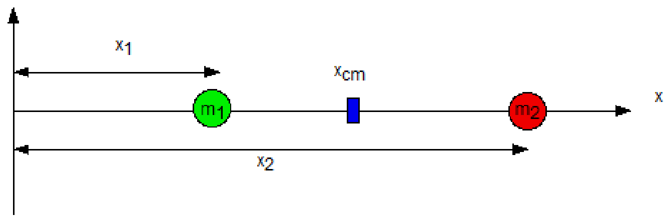
Chapter 9

1 Center of mass

Newton's laws of motion are formulated for single particle. However, they can be extended without difficulty to systems of particles and bodies of finite dimensions.

We now consider two particles located on the *x* axis.

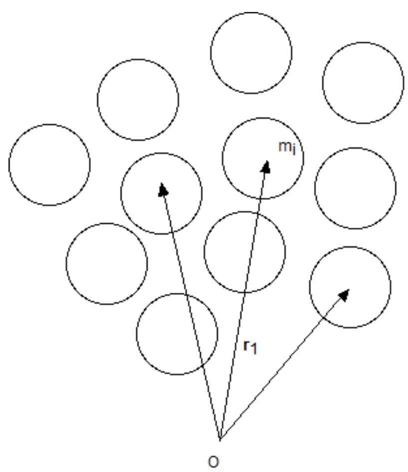


The position of the center of mass is defined by

$$x_{cm} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}$$

where m_1 and m_2 are the masses of particle 1 and particle 2.

For the system with many particles,



the position of center of mass is given by

$$\boldsymbol{R}_{cm} = \frac{m_1\boldsymbol{r}_1 + m_2\boldsymbol{r}_2 + m_3\boldsymbol{r}_3 + \ldots..}{m_1 + m_2 + m_3 + \ldots..} = \frac{\sum_i m_i\boldsymbol{r}_i}{\sum_i m_i}$$

where $M = \sum_i m_i$ is the total mass of the system. Physically, the center of mass can be interpreted as a weighted average position of the system of the particles. In the special case of a uniform gravitational field the center of mass coincides with the center of gravity.

For the continuous distribution of particles, the position of center of mass is given by

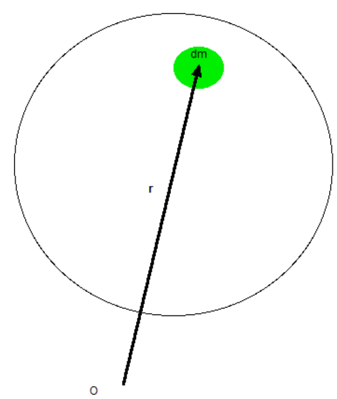
$$\boldsymbol{R}_{cm} = (x_{cm}, y_{cm}, z_{cm}) = \frac{\int \boldsymbol{r} dm}{\int dm}$$

or

$$x_{cm} = \frac{\int x dm}{\int dm}$$

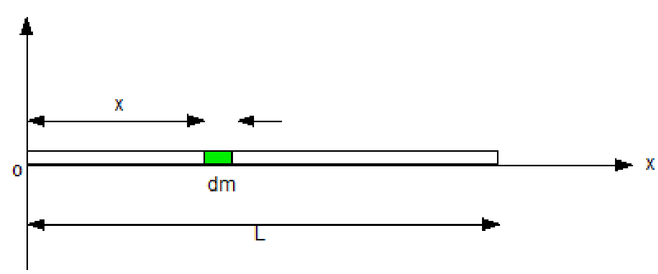
$$y_{cm} = \frac{\int y dm}{\int dm}$$

$$z_{cm} = \frac{\int z dm}{\int dm}$$



2. Calculation of center of mass

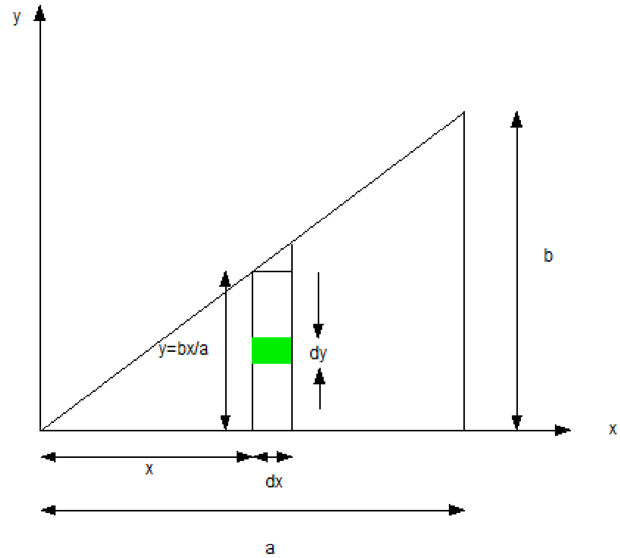
2.1. Center of the mass for the uniform rod



$$x_{cm} = \frac{\int x dm}{\int dm} = \frac{\lambda \int_0^L x dx}{\lambda \int_0^L dx} = \frac{\frac{L^2}{2}}{L} = \frac{L}{2}$$

where λ is the line density (mass/length): $\lambda = M/L$

2.2 Center of the mass for the uniform triangle

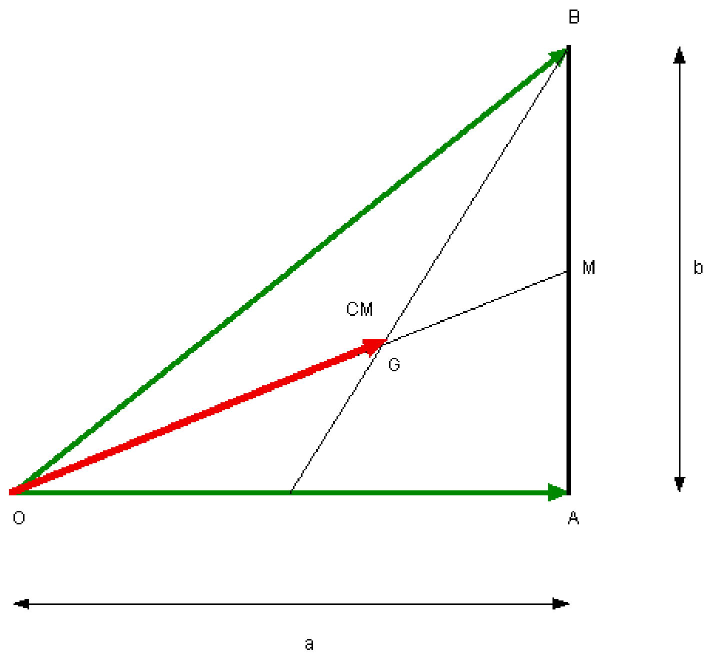


$$x_{cm} = \frac{\int x dm}{\int dm} = \frac{\sigma \iint x dx dy}{\sigma \iint dx dy} = \frac{\int_0^a x dx \int_o^{bx/a} dy}{\int_0^a dx \int_o^{bx/a} dy} = \frac{\int_0^a x \frac{bx}{a} dx}{\int_0^a \frac{bx}{a} dx} = \frac{\frac{ba^3}{3a}}{\frac{ba^2}{2a}} = \frac{2a}{3}$$

$$y_{cm} = \frac{\int y dm}{\int dm} = \frac{\sigma \iint y dx dy}{\sigma \iint dx dy} = \frac{\int_0^a dx \int_o^{bx/a} y dy}{\int_0^a dx \int_o^{bx/a} dy} = \frac{\int_0^a dx \frac{1}{2} \frac{b^2 x^2}{a^2}}{\int_0^a \frac{bx}{a} dx} = \frac{\frac{1}{2} \frac{b^2 a^3}{3a^2}}{\frac{ba^2}{2a}} = \frac{b}{3}$$

