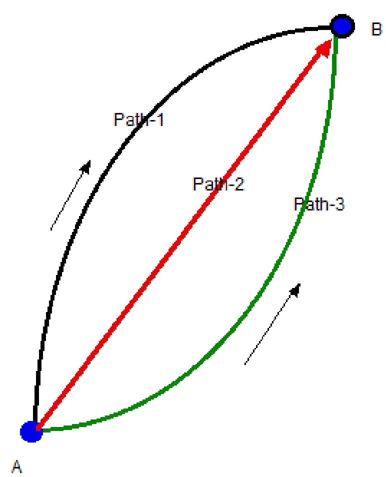
Lecture Note
Chapter 8
Potential energy and conservation of energy

1 Conservative force

1.1 Path integral

The work done by a conservative force on a particle moving between any two points is independent of the path taken by the particle.

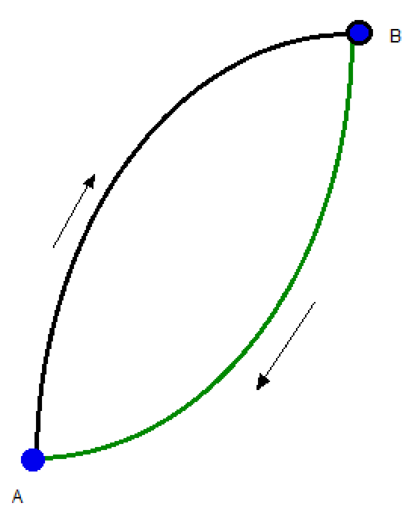


$$\int_{Path-1} \boldsymbol{F}_c \cdot d\boldsymbol{r} = \int_{Path-2} \boldsymbol{F}_c \cdot d\boldsymbol{r} = \int_{Path-3} \boldsymbol{F}_c \cdot d\boldsymbol{r}$$

for any path connecting two points A and B.

1.2 Path integral along the closed path

The work done by a conservative force on a particle moving through any closed path is zero. (A closed path is one for which the beginning point and the endpoint are identical).



$$\oint \boldsymbol{F}_c \cdot d\boldsymbol{r} = 0, \qquad \text{for any closed path}$$

2 Potential energy U

The quantity $\boldsymbol{F}_c \cdot d\boldsymbol{r}$ can be expressed in the form of a perfect differential

$$dU = -W_c = -\boldsymbol{F}_c \cdot d\boldsymbol{r}$$

where the function $U(\boldsymbol{r})$ depends only on the position vector r and does not depend explicitly on the velocity and time. A force $\boldsymbol{F}_c$ is conservative and U is known as the potential energy.

$$\int_A^B \boldsymbol{F}_c \cdot d\boldsymbol{r} = -\int_A^B dU = U(A) - U(B)$$

which does not depend on the path of integration but only on the initial and final positions. It is clear that the integral over a closed path is zero

$$\oint \boldsymbol{F}_c \cdot d\boldsymbol{r} = 0 \qquad (1)$$

which is a different way of saying that the force field is conservative

Using Stoke's theorem;

For any vector $\boldsymbol{A}$,

$$\oint \boldsymbol{A} \cdot d\boldsymbol{r} = \int (\nabla \times \boldsymbol{A}) \cdot d\boldsymbol{a}\,.$$

Since

$$\oint \boldsymbol{F}_c \cdot d\boldsymbol{r} = \int (\nabla \times \boldsymbol{F}_c) \cdot d\boldsymbol{a} = 0$$

we have

$$\nabla \times \boldsymbol{F}_c = 0, \qquad (2)$$

where ∇ is a differential operator called del or nabla. The operator can be written in n terms of the Cartesian components x, y, z in the form

$$\nabla = \hat{\boldsymbol{i}}\frac{\partial}{\partial x} + \hat{\boldsymbol{j}}\frac{\partial}{\partial y} + \hat{\boldsymbol{k}}\frac{\partial}{\partial z}$$

In this case, $\boldsymbol{F}_c$ can be expressed by

$$\boldsymbol{F}_c = -\nabla U$$

or

$$\boldsymbol{F}_c = \left(-\frac{\partial U}{\partial x}, -\frac{\partial U}{\partial y}, -\frac{\partial U}{\partial z} \right)$$

which leads to the relation

$$\begin{aligned} \frac{\partial F_{cx}}{\partial y} &= \frac{\partial F_{cy}}{\partial x} \\ \frac{\partial F_{cy}}{\partial z} &= \frac{\partial F_{cz}}{\partial y} . \\ \frac{\partial F_{cz}}{\partial x} &= \frac{\partial F_{cx}}{\partial z} \end{aligned}$$

This relation can be used to decide whether a force is conservative or not on physical grounds.

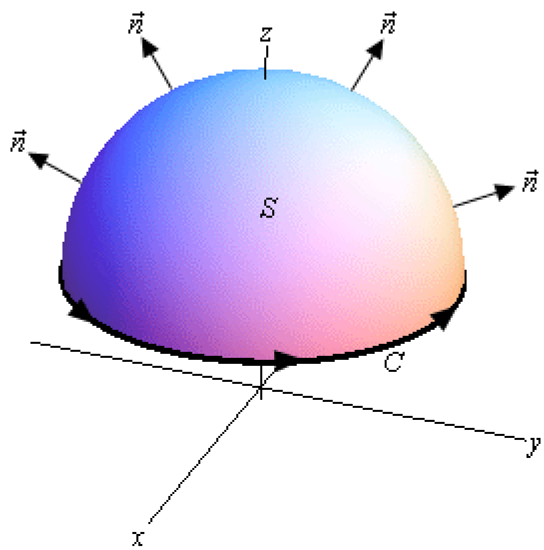
We note that

$$\begin{aligned} \nabla \times \boldsymbol{F}_c &= \begin{vmatrix} \hat{\boldsymbol{i}} & \hat{\boldsymbol{j}} & \hat{\boldsymbol{k}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_{cx} & F_{cy} & F_{cz} \end{vmatrix} = \hat{\boldsymbol{i}} \begin{vmatrix} \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_{cy} & F_{cz} \end{vmatrix} - \hat{\boldsymbol{j}} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial z} \\ F_{cx} & F_{cz} \end{vmatrix} + \hat{\boldsymbol{k}} \begin{vmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \\ F_{cx} & F_{cy} \end{vmatrix} \\ &= \hat{\boldsymbol{i}} \left(\frac{\partial F_{cz}}{\partial y} - \frac{\partial F_{cy}}{\partial z} \right) - \hat{\boldsymbol{j}} \left(\frac{\partial F_{cz}}{\partial x} - \frac{\partial F_{cx}}{\partial z} \right) + \hat{\boldsymbol{k}} \left(\frac{\partial F_{cy}}{\partial x} - \frac{\partial F_{cx}}{\partial y} \right) \end{aligned}$$

3 Stoke's theorem

Stokes' theorem (or Stokes's theorem) in differential geometry is a statement about the integration of differential forms which generalizes several theorems from vector calculus. It is named after Sir George Gabriel Stokes (1819–1903), although the first known statement of the theorem is by William Thomson (Lord Kelvin) and appears in a letter of his to Stokes. The theorem acquired its name from Stokes' habit of including it in the Cambridge prize examinations. In 1854, he asked his students to prove the theorem on an examination. It is unknown if anyone was able to do so.

Let's start off with the following surface with the indicated orientation.



Around the edge of this surface we have a curve *C*. This curve is called the **boundary curve**. The orientation of the surface *S* will induce the **positive orientation of *C***. To get the positive orientation of *C* think of yourself as walking along the curve. While you are walking along the curve if your head is pointing in the same direction as the unit normal vectors while the surface is on the left then you are walking in the positive direction on *C*. Now that we have this curve definition out of the way we can give Stokes' Theorem.

Stokes' Theorem

Let *S* be an oriented smooth surface that is bounded by a simple, closed, smooth boundary curve C with positive orientation. Also let ***F*** be a vector field, then

$$\oint \boldsymbol{F} \cdot d\boldsymbol{r} = \int (\nabla \times \boldsymbol{F}) \cdot d\boldsymbol{a} = \int (\nabla \times \boldsymbol{F}) \cdot \boldsymbol{n} da$$

In this theorem note that the surface *S* can actually be any surface so long as its boundary curve is given by *C*. This is something that can be used to our advantage to simplify the surface integral on occasion.

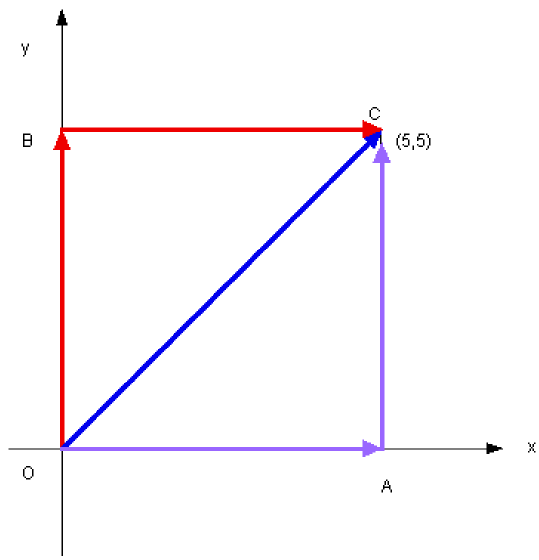
4. Path integral and conservative force

((**Example**)) **Serway Problem 7-15**

A force acting on a particle moving in the *x y* plane is given by

$$\boldsymbol{F} = (2y\hat{i} + x^2\hat{j})N ,$$

where *x* and *y* are in meters. The particle moves from the origin to a final position having coordinates *x* = 5.00 m and *y* = 5.00 m as in Fig. Calculate the work done by *F* along (a) OAC, (b) OBC, (c) OC. (d) Is *F* conservative or nonconservative? Explain.



