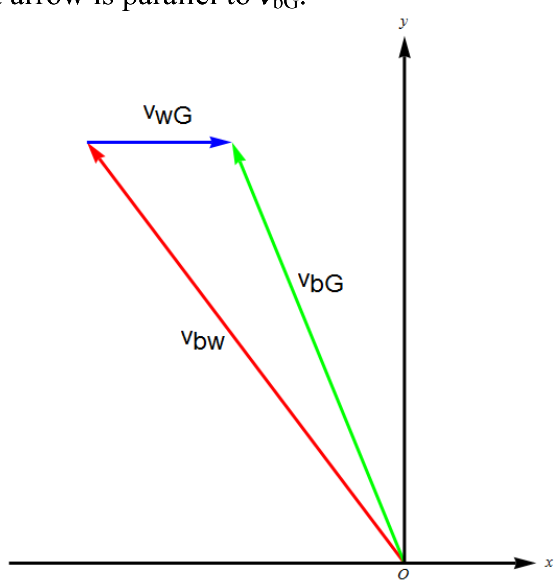
A 200-m-wide river has a uniform flow speed of 1.1 m/s through a jungle and toward the east. An explorer wishes to leave a small clearing on the south bank and cross the river in powerboat that moves at a constant speed of a 4.0 m/s with respect to the water. There is a clearing on the north bank 82 m from a point directly opposite the clearing on the south bank. (a) In what direction must the boat be pointed in order to travel in a straight line and land in the clearing on the north bank? (b) How long will the boat take to cross the river and land in the clearing?

((Solution))



BC = 82.0 m

The vector with the red arrow is parallel to $\boldsymbol{v}_{bG}$.



((Note)) We use the following abbreviation b: boat, G: ground, and w: water.

$$\tan\theta = \frac{82}{200}$$
$$\theta = 22.3°$$

$$\mathbf{v}_{bG} = \mathbf{v}_{bw} + \mathbf{v}_{wG}$$

$$\mathbf{v}_{wG} = (1.1,0)$$
$$\mathbf{v}_{bw} = (v_x, v_y)$$

where

$$v_{bw} = \sqrt{v_x{}^2 + v_y{}^2} = 4.0$$

For $\mathbf{v}_{bG}$,

$$\mathbf{v}_{bG} = (v_x + 1.1, v_y)$$

with

$$\frac{v_x + 1.1}{v_y} = -\tan\theta = -\frac{82}{200}$$

Then we have

v_x = -2.41 m/s

v_y = 3.19 m/s

The time required for crossing the river is

$$t = \frac{200}{v_y} = 62.84s$$

((Mathematica))

```
eq1 = vx^2 + vy^2 == 16
```

$vx^2 + vy^2 == 16$

```
eq2 = 200 (1.1 + vx) == -82 vy
```

$200\ (1.1 + vx) == -82\ vy$

```
eq3 = Solve[{eq1, eq2}, {vx, vy}]
```

{{vx → −2.40917, vy → 3.1931}, {vx → 0.525771, vy → −3.9653}}

$$\theta = \frac{180}{\pi} \text{ArcTan}\left[\frac{82}{200}\right] \text{ // N}$$

22.2936

$$t1 = \frac{200}{vy} \text{ /. eq3[[1]]}$$

62.635

8. Advanced problem

8.1

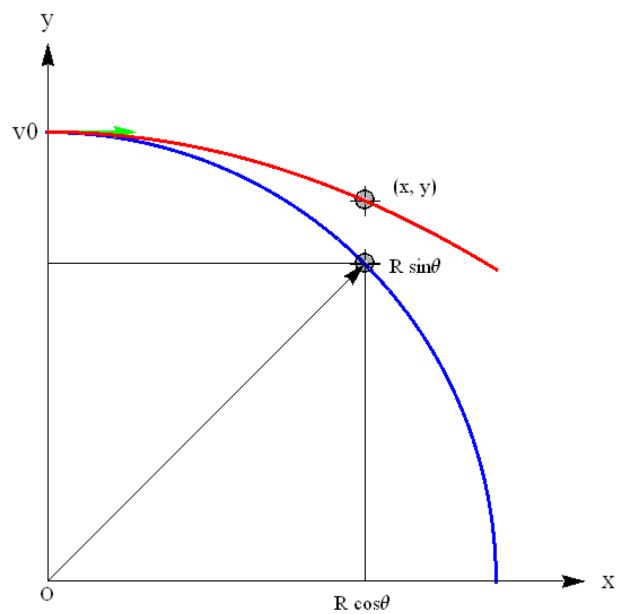
Problem 4-62 (Serway and Jewett) (Advanced problem)

A person standing at the top of a hemisphere rock of radius R kicks a ball (initially at rest on the top of the rock) to give it horizontal velocity v_0 as shown in Fig.