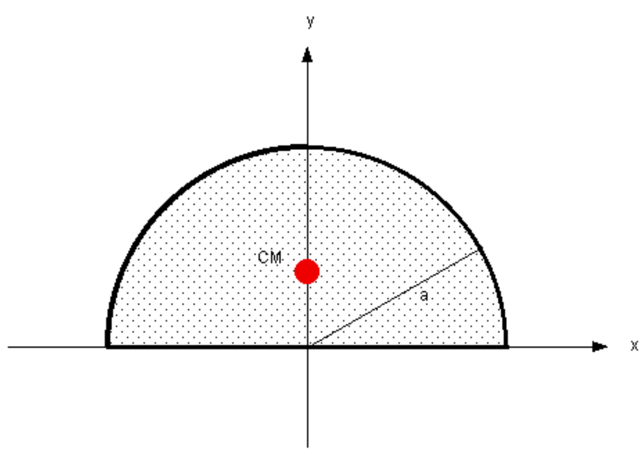
where σ is the area density (mass/area).

((**Geometry**))



$$\overrightarrow{OG} = \frac{2}{3} \cdot \left(\frac{\overrightarrow{OA} + \overrightarrow{OB}}{2} \right) = \frac{1}{3}[(a,0) + (a,b)] = (\frac{2a}{3}, \frac{b}{3})$$

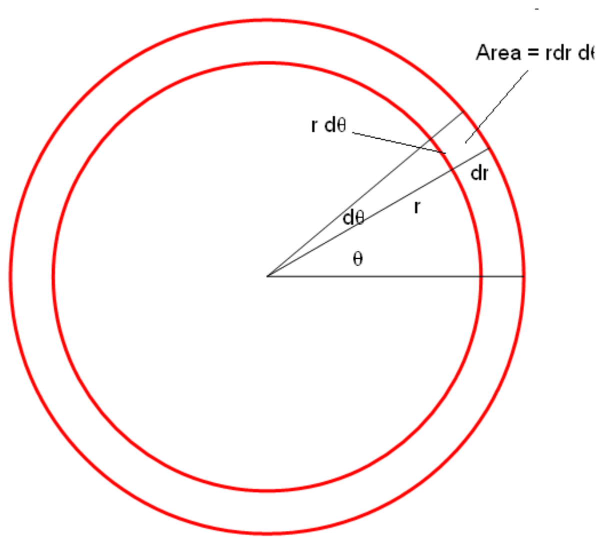
2.3 Center of mass for the half circle with radius *a*



((**Solution**))

$x_{cm} = 0$ from the symmetry.

$$y_{cm} = \frac{\int y dm}{\int dm} = \frac{\iint r\sin\theta r dr d\theta}{\iint r dr d\theta} = \frac{\int_0^a r^2 dr \int_0^\pi \sin\theta d\theta}{\int_0^a r dr \int_0^\pi d\theta} = \frac{2\frac{1}{3}a^3}{\pi(\frac{a^2}{2})} = \frac{4a}{3\pi} = 0.424a$$



3 Linear momentum

3.1 Definition of linear momentum

The linear momentum of the particle with a mass *m* and a velocity **v**, is defined by

$$\boldsymbol{p} = m\boldsymbol{v}$$

The general statement of Newton's second law

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

or

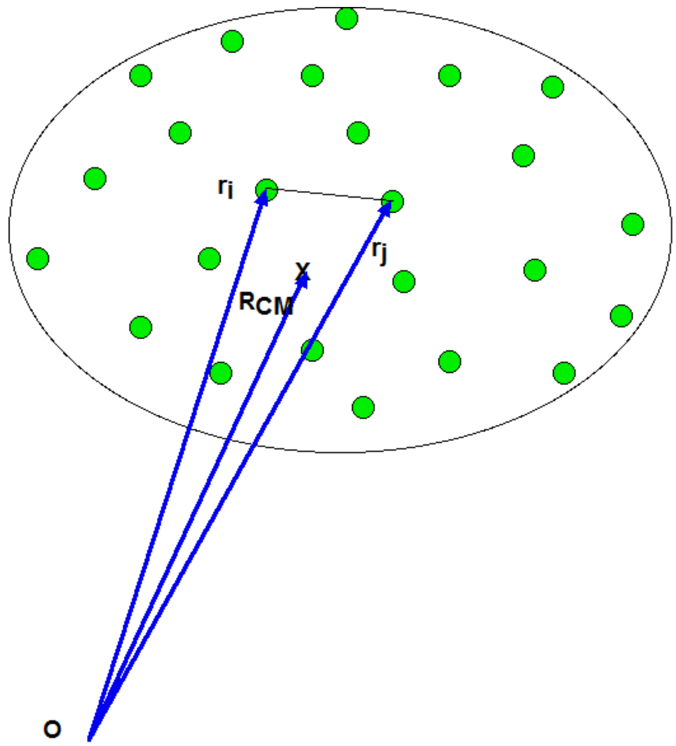
$$\boldsymbol{F} = \frac{d}{dt}(m\boldsymbol{v}) = m\frac{d\boldsymbol{v}}{dt} + \frac{dm}{dt}\boldsymbol{v}$$

For a constant mass *m*, we have

$$\boldsymbol{F} = m\frac{d\boldsymbol{v}}{dt} = m\boldsymbol{a} \qquad \text{(conventional law)}$$

3.2 System consisting of many particles

Now we apply the Newton's second law to the motion of many particles shown in Fig.



Newton's second law:

$$\dot{\boldsymbol{p}}_i = \sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)}$$

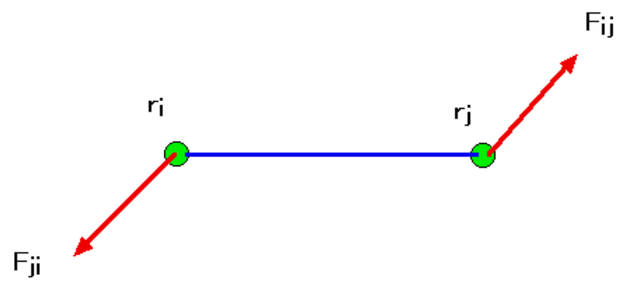
where $\boldsymbol{F}_{ji}$ is the internal force applied on the i-th particle from the j-th particle ($i \neq j$) and $\boldsymbol{F}_i^{(e)}$ is the external force applied on the i-th particle.

We note that

$$\boldsymbol{F}_{ji} = -\boldsymbol{F}_{ij} \qquad \text{(Newton's third law)}$$

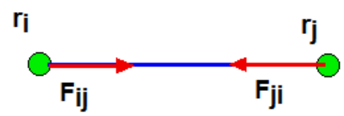
The forces two particles exert on each other are equal and opposite.

(a) Weak law of action and reaction



Weakg law of action and reaction

(b) Definition for the strong law of action and reaction.