((**Solution**))

$$\boldsymbol{F}\cdot d\boldsymbol{r} = F_x dx + F_y dy = 2ydx + x^2 dy$$

<u>Path OAC</u>
On the path OA; $x = 0 - 5$, $y = 0$; $\boldsymbol{F} = x^2\hat{j}$ and dy = 0.

$$\int_O^A \boldsymbol{F}.\cdot d\boldsymbol{r} = 0$$

On the path AC; $x = 5$, $y = 0$ - 5; $\boldsymbol{F} = 5^2\hat{j} = 25\hat{j}$

$$\int_A^C \boldsymbol{F}.\cdot d\boldsymbol{r} = \int_0^5 25dy = 125$$

Then we have

$$\int_O^C \boldsymbol{F}.\cdot d\boldsymbol{r} = 125J \text{ for the path OAC}$$

<u>Path OBC</u>
On the path OB; $x = 0$, $y = 0$ - 5;

$$\boldsymbol{F}\cdot d\boldsymbol{r} = 2ydx \text{ and dx} = 0$$

or

$$\int_O^B \boldsymbol{F} \cdot d\boldsymbol{r} = 0$$

On the path BC; $x = 0$ - 5, $y = 5$; $\boldsymbol{F} \cdot d\boldsymbol{r} = 2ydx = 10dx$

$$\int_B^C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^5 10dx = 50$$

Then we have

$$\int_O^C \boldsymbol{F} \cdot d\boldsymbol{r} = 50 \text{ J for the path OBC}$$

<u>Path OC</u>

$x = t$, y = t for $t = 0 – 5$.

$$2ydx + x^2dy = 2tdt + t^2dt = (t^2 + 2t)dt$$

$$\int_O^C \boldsymbol{F} \cdot d\boldsymbol{r} = \int_0^5 (t^2 + 2t)dt = \frac{125}{3} + 2\frac{5^2}{2} = \frac{200}{3} J$$

for the path OC line. Then the force is non-conservative.

((**Note**))

$$\boldsymbol{F} = (2y\hat{\boldsymbol{i}} + x^2\hat{\boldsymbol{j}})N$$

$$\frac{\partial F_x}{\partial y} = 2$$

$$\frac{\partial F_y}{\partial x} = 2x$$

This implies that the force is not conservative.

5 Energy conservation law

The work-energy theorem

$$\Delta K = W = W_{nc} + W_c$$

Since $W_c = -\Delta U$,

$\Delta K = W_{nc} - \Delta U$ or $\Delta(K + U) = W_{nc}$

or

$$\Delta E = W_{nc}$$

where

$$E = K + U$$

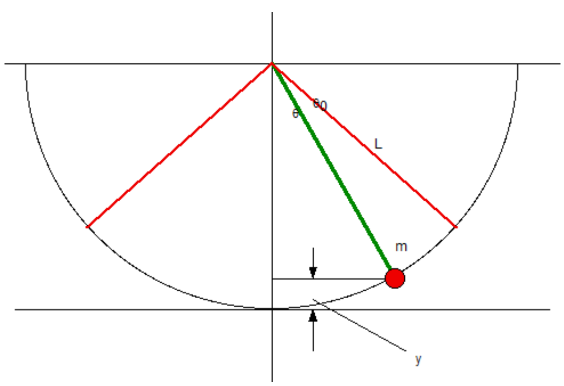
When $W_{nc} = 0$, the energy is conserved.

$$\Delta E = 0$$

(Energy conservation law)

6 Potential energy

6.1 Pendulum



The potential energy U for the pendulum is given by

$$U = mgy = mgL(1 - \cos\theta)$$

since

$$F_c = -mg$$

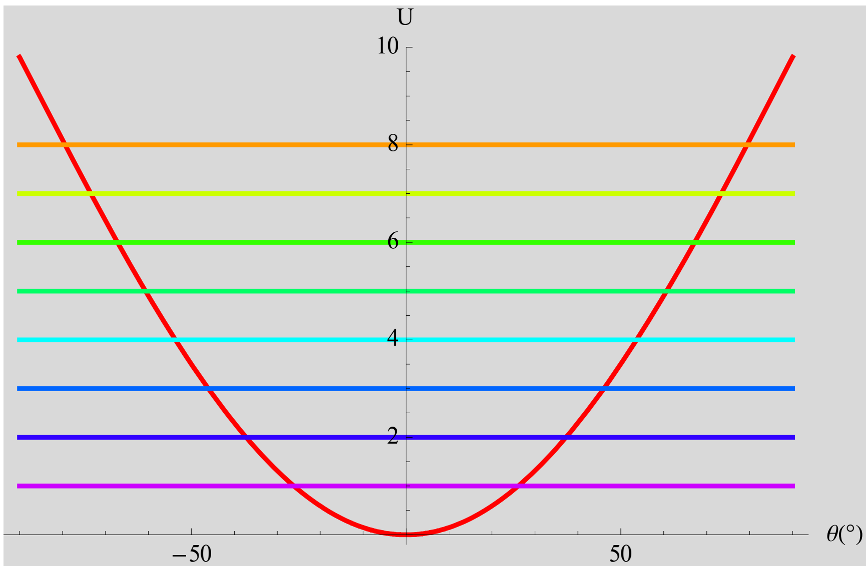
$$U = -\int_0^y F_c dy = mg\int_o^y dy = mgy$$

((Mathematica))

```
Clear["Global`*"]; U = m g L (1 - Cos[θ]);

rule1 = {L → 1, m → 1, g → 9.8}; U1 = U /. rule1 /. θ → π x/180;

Plot[{U1, 8, 7, 6, 5, 4, 3, 2, 1}, {x, -90, 90},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 AxesLabel → {"θ(°)", "U"}, Background → LightGray]
```



((Note))

The total energy for the simple pendulum is given by

$$E = \frac{1}{2}m(L\dot{\theta})^2 + mgL(1-\cos\theta) \qquad \text{(energy conservation law)}$$

where $\omega^2 = \dfrac{g}{L}$

$$\frac{dE}{dt} = mL^2(\ddot{\theta} + \omega^2 \sin\theta)\dot{\theta} = 0$$

leading to the differential equation for the simple pendulum.

$$\ddot{\theta} + \omega^2 \sin\theta = 0$$

In the limit of small angle, we have a simple harmonic with

$$\ddot{\theta} + \omega^2 \theta = 0$$

When $\dot{\theta} = 0$, we have $\theta = \theta_{max}$.

$$E = mgL(1 - \cos\theta_{max}) = 2mgL\sin^2(\frac{\theta_{max}}{2})$$

When $\theta = 0$, we have $\dot{\theta} = \dot{\theta}_{max}$;

$$E = \frac{1}{2}m(L\dot{\theta}_{max})^2$$

Then we have

$$2mgL\sin^2(\frac{\theta_{max}}{2}) = \frac{1}{2}m(L\dot{\theta}_{max})^2$$

or

$$(\dot{\theta}_{max})^2 = 4\omega^2\sin^2(\frac{\theta_{max}}{2}).$$

6.2 Simple harmonics

The potential energy U for the simple harmonics is given by

$$U = \frac{1}{2}kx^2$$

since

$$F_c = -kx$$

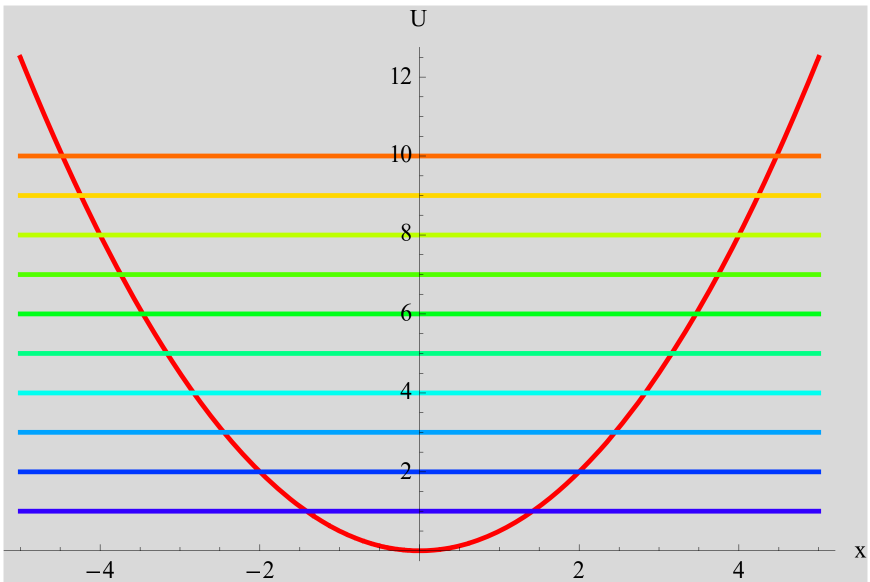
$$U = -\int_0^x F_c dx = k\int_o^x x dx = \frac{1}{2}kx^2$$

Note that F_c is given by

$$F_c = -\frac{dU}{dx}$$

((**Mathematica**))

```
Clear["Global`*"]; U = 1/2 k x^2; rule1 = {k → 1};
U1 = U /. rule1;
Plot[{U1, 10, 9, 8, 7, 6, 5, 4, 3, 2, 1}, {x, -5, 5},
 PlotStyle → Table[{Hue[0.07 i], Thick}, {i, 0, 10}],
 AxesLabel → {"x", "U"}, Background → LightGray]
```



((**Note**))
Simple harmonics:

$$m\ddot{x} + kx = 0 .$$

where $$\omega^2 = \frac{k}{m} .$$

Using the relation given by

$$m\ddot{x}\dot{x} + kx\dot{x} = 0 ,$$

the expression for the total energy is given by

$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \qquad \text{(energy conservation law)}$$

since

$$\frac{dE}{dt} = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = 0$$

At $\dot{x} = 0$, $x = x_{max}$.

$$E = \frac{1}{2}kx_{max}{}^2$$

At $x = 0$, $\dot{x} = v_{max}$

$$E = \frac{1}{2}mv_{max}{}^2$$

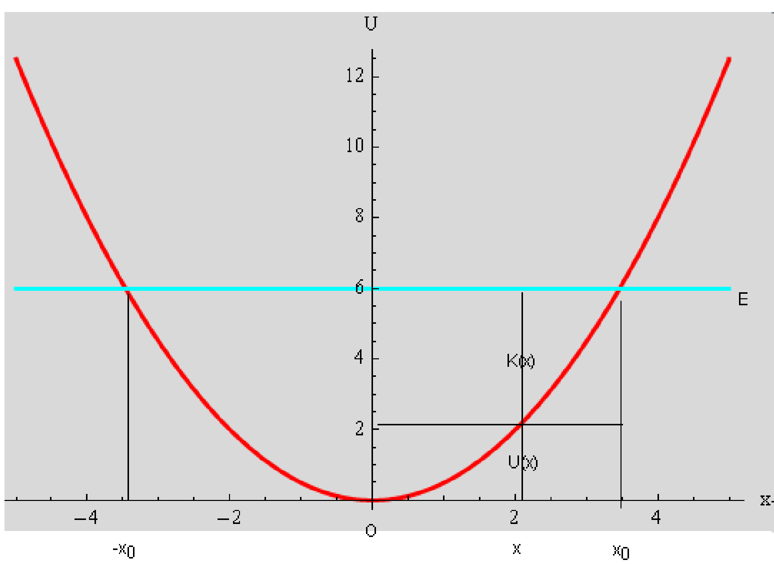
Then we have

$$v_{max} = \omega x_{max}\,.$$

7 Physical meaning of the potential energy

$$\boxed{\begin{aligned} F(x) &= -\frac{dU}{dx} \\ E &= K(x) + U(x) \end{aligned}}$$