(a) What must be its minimum initial speed if the ball is never to hit the rock after it is kicked?

(b) With its initial speed, how far from the base of the rock does the ball hit the ground?

((Solution))



$x = v_0 t$

$$y = R - \frac{1}{2}gt^2 = R - \frac{gx^2}{2{v_0}^2}$$

The coordinate (x, y) should be outside the hemisphere with radius R.

$$x = R\cos\theta$$

$$y = R - \frac{gR^2}{2{v_0}^2}\cos^2\theta \geq R\sin\theta$$

for any θ ($0<\theta<\pi/2$)

Then we have

$$R(1-\sin\theta) \geq \frac{gR^2}{2{v_0}^2}\cos^2\theta = \frac{gR^2}{2{v_0}^2}(1-\sin^2\theta)$$

or

$$R \geq \frac{gR^2}{2{v_0}^2}(1+\sin\theta)$$

The right-hand side of the inequality has a maximum $\frac{gR^2}{{v_0}^2}$ at $\theta = \pi/2$. Then it is concluded that

$$R \geq \frac{gR^2}{v_0^{\ 2}}$$

or

$$v_0 \geq \sqrt{gR}$$

(b)

When $v_0 = \sqrt{gR}$,

$$y = R - \frac{gx^2}{2v_0^{\ 2}} = R - \frac{gx^2}{2(gR)} = R - \frac{x^2}{2R}$$