Strong law of action and reaction

From the above equation, we have

$$\dot{\boldsymbol{p}}_i = \frac{d^2}{dt^2}(m_i \boldsymbol{r}_i) = \sum_j \boldsymbol{F}_{ji} + \boldsymbol{F}_i^{(e)}$$

or

$$\begin{aligned}\dot{\boldsymbol{P}} &= \sum \dot{\boldsymbol{p}}_i \\ &= \sum_i \frac{d^2}{dt^2}(m_i \boldsymbol{r}_i) \\ &= \sum_j \sum_j \boldsymbol{F}_{ji} + \sum_i \boldsymbol{F}_i^{(e)} \\ &= \sum_i \boldsymbol{F}_i^{(e)}\end{aligned}$$

or

$$\dot{\boldsymbol{P}} = \frac{d^2}{dt^2}\sum_i m_i \boldsymbol{r}_i = \sum_i \boldsymbol{F}_i^{(e)} = \boldsymbol{F}^{(e)},$$

since $\boldsymbol{F}_{ji} = -\boldsymbol{F}_{ij}$, where $\boldsymbol{F}^{(e)}$ is the total external force. The center of mass $\boldsymbol{R}_{CM}$ is the average of the radii vectors, weighed in proportion to their mass,

$$\sum_i m_i \boldsymbol{r}_i = M\boldsymbol{R}_{CM}$$

where M is the total mass defined by

$$M = \sum_i m_i \ .$$

Then we have

$$\dot{\boldsymbol{P}} = \frac{d^2}{dt^2}(M\boldsymbol{R}_{CM}) = \boldsymbol{F}^{(e)} \ ,$$

or

$$\dot{\boldsymbol{P}} = M\frac{d^2\boldsymbol{R}_{CM}}{dt^2} = \boldsymbol{F}^{(e)} \ .$$

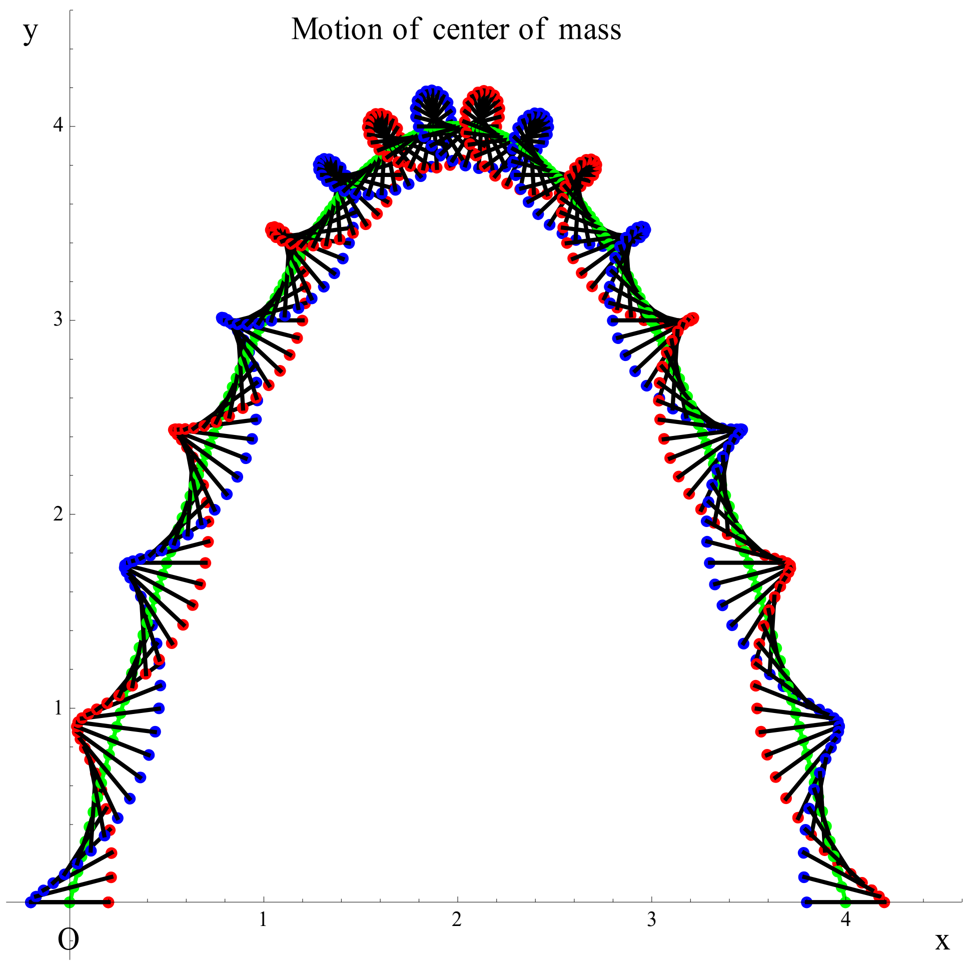


Fig. Typical motion of two bodies (with the same mass, red and blue circles) connecting by a uniform rod. The center of the mass is the midpoint of the two bodies. It shows a parabola-motion in the *x*-*y* plane.

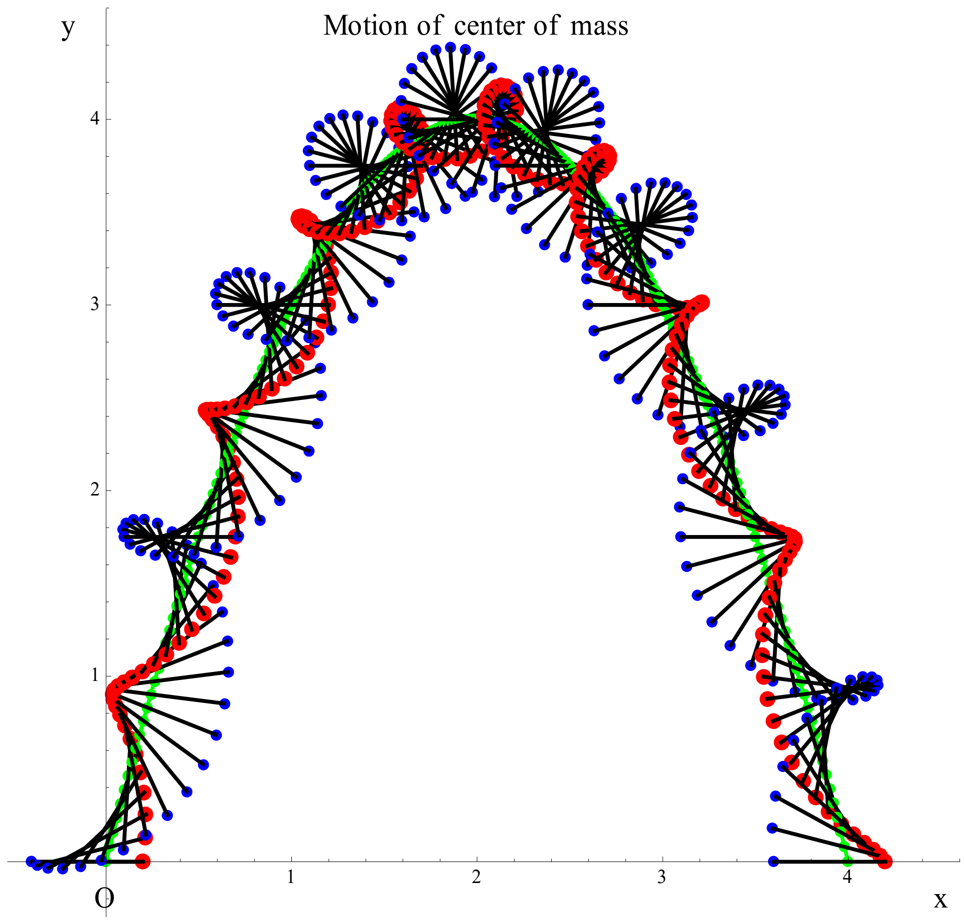


Fig. Typical motion of two bodies (mass 2*m* for the red circles and mass *m* for blue circles) connecting by a uniform rod. The center of the mass is deviated from the midpoint of the two bodies. It shows a parabola-motion in the *x*-*y* plane.