E: mechanical energy
K: kinetic energy



The energy conservation
For a given *E* (total energy), we have

$$E = K(x) + U(x)$$

At $x = \pm x_0$, $U(x) = E$ and $K = 0$.
At $x = 0$, $U(x) = 0$ and $F = 0$. $K = E$.
For $x>0$, $F < 0$
For $x<0$, $F > 0$

(a) x_0 is a turning point
$K = 0$. The particle changes
(b) $x = 0$ is an equilibrium point
The point at which $U(x)$ has a local minimum

8. Force and energy on a atomic scale: Lennard & Jones

8.1 Model

The potential energy associated with the force two neutral atoms in a molecule can be modeled by the Lennard-Jones potential energy function

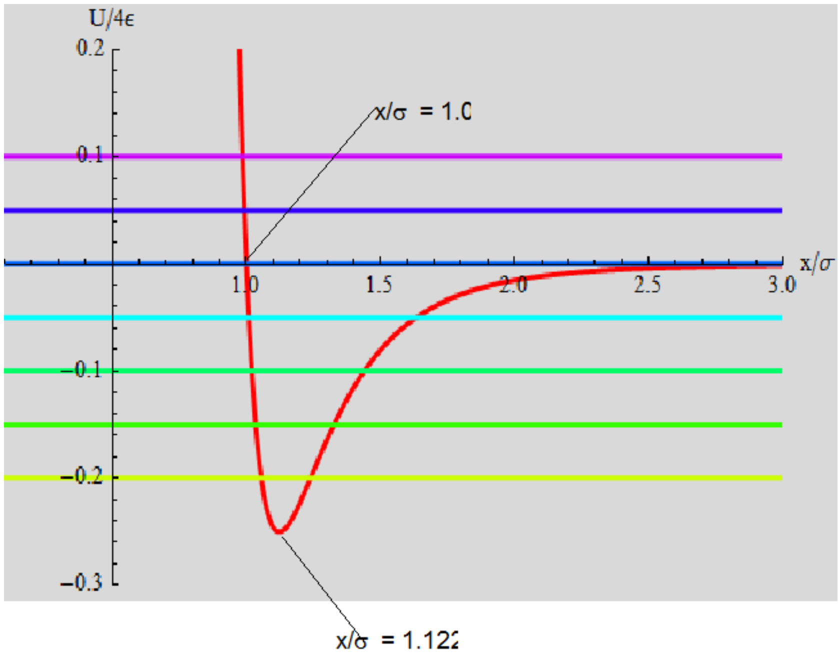
$$U(x) = 4\varepsilon[(\frac{\sigma}{x})^{12} - (\frac{\sigma}{x})^{6}]$$

The function $U(x)$ contains two parameters σ and ε that are determined from experiments: typically $\sigma = 0.263$ and $\varepsilon = 1.51 \times 10^{-22}$ J.

We use the scaling function.

$$u(x) = \frac{U(x)}{4\varepsilon} = [(\frac{\sigma}{x})^{12} - (\frac{\sigma}{x})^{6}]$$

$u(x)$ has a local minimum at $x = x_{eq} = 1.122\ \sigma$.



The force is given by

$$f(x) = -\frac{d}{dx}[\frac{U(x)}{4\varepsilon}] = -\frac{6\sigma^6}{x^{13}}(x^6 - 2\sigma^6)$$

Turning points

(a) $U/4\varepsilon$ = -0.1
x/σ = 1.02013, 1.43884

(b) $U/4\varepsilon$ = -0.05
x/σ = 1.0091, 1.63272

((**Mathematica**))

Lennard - Jones potential

```
u1 = (σ/x)^12 - (σ/x)^6
```

$$-\frac{\sigma^6}{x^6}+\frac{\sigma^{12}}{x^{12}}$$

```
u11 = D[u1, x] // Simplify
```

$$\frac{6\,\sigma^6\left(x^6-2\,\sigma^6\right)}{x^{13}}$$

```
F1 = -u11
```

$$-\frac{6\,\sigma^6\left(x^6-2\,\sigma^6\right)}{x^{13}}$$

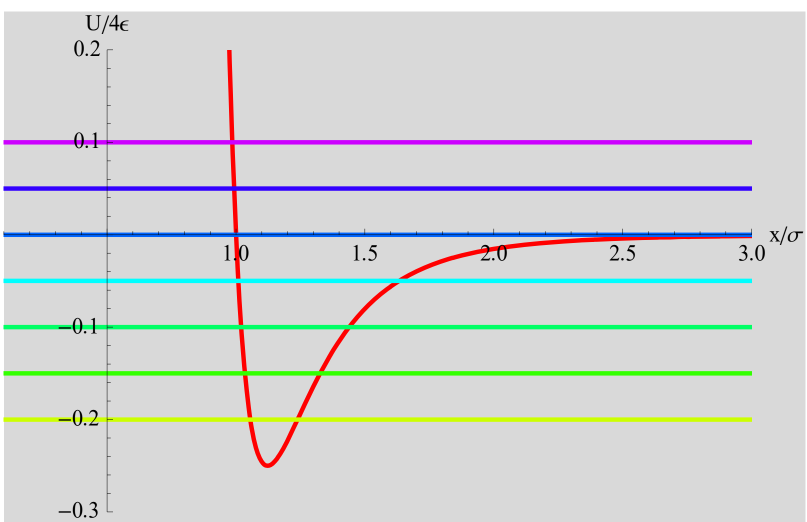
```
Simplify[Solve[u11 == 0, x], σ > 0] // N
```

```
{{x → -1.12246 σ}, {x → 1.12246 σ},
 {x → (-0.561231 - 0.972081 i) σ},
 {x → (0.561231 + 0.972081 i) σ},
 {x → (0.561231 - 0.972081 i) σ},
 {x → (-0.561231 + 0.972081 i) σ}}
```

```
u2 = u1 /. σ → 1
```

$$\frac{1}{x^{12}}-\frac{1}{x^6}$$

```
Plot[{u2, , -0.2, -0.15, -0.1, -0.05, 0, 0.05, 0.1},
 {x, 0.1, 6}, PlotRange → {{0.1, 3}, {-0.3, 0.2}},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 Background → LightGray, AxesLabel → {"x/σ", "U/4ϵ"}]
```



Turning points at U/4ϵ = - 0.1

```
FindRoot[u2 == -0.1, {x, 1.2, 1.7}]
```

```
{x → 1.43884}
```

```
FindRoot[u2 == -0.1, {x, 0.1, 1.2}]
```

```
{x → 1.02013}
```

Turning points at U/4 ϵ = - 0.05

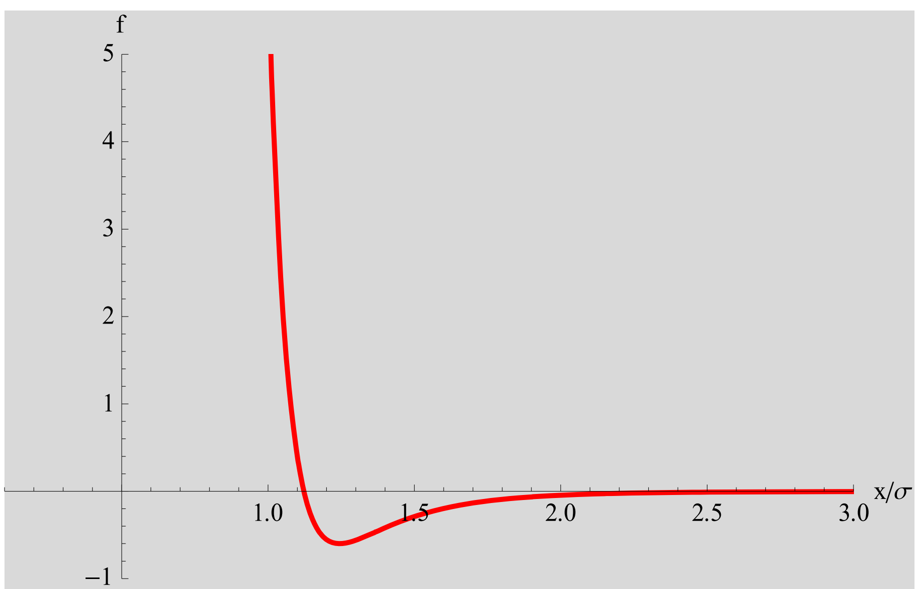
```
FindRoot[u2 == -0.05, {x, 1.2, 1.7}]
```

```
{x → 1.63272}
```

```
FindRoot[u2 == -0.05, {x, 0.1, 1.2}]
```

```
{x → 1.00908}
```

```
Plot[F1 /. σ → 1, {x, 0.1, 6},
 PlotRange → {{0.1, 3}, {-1, 5}}, PlotStyle → {Red, Thick},
 Background → LightGray, AxesLabel → {"x/σ", "f"}]
```



8.2 Lennard-Jones potential (from Wikipedia)

A pair of neutral atoms or molecules is subject to two distinct forces in the limit of large separation and small separation: an attractive force at long ranges (van der Waals force, or dispersion force) and a repulsive force at short ranges (the result of overlapping electron orbitals, referred to as Pauli repulsion from Pauli exclusion principle). The

Lennard-Jones potential (also referred to as the L-J potential, 6-12 potential or, less commonly, 12-6 potential) is a simple mathematical model that represents this behavior. It was proposed in 1924 by John Lennard-Jones.

The Lennard-Jones potential is an approximation. Its physical origin is related to the Pauli principle: when the electronic clouds surrounding the atoms start to overlap, the energy of the system increases abruptly. The exponent 12 was chosen exclusively because of ease of computation.