When $y = 0$, we have

$$x = \sqrt{2}R$$

((Mathematica))

```
Clear["Global`*"]
```

$$y1 = R - \frac{1}{2}\frac{g\,x^2}{v0^2};$$

```
rule1 = {R → 1, g → 9.8}
```

{R → 1, g → 9.8}

```
y11 = y1 /. rule1
```

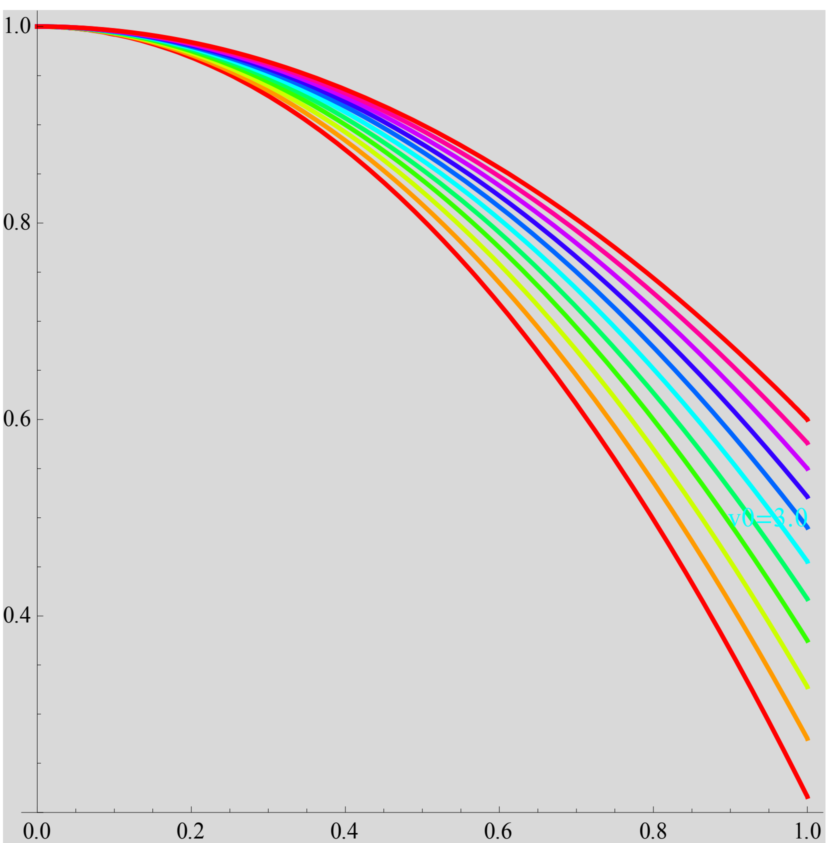
$$1 - \frac{4.9\,x^2}{v0^2}$$

$$y2 = \sqrt{R^2 - x^2};$$

```
y22 = y2 /. rule1
```

$$\sqrt{1 - x^2}$$

```
p1 = Plot[Evaluate[Table[y11, {v0, 2.5, 3.5, 0.1}]], {x, 0, 1},
   AspectRatio → 1, PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 10}],
   Background → LightGray,
   Epilog → {Hue[0.5], Thick, Text[Style["v0=3.0", 12], {0.95, 0.5}]}];
```




```
p2 = Plot[y22, {x, 0, 1}, AspectRatio → 1, PlotStyle → {Thick, Black},
   Background → LightGray, AxesLabel → {"x/R", "y"}];
```

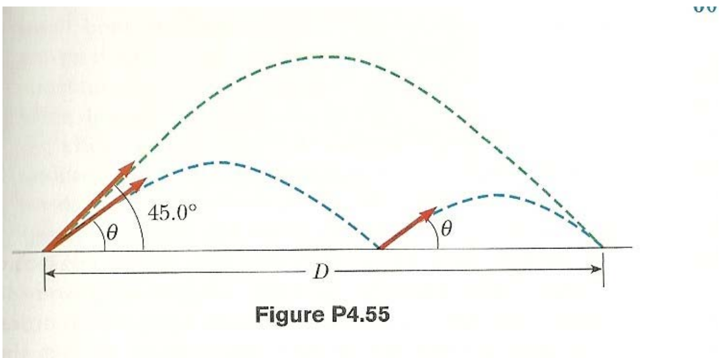
```
Show[p1, p2]
```

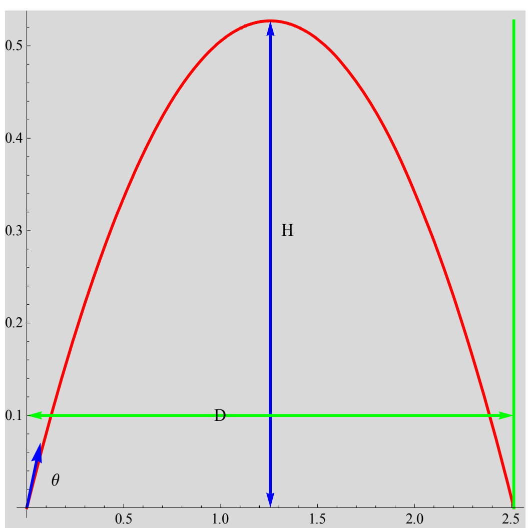
8.2
Problem 4-55 (Serway and Jewett)

When baseball players throw the ball in from the outfield, they usually allow it to take one bounce before it reaches the infield, on the theory that the ball arrives sooner that way. Suppose that the angle at which a bounced ball leaves the ground is the same as the angle at which the outfielder threw it, as in Fig., but that the ball's speed after the bounce is one half of what it was before the bounce.

(a) Assuming the ball is always thrown with the same initial speed, at what angle θ should the fielder throw the ball to make it go the same distance D with one bounce (blue path) as a ball thrown upward at 45.0º with no bounce (green path)?

(b) Determine the ratio of the times for the one-bounce and no-bounce throws.





$$H = \frac{v_0^{\ 2}\sin^2\theta}{2g}$$ (the maximum height of the ball)

$$D = \frac{v_0^{\ 2}}{g}\sin(2\theta),$$ (the distance where the ball flies)

$$T = \frac{2v_0}{g}\sin\theta$$ (total times when the ball arrives at the ground again)

(a)
For no bounce throw, the total distance D_1 along the x axis is

$$D_1 = \frac{v_0^2}{g}\sin(2\times 45°) = \frac{v_0^2}{g}$$

For one bounce throw, the total distance along the x axis is

$$D_2 = \frac{v_0^2}{g}\sin(2\theta) + \frac{v_0^2}{4g}\sin(2\theta) = \frac{5v_0^2}{4g}\sin(2\theta)$$