4. Conservation law of linear momentum

If $\boldsymbol{F}^{(e)} = 0$, we have

$$\dot{\boldsymbol{P}} = F^{(e)} = 0$$

leading to

$$\boldsymbol{P} = M\frac{d\boldsymbol{R}_{CM}}{dt} = \text{conserved}$$

This means that

$$\Delta \boldsymbol{P} = \boldsymbol{P}_f - \boldsymbol{P}_i = 0 \text{ (Momentum conservation law)}$$

where $\boldsymbol{P}_i$ and $\boldsymbol{P}_f$ are the linear momentum in the initial state and the final state.

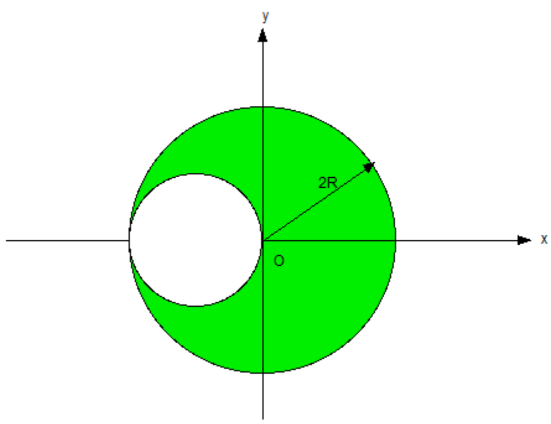
((Note))

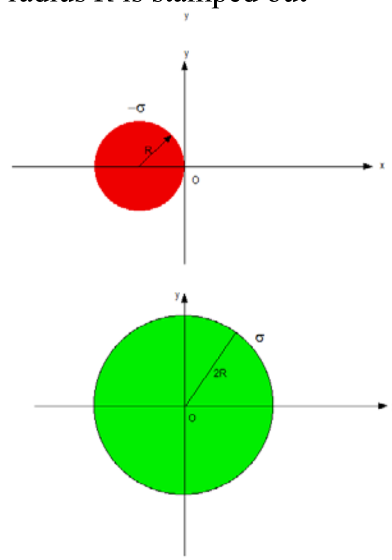
When $\dfrac{d\mathbf{R}_{CM}}{dt} = 0$ at $t = 0$, R_{CM} is independent of t.

5. Sample problems

5.1 Sample problem 9-2

A uniform metal plate (radius 2R) from which a disk of radius R is stamped out





We consider two circles (linear combination)
Circle (radius 2R and the area density σ) centered at the origin (0, 0)
Circle (radius R and the area density -σ) centered at (-R,0).

$$x_{cm} = \frac{\sigma[\pi(2R)^2]0 + [-\sigma(\pi R^2)](-R)}{\sigma[\pi(2R)^2] + [-\sigma(\pi R^2)]} = \frac{\sigma(\pi R^2)R}{3\sigma(\pi R^2)} = \frac{R}{3}$$

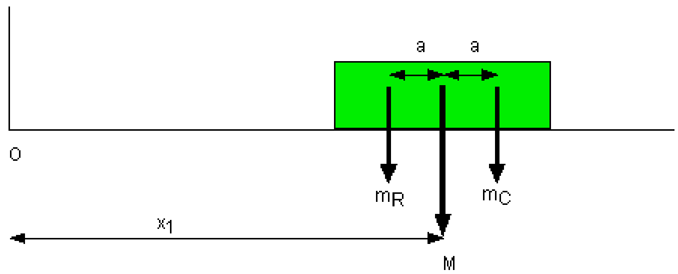
$$y_{cm} = 0$$

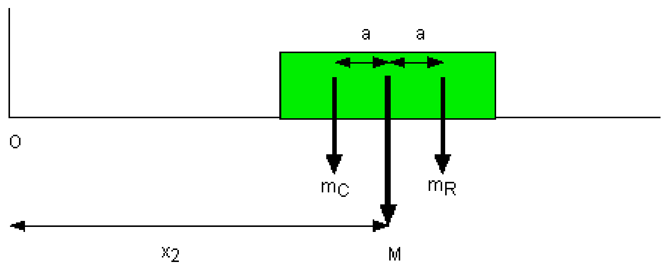
5.2 Typical problem

Problem 9-16* (SP-09) (10-th edition)**

Ricardo, of mass 80 kg, and Carmelita, who is lighter, are enjoying Lake Merced at dusk in a 30 kg canoe. When the canoe is at rest in the placid water, they exchange seats, which are 3.0 m apart and symmetrically located with respect to the canoe's center. If the canoe moves 40 cm horizontally relative to a pier post, what is Carmelita's mass.

((**Solution**))





$m_R = 80$ kg
$l = 2a = 3.0$ m.
$M = 30$ kg

The position of the center of mass x_{cm} does not change since there is no external force.

$$x_{cm} = \frac{m_R(x_1 - a) + Mx_1 + m_C(x_1 + a)}{m_R + M + m_C} = \frac{m_C(x_2 - a) + Mx_2 + m_R(x_2 + a)}{m_R + M + m_C}$$

or

$$m_R(x_1 - a) + Mx_1 + m_C(x_1 + a) = m_C(x_2 - a) + Mx_2 + m_R(x_2 + a)$$

We put

$$x_1 - x_2 = y$$

Then we have

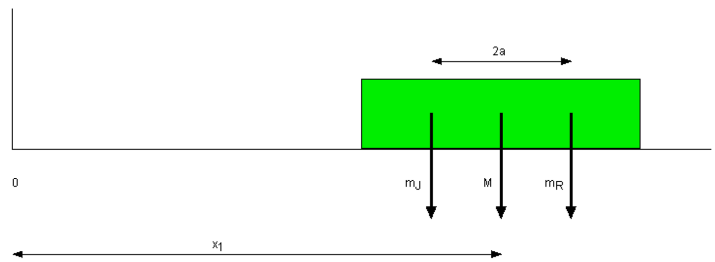
$$y = \frac{2a(m_R - m_C)}{m_R + M + m_C}$$

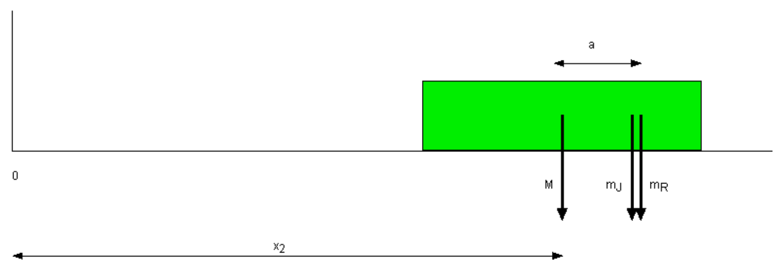
When $y = 0.4$m, we have $m_C = 57.6$ kg.

5.3 Problem from Serway 8-39

Romeo (77.0 kg) entertains Juliet (55.0 kg) by playing his guitar from the rear of their boat at rest in still water, 2.70 m away from Juliet who is in the front of the boat. After the serenade, Juliet carefully moves to the rear of the boat (away from shore) to plant a kiss on Romeo's cheek. How far does the 80.0-kg boat move toward the shore it is facing?