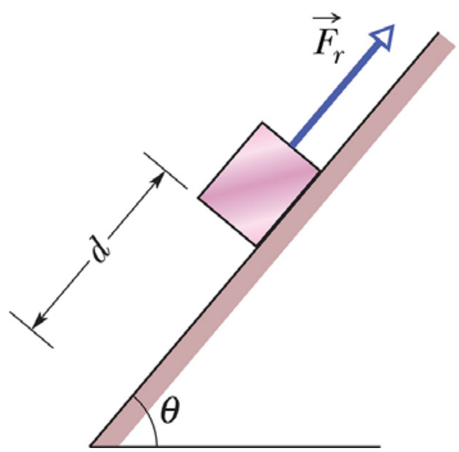
The attractive long-range potential, however, is derived from dispersion interactions. The L-J potential is a relatively good approximation and due to its simplicity is often used to describe the properties of gases, and to model dispersion and overlap interactions in molecular models. It is particularly accurate for noble gas atoms and is a good approximation at long and short distances for neutral atoms and molecules.

9. Selected problem

9.1

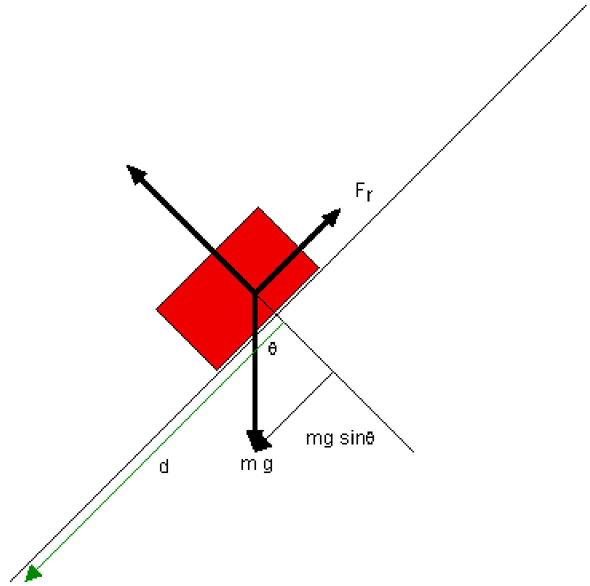
Problem 7-19 (10-th edition)**

In Fig., a block of ice slides down a frictionless ramp at angle $\theta = 50^\circ$ while an ice worker pulls on the block (via a rope) with a force $\mathbf{F}_r$ that has a magnitude of 50 N and is directed up the ramp. As the block slides through distance $d = 0.50$ m along the ramp, its kinetic energy increases by 80 J. How much greater would its kinetic energy has been if the rope had not been attached to the block?



$d = 0.5$ m, $\theta = 50°$, $F_r = 50$ N, $\Delta K = 80$ J.

Free-body diagram



$$\Delta E = E_f - E_i = W_{nc}$$

or

$$\Delta E = E_f - E_i = \frac{1}{2}mv^2 - mgd\sin\theta = -F_r d$$

where

$$E_i = mgh = mgd\sin\theta$$
$$E_f = \frac{1}{2}mv^2$$
$$W_{nc} = -F_r d$$

(1) $F_r = 0.$

$$\frac{1}{2}mv_1^{\ 2} = mg(d\sin\theta),$$

(2) $F_r \neq 0.$

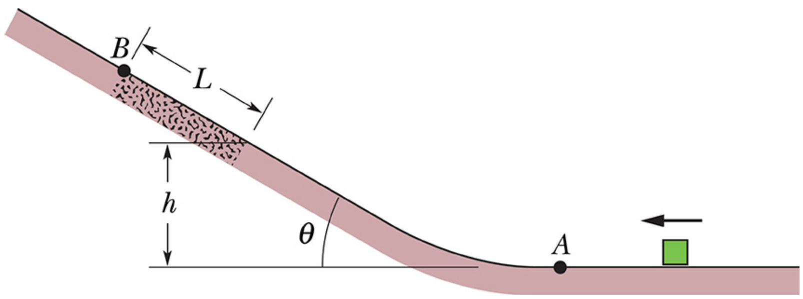
$$\frac{1}{2}mv_2^{\ 2} = -F_r d + mg(d\sin\theta)$$

Then we have

$$\frac{1}{2}m(v_2^{\,2} - v_1^{\,2}) = -F_r d = -25J$$

9.2
Problem 8-62* (SP-08) (10-th edition)**

In Fig. a block slides along a path that is without friction until the block reaches the section of length L = 0.75 m, which begins at height h = 0.2 m on a ramp of angle θ = 30º. In that section, the coefficient of kinetic friction is 0.40. The block passes through point A with a speed of 8.0 m/s. If the block can reach point B (where the friction ends), what is its speed there, and if it cannot, what is its greatest height above A?



((**My solution**))
h = 2.0 m, L = 0.75 m, θ = 30°, v_A = 8.0 m/s, g = 9.8 m/s^2
μ_K = 0.4

Suppose that the block arrives at the point B at the velocity v.

$$\Delta E = E_f - E_i = W_{nc} = -\mu_k mgL\cos\theta$$

Where

$$E_f = \frac{1}{2}mv^2 + mg(h + L\sin\theta)$$

$$E_i = \frac{1}{2}mv_A^{\,2}$$

Using the Mathematica (shown below), we have

v = 3.5 m/s

((**Mathematica**))

$$eq1 = \frac{1}{2} m\, v^2 + m\, g\, (h + L\, \mathrm{Sin}[\theta]) - \frac{1}{2} m\, vA^2 + \mu k\, m\, g\, L\, \mathrm{Cos}[\theta]$$

$$\frac{m\, v^2}{2} - \frac{m\, vA^2}{2} + g\, L\, m\, \mu k\, \mathrm{Cos}[\theta] + g\, m\, (h + L\, \mathrm{Sin}[\theta])$$

```
rule1 = {h → 2, L → 0.75, θ → 30 °, vA → 8, m → 1, g → 9.8, μk → 0.4}
{h → 2, L → 0.75, θ → 30 °, vA → 8, m → 1, g → 9.8, μk → 0.4}
```

```
eq11 = eq1 /. rule1
```

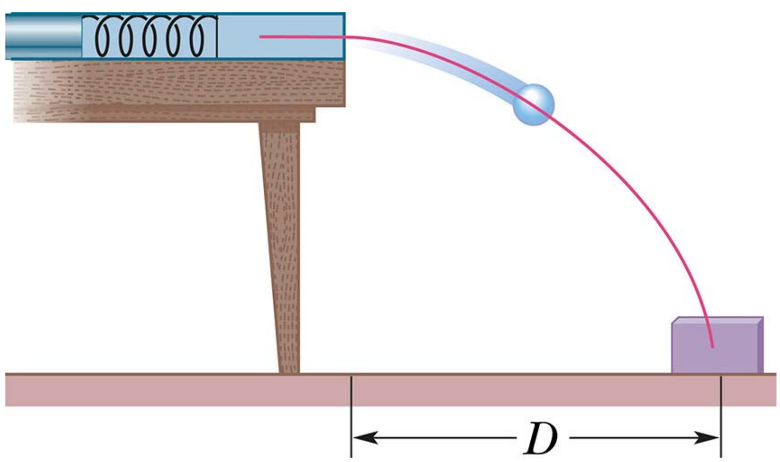
$$-6.17889 + \frac{v^2}{2}$$

```
Solve[eq11 == 0, v]
{{v → -3.51536}, {v → 3.51536}}
```

9.3

Problem 8-36* (SP-8) (10-th edition)**

Two children are playing a game in which they try to hit a small box on the floor with a marble fired from a spring-loaded gun that is mounted on a table. The target box is horizontal distance D = 2.20 m from the edge of the table; see Fig. Bobby compresses the spring 1.10 cm, but the center of the marble falls 27.00 cm short of the center of the box. How far should Rhoda compress the spring to score a direct hit? Assume that neither the spring nor the ball encounters friction in the gun.



((**My solution**))
D = 2.20m

Energy conservation:

$$\frac{1}{2}kx_0^{\ 2} = \frac{1}{2}mv_0^{\ 2}$$

$$v_0 = \sqrt{\frac{k}{m}}x_0$$

<u>Motion of the free fall for the system</u>

$$x = v_0 t$$

$$y = h - \frac{1}{2}gt^2$$

When $y = 0$, we have

$$t = \sqrt{\frac{2h}{g}}$$

$$x = d = v_0\sqrt{\frac{2h}{g}} = \sqrt{\frac{2h}{g}}\sqrt{\frac{k}{m}}x_0$$

In other words, it follows that d/x_0 = constant.

When x_0 = 1.10 cm, $d = D - 27$ cm = 193 cm. Then we have the relation

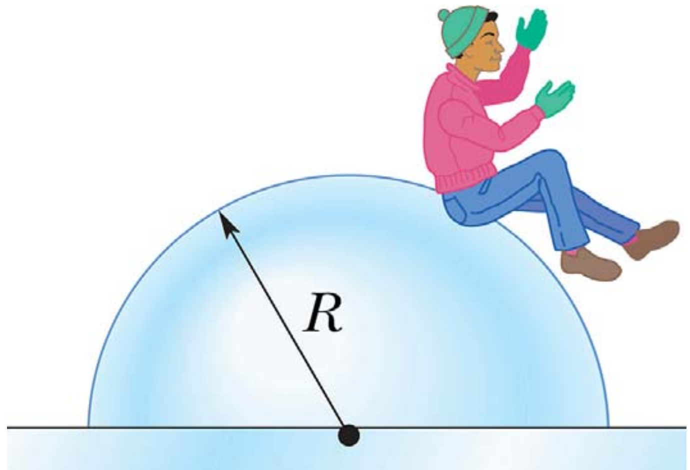
$$\frac{x_0}{d} = \frac{1.10}{193}$$

When $d = D$ = 220 cm, the value of x_0 is obtained as

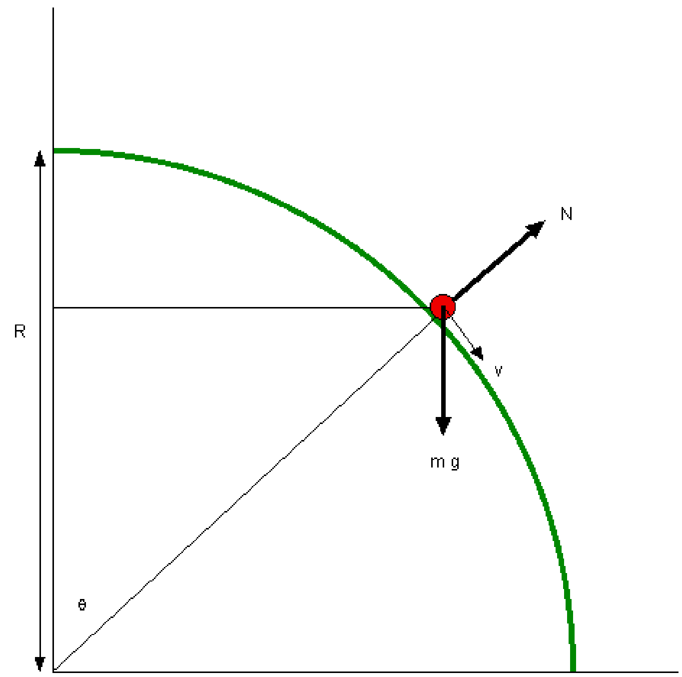
$$x_0 = \frac{1.10}{193}D = 1.25cm$$

9.4
Problem 8-34* (SP-07) (10-th edition)**

A boy is initially seated on the top of a hemispherical ice mound of radius R = 13.8 m. He begins to slide down the ice, with a negligible initial speed (Fig.). Approximate the ice as being frictionless. At what height does the boy lose contact with the ice?