When $D_1 = D_2$, we have

$$\sin(2\theta) = \frac{4}{5} \qquad \text{or} \qquad \theta = 25.6^\circ$$

(b)
For no bounce throw, the total time T_1 is

$$T_1 = \frac{2v_0}{g}\sin(45°)$$

For one bounce throw, the total time T_2 is

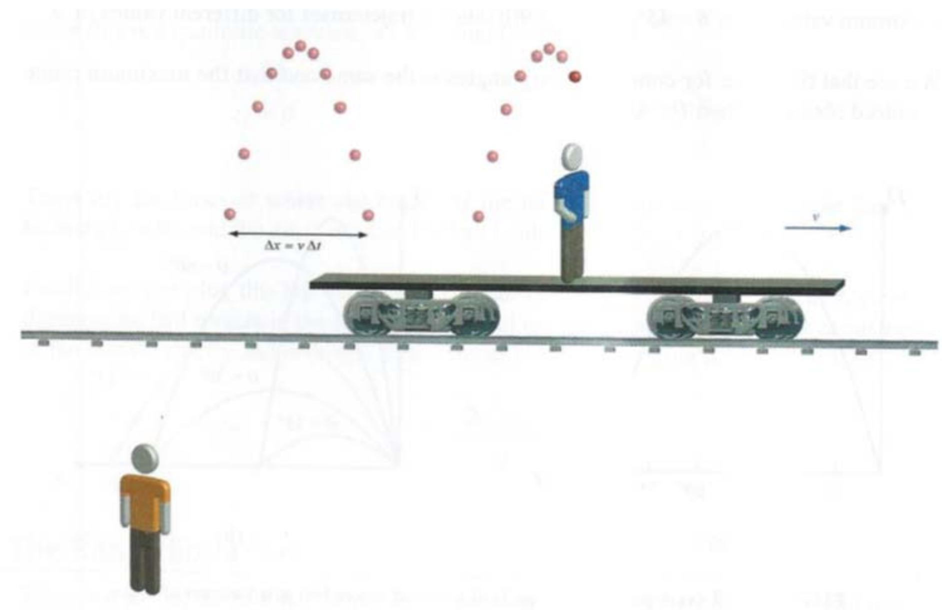
$$T_2 = \frac{2v_0}{g}\sin(\theta) + \frac{2\frac{v_0}{2}}{g}\sin(\theta) = \frac{3v_0}{g}\sin(\theta) = \frac{3v_0}{g}\sin(26.6^\circ)$$

The ratio is

$$\frac{T_2}{T_1} = \frac{3\sin(26.6°)}{2\sin(45°)} = 0.949$$

APPENDIX-I
((**Example**)) Relative motion
Ball shot upward from moving cart (Howitzer cart)

The trajectory of a ball thrown straight up by a man (at the initial velocity v_0) at rest on a train moving at constant velocity v_1 with respect to an observer on the ground. In the time Δt it takes the ball to go straight up and down with respect to the man on the train, the train has traveled a distance $\Delta x = v_1 \Delta t$. the ball is always directly above the man on the train and therefore appears to have the trajectory shown to the observer on the ground.



We consider using the superposition for the relative motion.

$$v_{bg} = v_{bt} + v_{tg} ,$$

or

$$\boldsymbol{r}_{bg} = \boldsymbol{r}_{bt} + \boldsymbol{r}_{tg} = (v_1 t, v_0 t - \frac{1}{2} g t^2),$$

where *b*: ball, *t*: train, and *g*: ground. Thus we have

$$x = v_1 t , \qquad y = v_0 t - \frac{1}{2} g t^2 .$$

The motion is equivalent the motion of ball with the horizontal velocity (v_1) and vertical velocity (initial velocity v_0).