((**Solution**))





m_R = 77.0 kg: weight of Romeo
m_J = 55.0 kg: weight of Juliet
M = 80.0 kg: mass of the canoe
$l = 2a$ = 2.70 m

The position of the center of mass x_{cm} does not change since there is no external force.

$$x_{cm} = \frac{m_J(x_1 - a) + Mx_1 + m_R(x_1 + a)}{m_R + M + m_J} = \frac{Mx_2 + (m_R + m_J)(x_2 + a)}{m_R + M + m_J}$$

or

$$m_J(x_1 - a) + Mx_1 + m_R(x_1 + a) = Mx_2 + (m_R + m_J)(x_2 + a)$$

We put

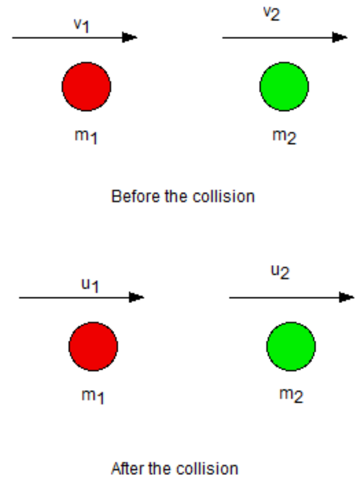
$$x_1 - x_2 = y$$

Then we have

$$y = \frac{2am_J}{m_R + M + m_C} = \frac{2.7 \times 55.0}{212} = 0.70m$$

6 Elastic collision (head-on collision): one-dimensional

In the absence of the external force, the total linear momentum is conserved before and after the collision of two particles with mass m_1 and m_2. We consider the elastic collision where the total energy is conserved before and after the collision.



<u>Momentum conservation law</u>

$$m_1 v_1 + m_2 v_2 = m_1 u_1 + m_2 u_2$$

<u>Energy conservation law (elastic collision)</u>

$$\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 = \frac{1}{2} m_1 u_1^2 + \frac{1}{2} m_2 u_2^2$$

or

$$u_1 = \frac{(m_1 - m_2) v_1 + 2 m_2 v_2}{m_1 + m_2}$$
$$u_2 = \frac{2 m_1 v_1 - (m_1 - m_2) v_2}{m_1 + m_2}$$

((**Mathematica**))

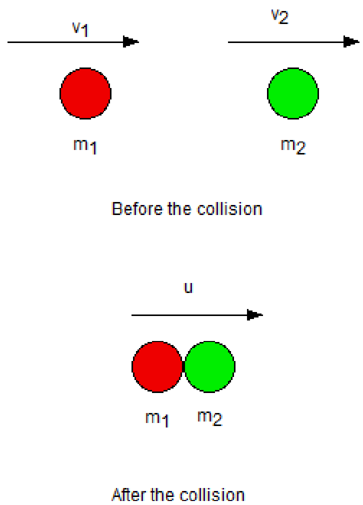
```
Clear["Global`*"];
eq1 = m1 v1 + m2 v2 == m1 u1 + m2 u2;
```

$$eq2 = \frac{1}{2} m1\, v1^2 + \frac{1}{2} m2\, v2^2 == \frac{1}{2} m1\, u1^2 + \frac{1}{2} m2\, u2^2;$$

```
eq3 = Solve[{eq1, eq2}, {u1, u2}] // Simplify
```

$$\left\{\{u1 \to v1,\ u2 \to v2\},\ \left\{u1 \to \frac{m1\, v1 - m2\, v1 + 2\, m2\, v2}{m1 + m2},\ u2 \to \frac{2\, m1\, v1 - m1\, v2 + m2\, v2}{m1 + m2}\right\}\right\}$$

7 Perfectly inelastic collision

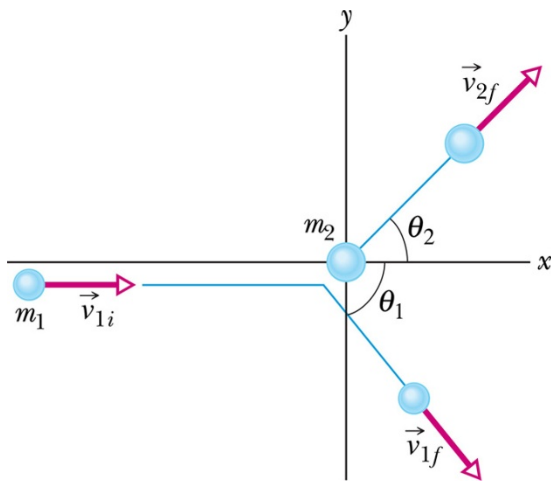


Momentum conservation law;

$$m_1 v_1 + m_2 v_2 = m_1 u + m_2 u$$

The energy is not conserved in this case.

8 2D collisions



$m_1\mathbf{v}_{1i} = m_1\mathbf{v}_{1f} + m_2\mathbf{v}_{2f}$ (Two-dimensional vectors)

From the above figure

$$m_1 v_{1i} = m_1 v_{1f}\cos\theta_1 + m_2 v_{2f}\cos\theta_2$$
$$m_1 v_{1f}\sin\theta_1 = m_2 v_{2f}\sin\theta_2$$

The solution is

$$v_{1f} = v_{1i}\frac{\sin\theta_2}{\cos(\theta_1+\theta_2)}$$
$$v_{2f} = \frac{m_1}{m_2}v_{1i}\frac{\sin\theta_1}{\cos(\theta_1+\theta_2)}$$

Here we assume the case of elastic collision (the energy conservation law)

$$\frac{1}{2}m_1{v_{1i}}^2 = \frac{1}{2}m_1{v_{1f}}^2 + \frac{1}{2}m_2{v_{2f}}^2$$

When $m_1 = m_2$, this condition is rewritten as

$$\theta_1 + \theta_2 = \frac{\pi}{2}$$

((Note))
In the general case (including the case of inelastic collision), the energy is not conserved.

$$\Delta E = E_f - E_i = K_f - K_i = \frac{1}{2} m_1 v_{1f}{}^2 + \frac{1}{2} m_2 v_{2f}{}^2 - \frac{1}{2} m_1 v_{1i}{}^2$$

$$= \frac{m_1 v_{1i}{}^2}{2m_2} \csc^2(\theta_1 + \theta_2) \sin\theta_1 [m_1 \sin\theta_1 - m_2 \sin(\theta_1 + 2\theta_2)]$$

When $m_1 = m_2$,

$$\frac{\Delta E}{\frac{1}{2} m_1 v_{1i}{}^2} = -2\cot(\theta_1 + \theta_2)\csc(\theta_1 + \theta_2)\sin\theta_1 \sin\theta_2$$