((**Solution**))



$R = 13.8$ m

Newton's second law

$$mg\cos\theta - N = m\frac{v^2}{R}$$

When $N = 0$, we have

$$v^2 = gR\cos\theta \qquad (1)$$

Energy conservation

$$mgR = \frac{1}{2}mv^2 + mgR\cos\theta$$

or

$$v^2 = 2gR(1-\cos\theta) \qquad (2)$$

From Eqs.(1) and (2), we have

$$2(1-\cos\theta) = \cos\theta$$

or

$$\cos\theta = \frac{2}{3} \qquad \theta = 48.2°$$

The height h is given by

$$h = R\cos\theta = 13.8\times\frac{2}{3} = 9.2m$$

10. Problems of Homework (Hint) and SP

10.1
Problem 8-26 (HW-08) (10-th edition)**

A conservative $F = (6.0x - 12)\mathbf{i}N$, where x is in meters, acts on a particle moving along an x axis. The potential energy U associated with this force is assigned a value of 27 J at x = 0. (a) Write an expression for U as a function of x, with U in joules and x in meters. (b)What is the maximum positive potential energy? At what (c) negative value and (d) positive value of x is the potential energy equal to zero?

((**Solution**))
$\boldsymbol{F}$ = (6x-12) $\boldsymbol{i}$.
$U(x=0)$ = 27 J.

(a)

$$F = -\frac{dU}{dx} = 6x - 12$$

$$U(x) = U(0) - \int_0^x (6x-12)dx$$

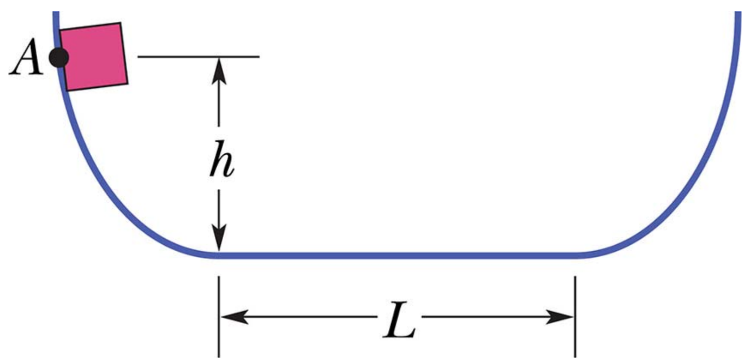
$$= -3x^2 + 12x + 27$$

(b)

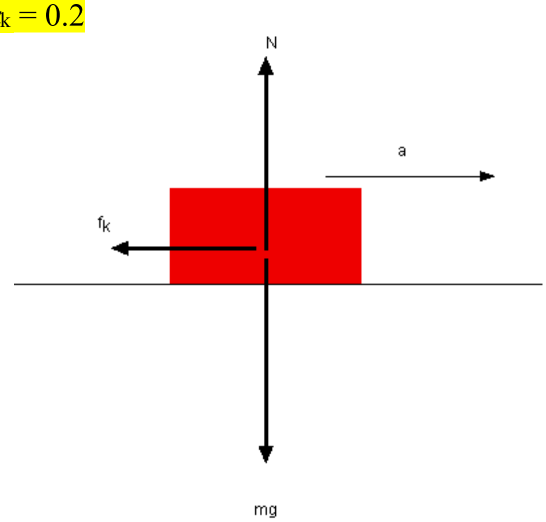
$$\frac{dU}{dx} = -6x + 12 = 0$$

10.2
Problem 8-65 (SP-08) (10-th edition)

A particle can slide along a track with elevated ends and a flat central part, as shown in Fig. The flat part has length L = 40 cm. The curved portions of the track are frictionless, but for the flat part the coefficient of kinetic friction is μ_k = 0.20. The particle is released from rest at point A, which is at height h = L/2. How far from the left edge of the flat part does the particle finally stop?



((**Solution**))
L = 0.4 m, h = 0.2 m, μ_k = 0.2



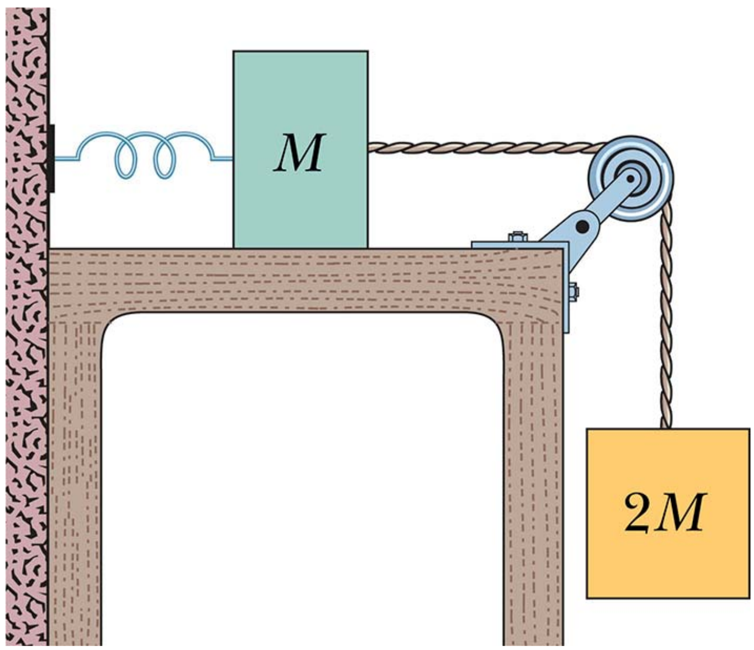
$$E_i = mgh = 0.2mg$$

$$\Delta E = 0 - 0.2mg = -\mu_k mgd_T = -0.2mgd_T$$

when the particle passes through the total distance d_T.

10.3
Problem 8-91 (HW-08) (10-th edition)

Two blocks, of mases M = 2.0 kg and 2M, are connected to a spring constant k = 200 N/m that has one end fixed, as shown in Fig. The horizontal surface and the pulley are frictionless, and the pulley has negligible mass. The blocks are released from rest with the spring relaxed. (a) What is the combined kinetic energy if the two blocks when the hanging block has fallen 0.090m? (b) What is the kinetic energy of the hanging block when it has fallen that 0.090 m? (c) What maximum distance does the hanging block fall before momentarily stopping?



((**Solution**))
k = 200 N/m. M = 2.0 kg.

The energyconservation: $E_f = E_i$ (= 0).

$$E_f = \frac{1}{2}kd^2 - 2Mgd + \frac{1}{2}(2M)v^2 + \frac{1}{2}Mv^2 = 0$$

The kinetic energy K_T is

$$K_T = \frac{1}{2}(2M)v^2 + \frac{1}{2}Mv^2 = \frac{3}{2}Mv^2$$

The maximum distance d_{max} is obtained when $v = 0$.

11. Advanced Problems

11.1 Young Problem 7-69

A Hooke's law force $-kx$ and a constant conservative force F in the x-direction act on an atomic ion.

(a) Show that a possible potential-energy function for this combination of forces is

$$U(x) = \frac{1}{2}kx^2 - Fx - \frac{F^2}{2k}$$

Is this the only possible function? Explain.

(b) Find the stable equilibrium position .

(c) Graph $U(x)$ (in units of F^2/k) versus x (in units of F/k) for values of x between -$5F/k$ to $5F/k$.

(d) Are there any unstable equilibrium position?

(e) If the total energy is $E = F^2/k$, what are the maximum and minimum values of x that the ion reaches in its motion?

(f) If the ion has mass m, find its maximum speed if the total energy is $E = F^2/k$. For what value of x is the speed maximum?

((**Solution**))

(a)

We start with

$$F(x) = -\frac{dU(x)}{dx} = -kx + F$$

or

$$\frac{dU(x)}{dx} = kx - F$$

$$U(x) = \frac{1}{2}kx^2 - Fx + C$$

where C is a constant.

(b) We choose $C = -\frac{F^2}{2k}$. Then we have

$$U(x) = \frac{1}{2}kx^2 - Fx - \frac{F^2}{2k}$$

Graph $U(x)$ (in units of F^2/k) versus x (in units of F/k)

$$\frac{U(x)}{\frac{F^2}{k}} = V(\xi) = \frac{k^2x^2}{2F^2} - \frac{kx}{F} - \frac{1}{2} = \frac{1}{2}\xi^2 - \xi - \frac{1}{2}$$

where

$$V(\xi) = \frac{U(x)}{\frac{F^2}{k}}$$

$$\xi = \frac{kx}{F}$$

(c) The stable equilibrium position is $\xi = 1$.