When $t = \dfrac{x}{v_1}$,

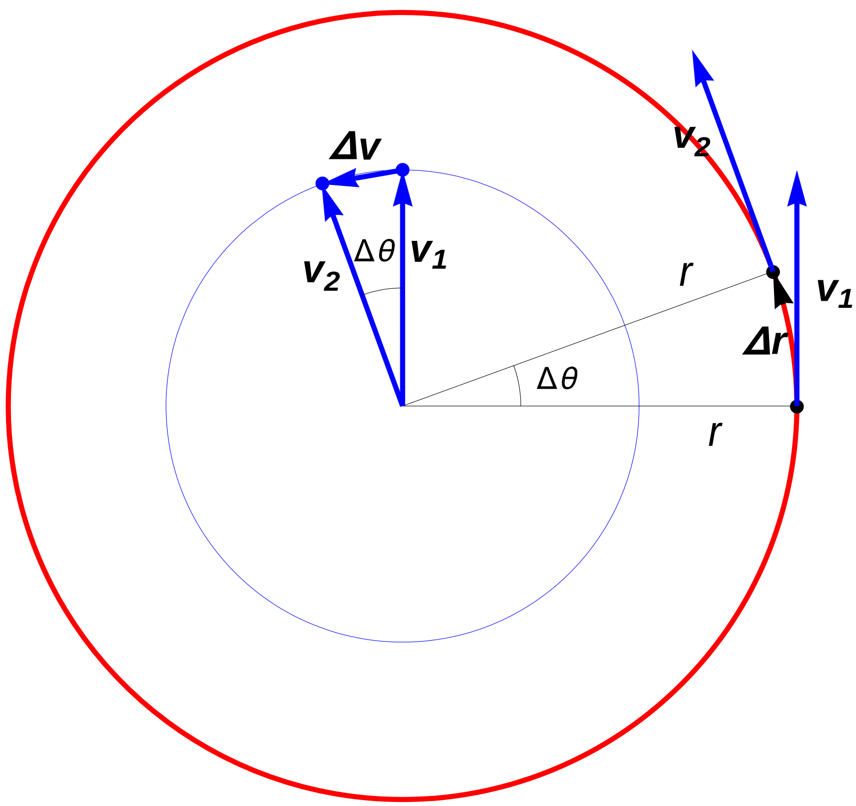
$$y = v_0 t - \frac{1}{2} g t^2 = v_0 \left(\frac{x}{v_1} \right) - \frac{1}{2} g \left(\frac{x}{v_1} \right)^2 = \frac{v_0}{v_1} x - \frac{1}{2} \frac{g}{v_1^2} x^2 ,$$

which is the motion of the ball observed from the ground. When $y = 0$, we have

$$x = v_1 t = \frac{2}{g} v_0 v_1 \qquad \text{and} \qquad t = \frac{x}{v_1} = \frac{2v_0}{g}.$$

He sees the train moving so that the horizontal positions of the ball when it leaves the man's hamd and when it returns are separated by the distance the train travels during the flight time of the ball. In fact, what he sees is the combined motions of constant acceleration in the vertical direction and constant velocity in the horizontal direction.

APPENDIX II Huygen's formula for the centripetal acceleration



We calculate the centripetal acceleration of a particle rotating on a circle of radius r, with constant speed. During a short time interval $\Delta t = t_2 - t_1$ from t_1 to t_2, the particle moves along the circle by a small distance $r\Delta\theta$, where $\Delta\theta$ is the rotation angle during the time interval Δt. The magnitude of the velocity is

$$v = \lim_{\Delta t \to 0} \frac{|\Delta \mathbf{r}|}{\Delta t} = \lim_{\Delta t \to 0} r \frac{\Delta\theta}{\Delta t} = r\omega.$$

Velocities of a particle moving on a red circle (with radius r) at two times, separated by a short time interval Δt. The particle undergoes a rotation at the constant angular velocity ω. The velocity vector $\boldsymbol{v}$ is always tangent to the circle. So it is perpendicular to the radial vector $\boldsymbol{r}$. The $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in the blue circle of Fig., are brought together into a triangle. For convenience, we draw the velocity space and position space in the same figure. The short side is the change of velocity in this interval, $\Delta\boldsymbol{v}$.

$$\Delta v = v\sin(\Delta\theta) \approx v(\Delta\theta),$$

where

$$v = \omega r.$$

Then the centripetal acceleration is obtained as

$$\boxed{a = \frac{\Delta v}{\Delta t} = v(\frac{\Delta\theta}{\Delta t}) = v\omega = \omega^2 r = \frac{v^2}{r}}.$$

This is Huygen's formula for the centripetal acceleration. In the limit of $\Delta t \to 0$,

$$\boldsymbol{a} = \frac{\Delta\boldsymbol{v}}{\Delta t}.$$

is perpendicular to the velocity $\boldsymbol{v}_1$ and is directed toward the center of the circle (called the centripetal acceleration).

REFERENCE

S. Weinberg, To Explain the World: The Discovery of Modern Science (HarperCollins, 2015).