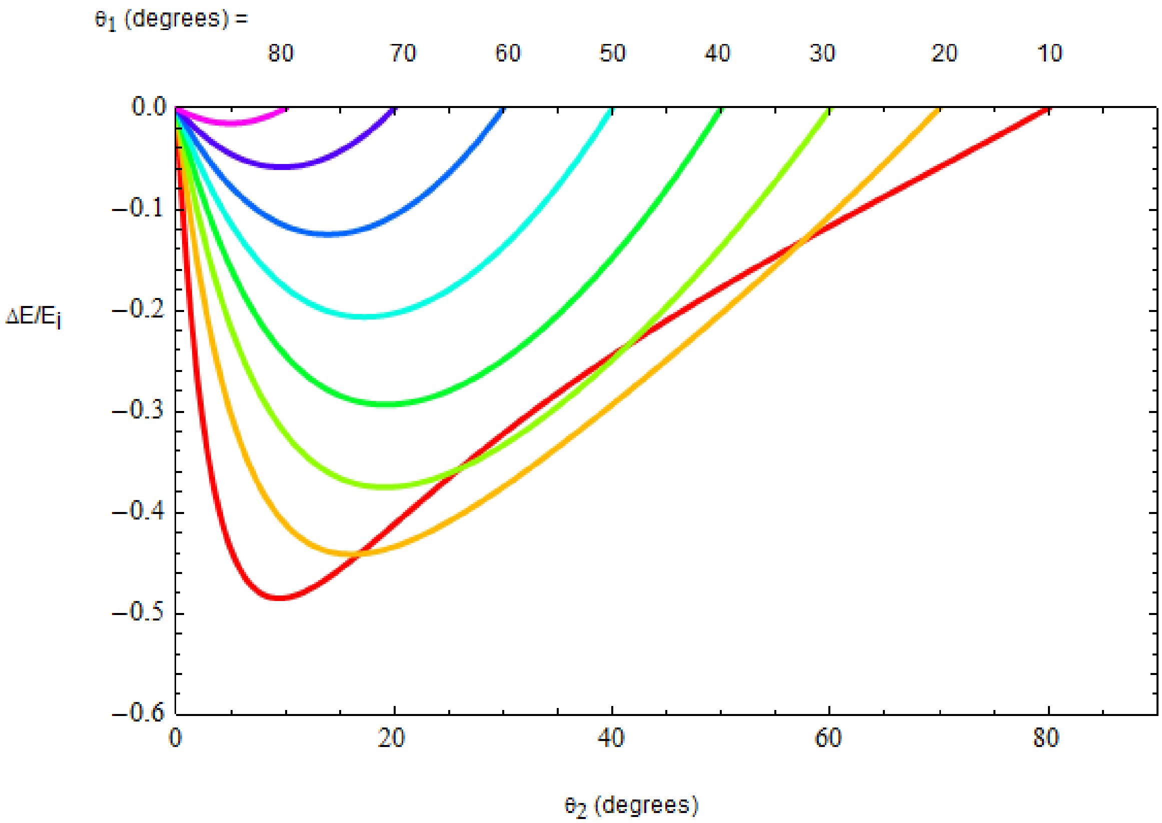


Figure Plot of $\Delta E/E_i$ as a function of θ_2, where $m_1 = m_2$. θ_1 is changed as a parameter: θ_1 = 10°, 20°, 30°, 40°, 50°, 60°, 70°, and 80°. $\Delta E/E_i$ becomes zero for $\theta_1 + \theta_2 = \pi/2$ (elastic scattering).

((**Mathematica**))

```
eq1 = m1 v1i == m1 v1f Cos[θ1] + m2 v2f Cos[θ2]
```

$$m1\ v1i == m1\ v1f\ \mathrm{Cos}[\theta 1] + m2\ v2f\ \mathrm{Cos}[\theta 2]$$

```
eq2 = m1 v1f Sin[θ1] == m2 v2f Sin[θ2]
```

$$m1\ v1f\ \mathrm{Sin}[\theta 1] == m2\ v2f\ \mathrm{Sin}[\theta 2]$$

```
eq3 = Solve[{eq1, eq2}, {v1f, v2f}] // Simplify
```

$$\left\{\left\{v1f \to v1i\ \mathrm{Csc}[\theta 1 + \theta 2]\ \mathrm{Sin}[\theta 2],\ v2f \to \frac{m1\ v1i\ \mathrm{Csc}[\theta 1 + \theta 2]\ \mathrm{Sin}[\theta 1]}{m2}\right\}\right\}$$

$$eq4 = \frac{1}{2}\ m1\ v1f^2 + \frac{1}{2}\ m2\ v2f^2 - \frac{1}{2}\ m1\ v1i^2$$

$$\frac{m1\ v1f^2}{2} - \frac{m1\ v1i^2}{2} + \frac{m2\ v2f^2}{2}$$

```
eq41 = eq4 /. eq3[[1]] // FullSimplify
```

$$\frac{m1\ v1i^2\ \mathrm{Csc}[\theta 1 + \theta 2]^2\ \mathrm{Sin}[\theta 1]\ (m1\ \mathrm{Sin}[\theta 1] - m2\ \mathrm{Sin}[\theta 1 + 2\ \theta 2])}{2\ m2}$$

When m1 = m2

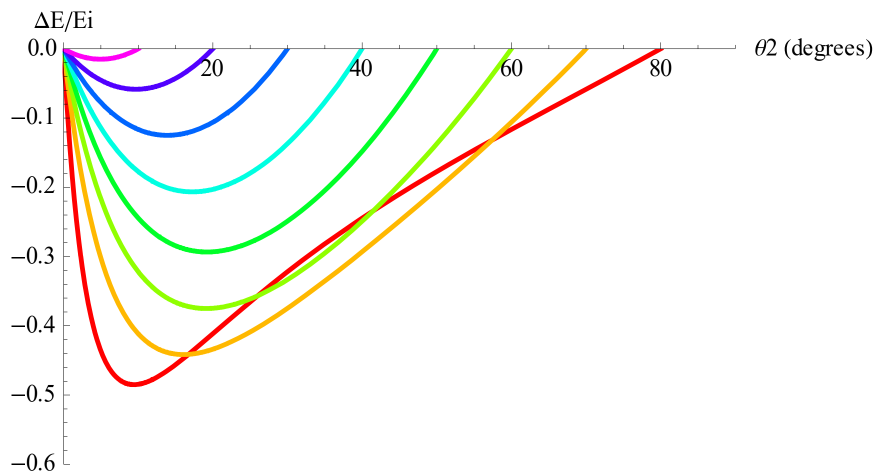
```
eq42 = eq41 /. {m2 → m1} // FullSimplify
```

$$-m1\ v1i^2\ \mathrm{Cot}[\theta 1 + \theta 2]\ \mathrm{Csc}[\theta 1 + \theta 2]\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 2]$$

$$eq5 = \frac{eq42}{\frac{1}{2}\ m1\ v1i^2}$$

$$-2\ \mathrm{Cot}[\theta 1 + \theta 2]\ \mathrm{Csc}[\theta 1 + \theta 2]\ \mathrm{Sin}[\theta 1]\ \mathrm{Sin}[\theta 2]$$

```
f = eq5 /. {θ1 → π/180 x1, θ2 → π/180 x2};
Plot[Evaluate[Table[f, {x1, 10, 80, 10}]], {x2, 0, 90},
 PlotStyle → Table[{Hue[0.12 i], Thick}, {i, 0, 7}],
 AxesLabel → {"θ2 (degrees)", "ΔE/Ei"},
 PlotRange → {{0, 90}, {-0.6, 0}}]
```



9 Elastic collision of particles with the same mass

Problem 9-75** (SP-09) (10-th edition)

A projectile proton with a speed of 500 m/s collides elastically with a target proton initially at rest. The two protons then move along perpendicular paths, with the projectile path at 60° from the original direction. After the collisions, what are the speeds of (a) the target proton and (b) the projectile proton?