Since

$$V(\xi) = \frac{1}{2}\xi^2 - \xi - \frac{1}{2}$$

$$\frac{dV(\xi)}{d\xi} = \xi - 1 = 0$$

(d)

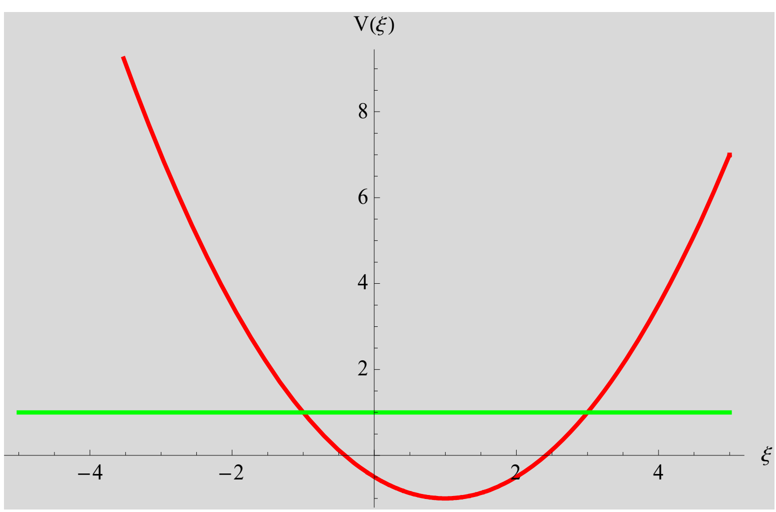


Fig. Plot of the normalized potential energy $V(\xi)$ as a function of ξ. The green line denotes $V(\xi) = 1$.

(e)
From

$$V(\xi) = \frac{1}{2}\xi^2 - \xi - \frac{1}{2} = 1$$
$$\xi^2 - 2\xi - 3 = 0$$
$$(\xi - 3)(\xi + 1) = 0$$

We have

ξ = -1 and 3.

(f)
From the energy conservation law

$$K = \frac{1}{2}mv^2 = E - U(x) = \frac{F^2}{k} - (\frac{1}{2}kx^2 - Fx - \frac{F^2}{2k}) = -\frac{1}{2}kx^2 + Fx + \frac{3F^2}{2k}$$

$$g(\xi) = \frac{K}{\frac{F^2}{k}} = -\frac{1}{2}\xi^2 + \xi + \frac{3}{2}$$

From $\frac{dg(\xi)}{d\xi} = 0$, we find that $g(\xi)$ has a local maximum at

$\xi = 1$ or $x = F/k$.

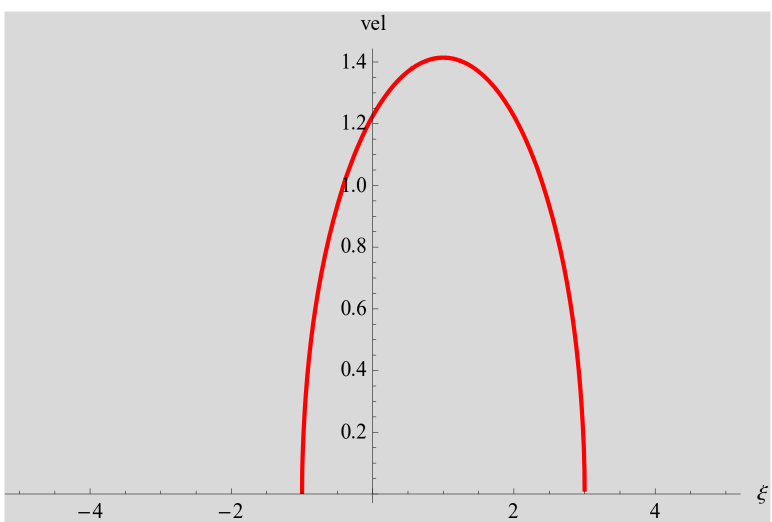
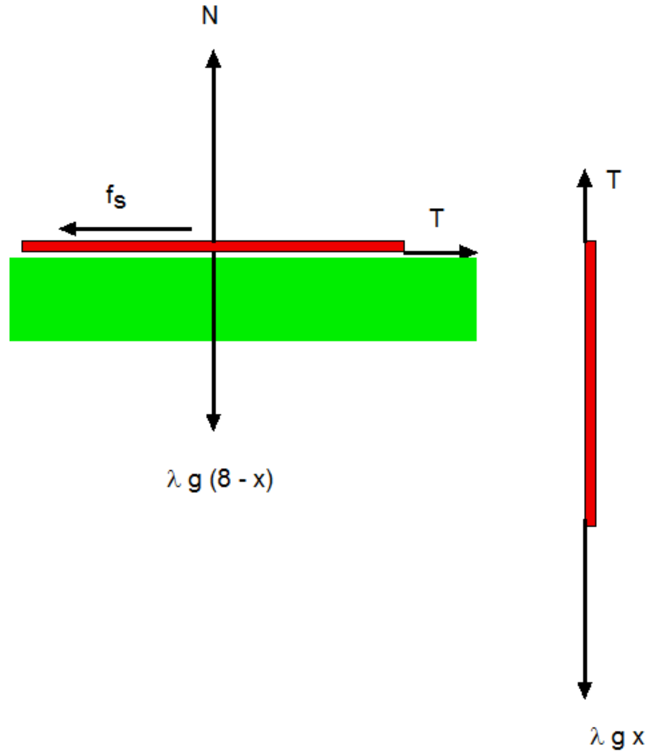


Fig. Plot of $[g(\xi)]^{1/2}$ (proportional to velocity) as a function of ξ.

11.2 Problem Serway 8-62

A uniform chain of length 8.00 m initially lies stretched out on a horizontal table.
(a) If the coefficient of static friction between chain and table is 0.60, show that the chain will begin to slide off the table if at least 3.00 m of it hangs over the edge of the table.
(b) Determine the speed of the chain as all of it leaves the table, given that the coefficient of kinetic friction between the chain and the table is 0.400.

(a)
Free-body diagram



$$N = \lambda g(8 - x)$$
$$f_s \le \mu_s N \qquad\qquad T = \lambda g x$$
$$T - f_s = 0$$

From these equations, we have

$$\lambda g x \le \mu_s \lambda g(8 - x)$$

or

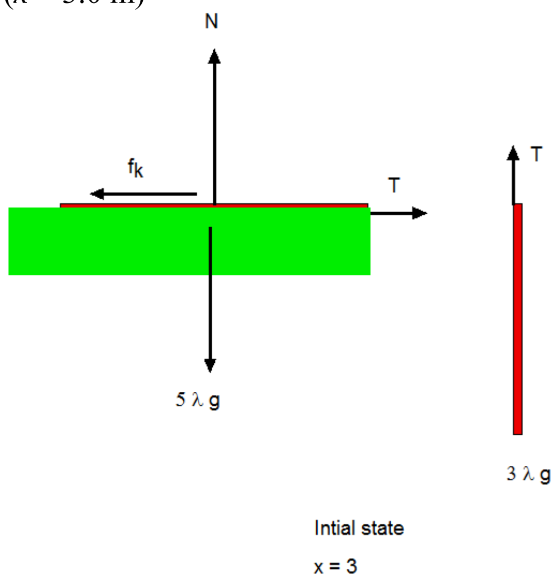
$$x \le 0.6(8 - x), \qquad \text{or} \qquad x \le 3$$

(b)
We apply the work-energy theorem

$$\Delta E = E_f - E_i = W_{nc}$$

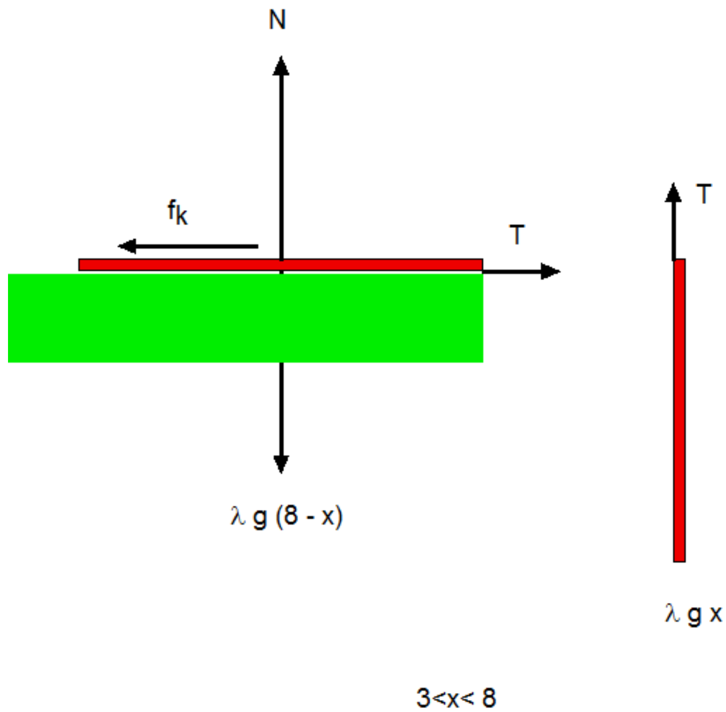
where E is the sum of the kinetic energy and potential energy. The reference height for the potential energy is at the table line. λ is the mass per unit length.

(i) Initial state ($x = 3.0$ m)



$$E_i = K_i + U_i = 0 + 3\lambda g(-\frac{3}{2}) = -4.5\lambda g$$

Intermediate state (3<x<8)

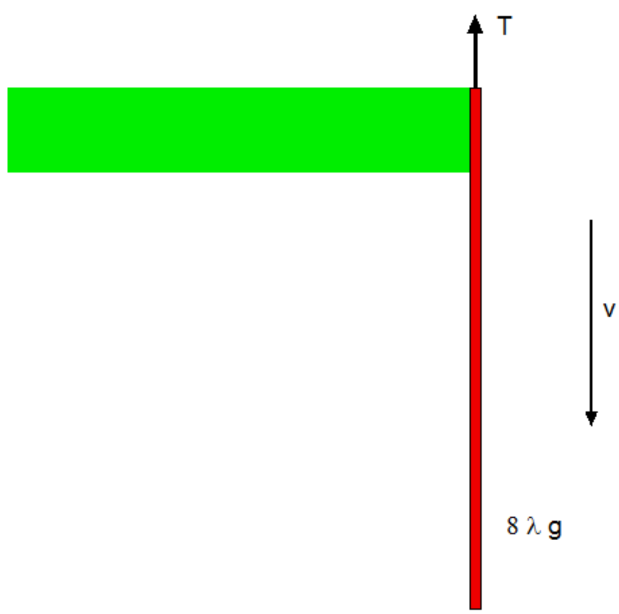


The friction force f_k is given as follows.

$$f_k = \mu_k \lambda g(8-x) = 0.4\lambda g(8-x)$$

where $3 \le x \le 8$.