((**Solution**))

$$\boldsymbol{v}_{1i} = \boldsymbol{v}_{1f} + \boldsymbol{v}_{2f}$$

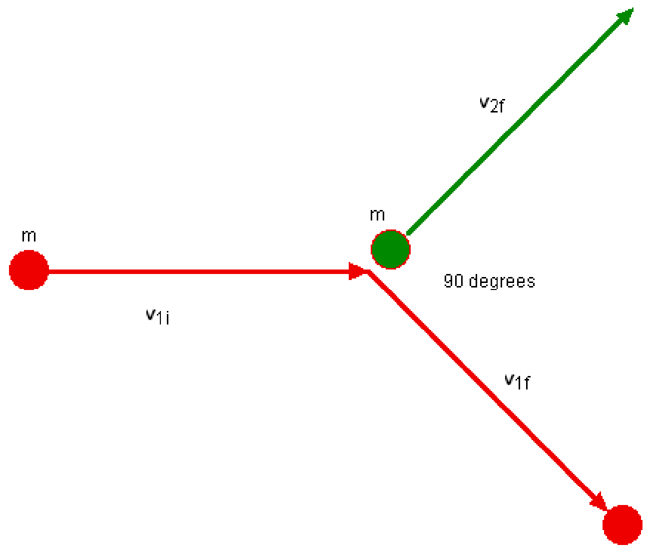
$$\boldsymbol{v}_{1i}^{2} = \boldsymbol{v}_{1f}^{2} + \boldsymbol{v}_{2f}^{2}$$

$$\boldsymbol{v}_{1i}^{2} = \boldsymbol{v}_{1f}^{2} + \boldsymbol{v}_{2f}^{2} = (\boldsymbol{v}_{1f} + \boldsymbol{v}_{2f})^{2} = \boldsymbol{v}_{1f}^{2} + \boldsymbol{v}_{2f}^{2} + 2\boldsymbol{v}_{1f}\cdot\boldsymbol{v}_{2f}$$

or $\boldsymbol{v}_{1f}\cdot\boldsymbol{v}_{2f} = 0$

In other words,

$\boldsymbol{v}_{1f}$ and $\boldsymbol{v}_{2f}$ are perpendicular to each other.

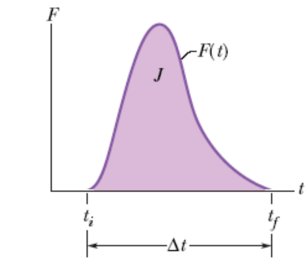


10 Impulse-linear momentum theorem

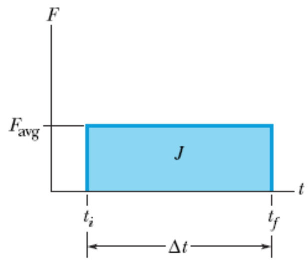
10.1 Theorem

$$\boldsymbol{F} = \frac{d\boldsymbol{p}}{dt}$$

$$\Delta\boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i = \int_{t_i}^{t_f} \boldsymbol{F}dt$$



(a)



(b)

Definition of impulse $\boldsymbol{J}$

$$\boldsymbol{J} = \Delta\boldsymbol{p} = \boldsymbol{p}_f - \boldsymbol{p}_i$$

$$\boldsymbol{J} = \int_{t_i}^{t_f} \boldsymbol{F}dt = \boldsymbol{F}_{av}(t_f - t_i) = \boldsymbol{F}_{av}\Delta t$$

or

$$\frac{\boldsymbol{J}}{\Delta t} = \frac{\Delta\boldsymbol{p}}{\Delta t} = \boldsymbol{F}_{av},$$

with

$$\boldsymbol{F}_{av} = \frac{1}{t_f - t_i}\int_{t_i}^{t_f} \boldsymbol{F}dt$$

The average force $\boldsymbol{F}_{av}$ can be evaluated from the values of $\boldsymbol{J} = \Delta\boldsymbol{p}$ and the time interval

((**Impulse-momentum theorem**))

The change in the momentum of a particle is equal to the impulse of the net force acting on the particle.

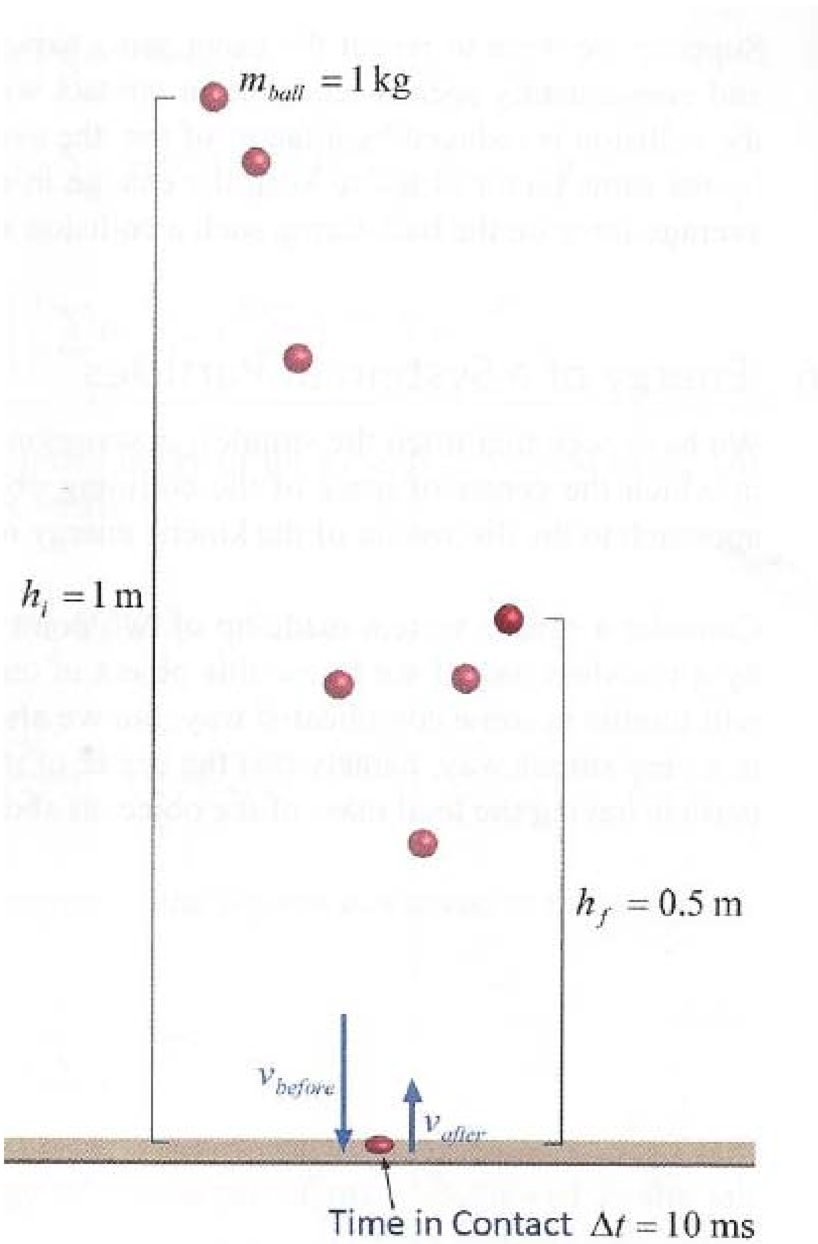
((**Note**))

A moving particle has momentum and kinetic energy, but it does not carry with it a force. The force required to cause a particle to stop depends on how big the momentum change is and on how quickly the momentum change occurs.

((**Example**))
Impulse example

The change in momentum of an object during a collision is equal to the product of the average force acting on an object and the time over which it acts.

((**Example**)) **SmartPhysics p.144-146**
A ball with a mass of m = 1 kg is released from rest from an initial height of (h_i = 1 m) above the floor. It bounces back to half its original height (h_f = 0.5 m). If we assume the ball is in contact with the floor for a time of Δt = 10 ms, what is the average force on the ball during the collision?



((Solution))

$h_i = 1$ m. $h_f = 0.5$ m. $m = 1$ kg. $\Delta t = 10$ ms.