(iii) The final state



$$E_f = K_f + U_f = \frac{1}{2} 8\lambda v^2 + 8\lambda g(-4) = 4\lambda v^2 - 32\lambda g$$

where v is the final velocity of the chain.

Using the work-energy theorem,

$$(4\lambda v^2 - 32\lambda g) - (-4.5\lambda g) = -\int_3^8 0.4\lambda g(8-x)dx$$

or

$$4v^2 - 27.5g = -g\int_3^8 0.4(8-x)dx$$

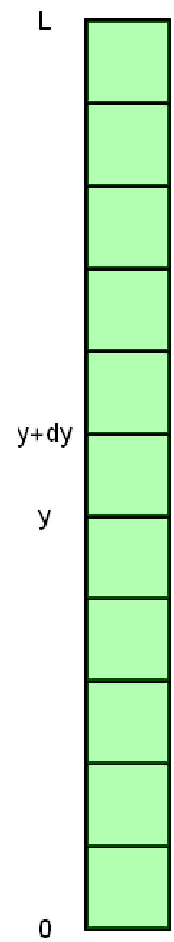
$v = 7.425$ m/s

((**Mathematica**))

```
eq1 = 4 v^2 - 27.5 g == -g ∫_3^8 0.4 (8 - x) dx
-27.5 g + 4 v^2 == -5. g

Solve[eq1 /. g → 9.8, v]
{{v → -7.42462}, {v → 7.42462}}
```

((**Note**))
The potential energy of a rod with mass having an uniform density



The potential energy of a rod with length (L) and a mass per unit length (λ). The total mass of the rod is given by

$$M = \lambda L\,.$$

The potential energy of the rod arising from the length between y and y+ dy is

$$dU = (\lambda dy)gy$$

Then the resultant potential energy U is

$$U = \int dU = \int_0^L (\lambda dy)gy = \lambda g \int_0^L ydy = \lambda g \frac{L^2}{2} = Mg\frac{L}{2}\,.$$

12. Measurement of period of the simple pendulum as a function of the initial angle

12.1 Energy conservation

We consider the physics of simple pendulum, where the bob is freely released from the intial angle θ_0. When θ_0 is very small, the system indergoes an ideal simple harmonics. However, if θ_0 becomes large, the non-linearity of the system is enhanced, leading to the deviation of the period from the ideal value.

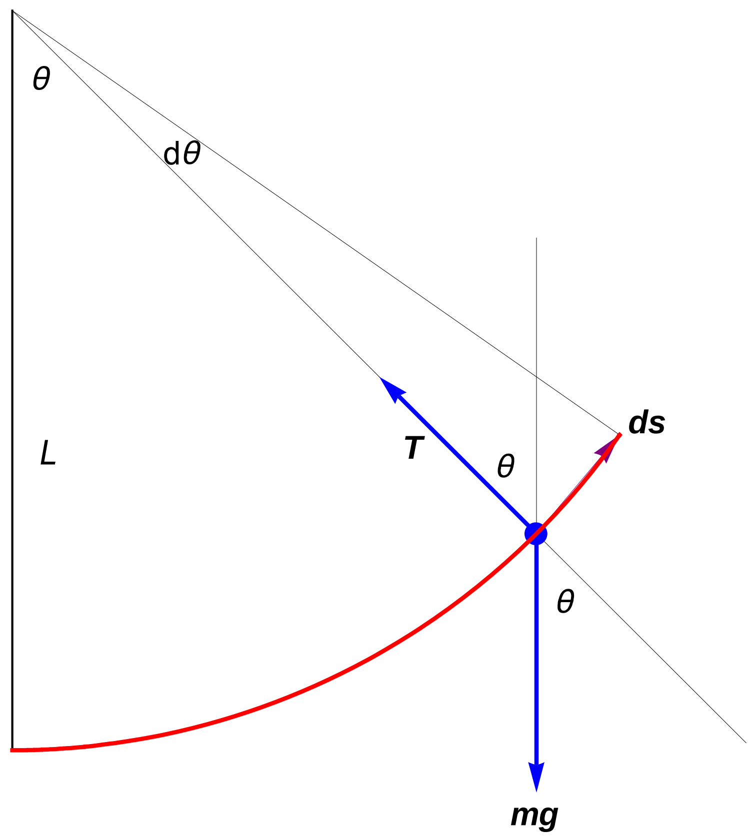


Fig. Free-body diagram for simple pendulum. No work is done by the tension *T*, since T is perpendicular to the circular path. L is the length of string (mass-less).

The kinetic energy is given by

$$K = \frac{1}{2} m (L\dot{\theta})^2 .$$

The potential energy is given by

$$U = mgL(1 - \cos\theta) .$$

The energy conservation

$$E = K + U = \frac{1}{2} mL^2\dot{\theta}^2 + mgL(1 - \cos\theta) .$$

Since $\frac{dE}{dt} = 0$, we get the differential equation;

$$mL^2\dot{\theta}(\ddot{\theta} + \frac{g}{L}\sin\theta) = 0,$$

leading to the equation of motion as

$$\ddot{\theta} + \omega^2 \sin\theta = 0,$$

with

$$\omega = \sqrt{\frac{g}{L}}.$$

The nonlinearity of the above equation arises from the term $\sin\theta$.

12.2. Newton's second law

From the free-body diagram shown above, we have

$$ma_\theta = mL\ddot{\theta} = -mg\sin\theta, \qquad \text{or} \qquad \ddot{\theta} = -\frac{g}{L}\sin\theta$$

$$T - mg\cos\theta = m\frac{(L\dot{\theta})^2}{L} \qquad \text{or} \qquad T = mg\cos\theta + mL\dot{\theta}^2.$$

The first equation is the same as derived from the energy conservation.

12.3. Work-energy theorem and energy conservation

We use the work-energy theorem,

$$\Delta K = W,$$

$$K = \frac{1}{2}mv_\theta{}^2 = \frac{1}{2}mL^2\dot{\theta}^2,$$

$$W = F_\theta(dr) = -mg\sin\theta(Ld\theta) = d(mgL\cos\theta).$$

There is no work from the tension T since the direction of the tension is perpendicular to the direction. Then we have

$$d(\frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta) = 0,$$

or

$$\frac{1}{2}mL^2\dot{\theta}^2 - mgL\cos\theta = \text{constant},$$

or

$$\frac{1}{2}mL^2\dot{\theta}^2 + mgL(1-\cos\theta) = E,$$

or

$$\frac{1}{2}\dot{\theta}^2 + \omega^2(1-\cos\theta) = E,$$

which is exactly the same as the result from the energy conservation law.

12.4 The expression for the period

We now discuss the period T for the energy conservation given by

$$E = \frac{1}{2}\dot{\theta}^2 + \omega^2(1-\cos\theta) = E = \omega^2(1-\cos\theta_0), \qquad (1)$$

where

$$\dot{\theta} = 0, \qquad \theta = \theta_0.$$

From Eq.(1), we have

$$\dot{\theta} = \frac{d\theta}{dt} = \pm 2\omega(\sin^2\frac{\theta_0}{2} - \sin^2\frac{\theta}{2})^{1/2},$$

where we use

$$\cos\theta = 2\cos^2\frac{\theta}{2} - 1 = 1 - 2\sin^2\frac{\theta}{2}.$$

The period T is obtained as

$$T=\frac{4}{2\omega}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}}=\frac{2}{\omega}\int_0^{\theta_0}\frac{d\theta}{\sqrt{\sin^2\frac{\theta_0}{2}-\sin^2\frac{\theta}{2}}},$$

when taking into account of the symmetry of the system. For simplicity we use

$$z=\frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_0}{2}}=\frac{1}{k}\sin\frac{\theta}{2}, \qquad \text{and} \qquad k=\sin\frac{\theta_0}{2}.$$

Then we get the final form of T as

$$T(k)=\frac{4}{\omega}\int_0^1\frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}.$$

We uase Mathematica for the calculation of T.

$$\begin{aligned}\frac{T(k)}{T(k=0)}&=\frac{\omega}{2\pi}\frac{4}{\omega}\int_0^1\frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}\\&=\frac{2}{\pi}\int_0^1\frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}\\&=\frac{2}{\pi}EllipticK[k^2]\end{aligned}$$

where

$$T(k=0)=\frac{4}{\omega}\int_0^1\frac{dz}{\sqrt{1-z^2}}=\frac{4}{\omega}\frac{\pi}{2}=\frac{2\pi}{\omega}$$

((**Note**)) EllipticK[k^2] is the complete elliptic integral of the first kind;

$$EllipticK[k^2]=\int_0^1\frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}}$$

We define

$$p=\frac{\Delta T(k)}{T(k=0)}=\frac{T(k)}{T(k=0)}-1$$

where

$$T(k) = T(\theta_0), \qquad T(k=0) = T(\theta_0 = 0)$$

We make a plot of p x 100 (%) as a function of the angle θ_0.

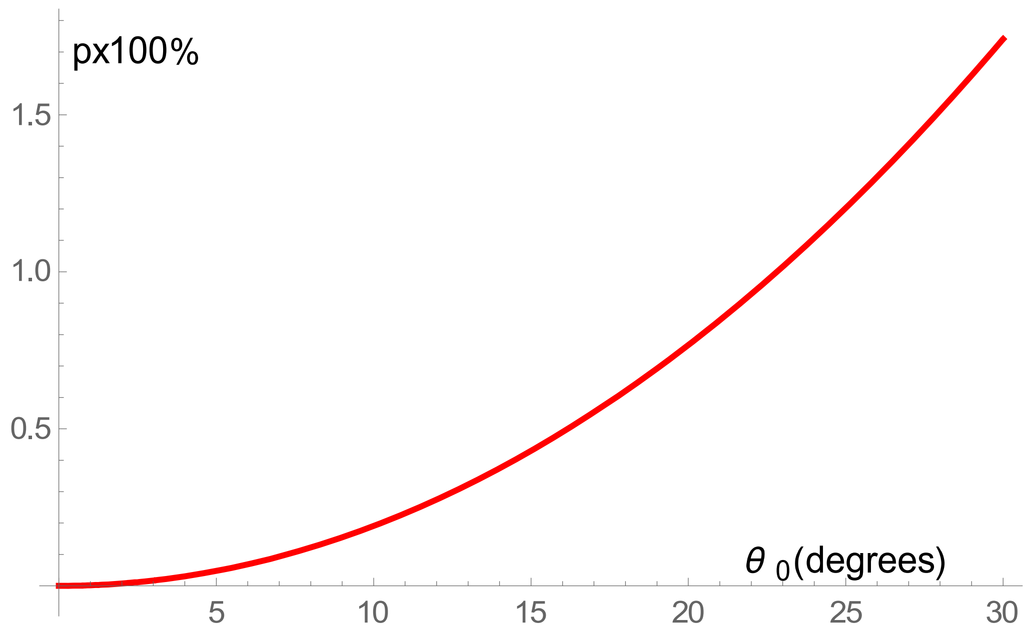


Fig. $p = \frac{T(k)}{T(k=0)} - 1$. Plot of p × 100 (%) as a function of the initial angle θ_0.

Series expansion in the vicinity of $k = 0$ is given by

$$\frac{T(k)}{T(k=0)} = 1 + \frac{k^2}{4} + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + \frac{53361}{1048576}k^{12} + ...$$

or

$$EllipticK[k^2] = \frac{\pi}{2}\frac{T(k)}{T(k=0)}$$
$$= \frac{\pi}{2}[1 + \frac{k^2}{4} + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + \frac{3969}{65536}k^{10} + ..]$$

where

$$k = \sin\frac{\theta_0}{2}.$$

12.5 Comment on the experiment on simple pendulum

Suppose that we have a measurement of the period of the simple harmonics. The bob is freely released from the initial angle θ_0. The period is measured as a function of the angle θ_0. In the limit $\theta_0 \to 0$, the system undergoes an ideal simple harmonic oscillation with the period $T = 2\pi\sqrt{l/g}$. However, if the angle θ_0 becomes large, the system shows nonlinear behavior. The period deviates from the ideal case. The theoretical deviation p (%) vs θ_0 is listed below the Table. The deviation percentage p increases with increasing θ_0. We note that for θ_0 = 6°, we have p = 0.069 %. This implies that the deviation p is assumed to be negligibly small when θ_0 is smaller than 5°.

Angle θ_0	Deviation p (%)
0	o
2	0.007
4	0.030
6	0.069
8	0.122
10	0.191
12	0.275
14	0.374
16	0.490
18	0.620
20	0.767
22	0.929
24	1.108
26	1.302
28	1.513
30	1.741
35	2.833
40	3.134
45	3.997
50	4.978
55	6.083
60	7.318
65	8.692
70	10.215
75	11.896
80	13.749
85	15.790
90	18.034

REFERENCES

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APPENDIX

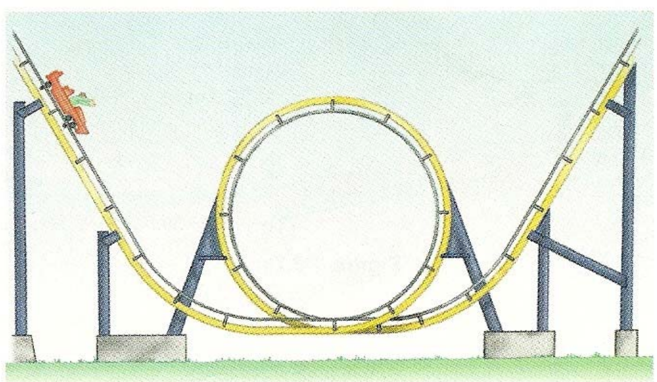
A Roller-coaster physics

Serway 8-73

A roller coaster car is released from rest at the top of the first rise and then moves freely with negligible friction. The roller coaster shown in Fig. has a circular loop of radius R in a vertical plane.

(a) Suppose first that the car barely makes it around the loop: at the top of the loop the riders are upside down and feel weightless. Find the required height of the release point above the bottom of the loop in terms of *R*.

(b) Now assume that the release point is at or above the minimum required height. Show that the normal force on the car at the bottom of the loop by six times the weight of the car. The normal force on each rider follows the same rule. Such a large normal force is dangerous and very uncomfortable for the rider. Roller coasters are therefore not built with circular loops in vertical planes.



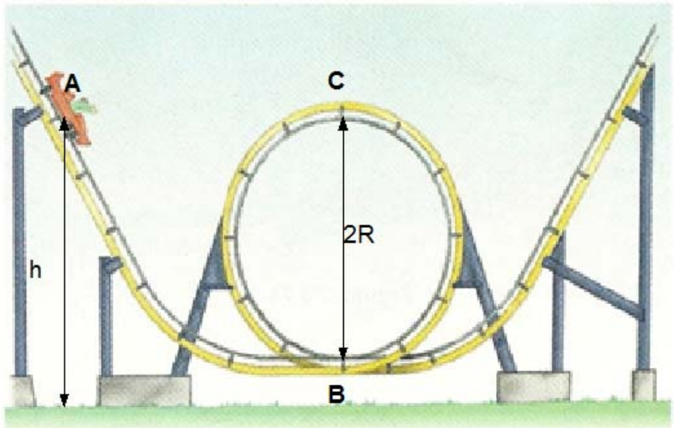
((**Solution**))

(a) The energy conservation law:

The total energy (consisting of the kinetic energy and potential energy) at the point A is equal to that at the point C,

$$mgh = mg(2R) + \frac{1}{2}mv_C{}^2 \qquad (1)$$

where v_C is the velocity at the point C.



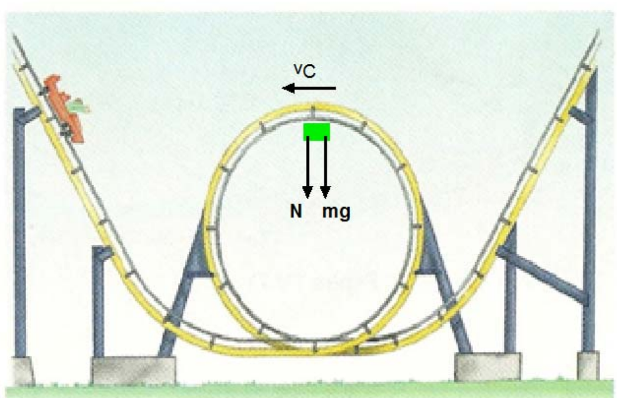
The condition that the car does not fall from the point C.

$$N + mg = m\frac{{v_C}^2}{R}$$

$$N = m(\frac{{v_C}^2}{R} - g) \geq 0$$

From the condition that $N \geq 0$, the minimum velocity at the point C is

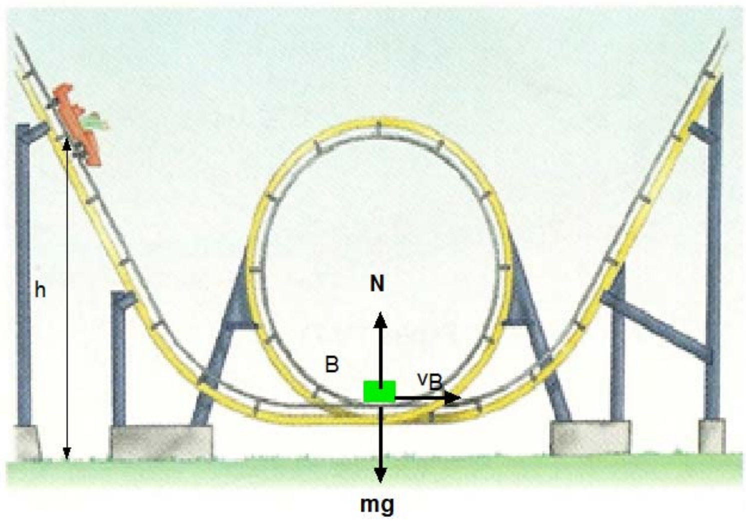
$$(v_C)_{\min} = \sqrt{gR} \tag{2}$$



From Eqs.(1) and (2), we have

$$h = 2R + \frac{1}{2g}{v_C}^2 \geq 2R + \frac{1}{2}R = \frac{5}{2}R$$

(b)



From the energy conservation law, we have

$$mgh = \frac{1}{2} m v_B{}^2 \qquad (3)$$

The normal force at the point B is calculated as

$$N - mg = m\frac{v_B{}^2}{R} = mg\frac{2h}{R}$$

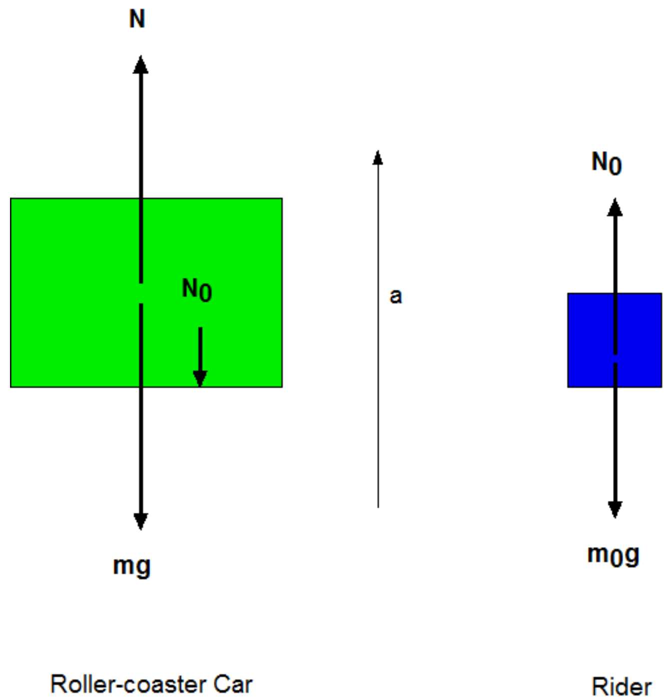
or

$$N = mg(1 + \frac{2h}{R})$$

When $h = 5R/2$, we have

$$N = 6\, mg.$$

((Note))
What is the normal force (N_0) for the rider with mass m_0?



$$N - N_0 - mg = m\frac{{v_B}^2}{R}$$

$$N_0 - m_0 g = m_0 \frac{{v_B}^2}{R}$$

The normal forces N (a roller-coaster and rider) and the normal force N_0 (a rider) are obtained as

$$N = (m + m_0)g(1 + \frac{2h}{R})$$

$$N_0 = m_0 g(1 + \frac{2h}{R}) \quad .$$