(i) v_1 is the velocity of the ball which falls from the height h_i and touchs on the floor

$$0^2 - v_1^2 = (-2gh_i), \quad \text{or} \quad v_1 = \sqrt{2gh_i} = 4.43 \text{ m/s}$$

(ii) v_2 is the velocity of the ball which bounces back from the floor

$$0^2 - v_2^{\ 2} = (-2gh_f)\,, \quad \text{or} \quad v_2 = \sqrt{2gh_f} = 3.13 \text{ m/s}$$

Impulse

$$J = \Delta p = F_{av}\Delta t = mv_2 - (-mv_1) = m(v_1 + v_2) = 7.56 \text{ N s}$$

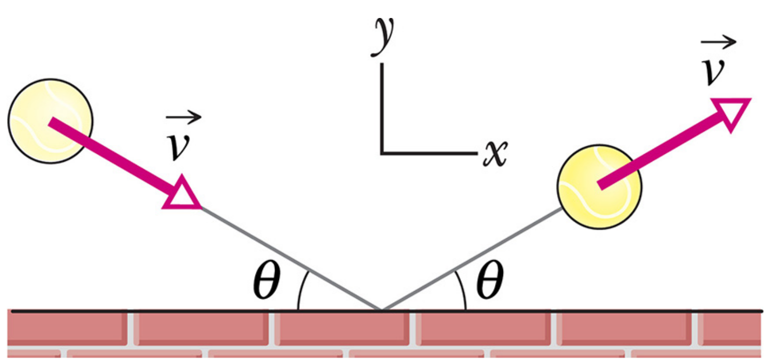
The average force F_{av} is obtained as

$$F_{av} = \frac{m(v_1 + v_2)}{\Delta t} = 756 \text{ N.}$$

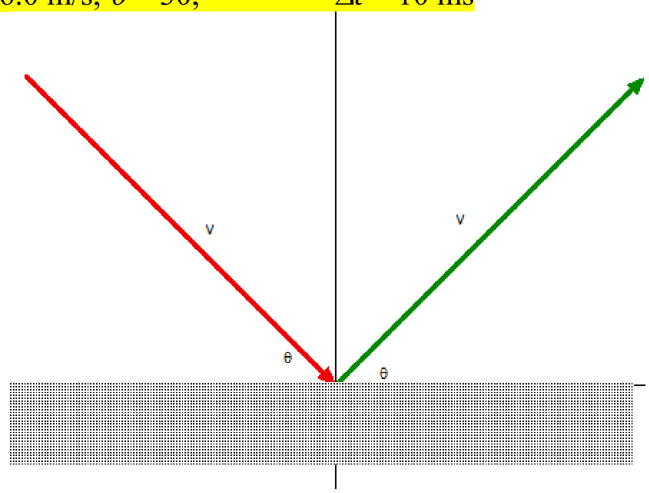
10.2 Example

Problem 9-38 (10-th edition)**

In the overhead view of Fig., a 300 g ball with a speed v of 6.0 m/s strikes a wall at an angle θ of 30° and then rebounds with the same speed and angle. It is in contact with the wall for 10 ms. In unit-vector notation, what are (a) the impulse on the ball from the wall and (b) the average force on the wall from the ball?



m = 0.3 kg, v = 6.0 m/s, θ = 30, Δt = 10 ms



$$J = \Delta p = F_{av}\Delta t$$

where

(a)

$$J = \Delta p = 2mv\sin\theta = 1.8Ns$$

(b)

$$F_{av} = \frac{J}{\Delta t} = 180N$$

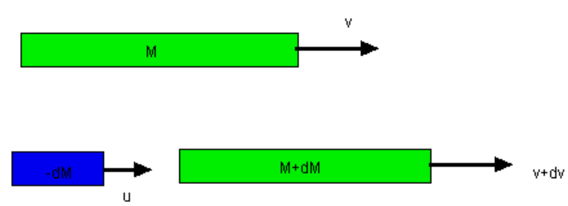
11 Systems with varying mass: A rocket

11.1 Formulation

We assume that we are at rest relative to an inertial reference frame (the Earth), watching a rocket accelerate through deep space with no gravitational or atmospheric drag force acting on it.

We consider a rocket plus fuel that is initially at rest. When fuel is ejected out the back of the rocket, it acquires momentum, and so the rocket must move forward to acquire opposite momentum to cancel the fuel's momentum, since the total momentum of the system remains constant.

Suppose that a rocket of mass M is moving at the velocity v with respect to the Earth. Now a mass of fuel (- dM) is ejected with the velocity u with respect to the Earth. The rocket now moves forward with mass (M+dM) and velocity (v+dv)



Momentum conservation law:

$$Mv = (-dM)u + (M + dM)(v + dv) . \qquad (1)$$

Note that

$$V_{\text{rocket-Earth}} = V_{\text{rocket-fuel}} + V_{\text{fuel-Earth}}$$

where

$V_{\text{rocket-Earth}}$ (= v + d v) is the velocity of rocket relative to the Earth.
$V_{\text{rocket-fuel}}$ (= v_{rel}) is the velocity of rocket relative to the exhaust fuel.
$V_{\text{fuel-Earth}}$ (= u) is the velocity of the exhaust fuel relative to the Earth.

We get the relation

$$u = v + dv - v_{rel}. \qquad (2)$$

Substituting Eq.(2) for u into Eq.(1), we get

$$Mv = -dM(v + dv - v_{rel}) + (M + dM)(v + dv)$$

or

$$Mv = -vdM - dMdv + v_{rel}dM + Mv + Mdv + vdM + dMdv.$$

or

$$0 = v_{rel}dM + Mdv,$$

or

$$0 = v_{rel}\frac{dM}{dt} + M\frac{dv}{dt}$$

Then we have

$$\boxed{\begin{aligned} M\frac{dv}{dt} &= -v_{rel}\frac{dM}{dt} \\ &= v_{rel}\left(-\frac{dM}{dt}\right) \\ &= Rv_{rel} \end{aligned}} \qquad \text{(first rocket equation).}$$

where

v_{rel}:	the **fuel's exhaust velocity relative to the rocket**.
$T = Rv_{rel}$	the **thrust of the rocket engine** (N)
R (=-dM/dt)	the **rate of fuel consumption**

Here we assume that *v_{rel} is constant*. Then we have

$$Mdv = -v_{rel}dM$$

$$dv = -v_{rel}\frac{dM}{M}$$

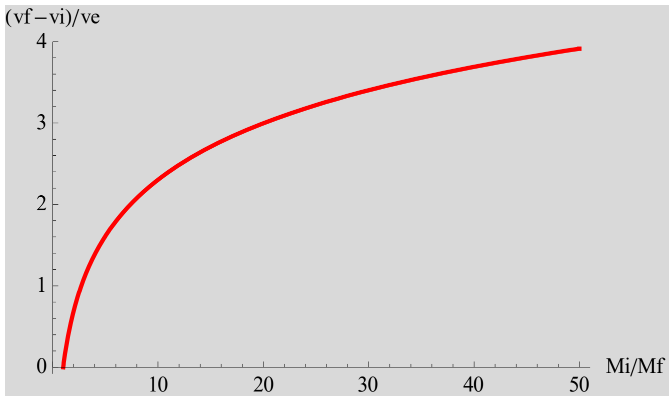
$$\boxed{v_f - v_i = \int dv = -v_{rel}\int_{M_i}^{M_f}\frac{dM}{M} = -v_{rel}\ln\left(\frac{M_f}{M_i}\right)},$$

or

$$v_f - v_i = v_{rel} \ln(\frac{M_i}{M_f})$$

(second rocket equation)

We make a plot of $\frac{v_f - v_i}{v_{rel}}$ as a function of $\frac{M_i}{M_f}$



11.2 The case for constant *R* (=-d*M*/d*t*) and v_{rel}.

Here we assume that R (= -dM/dt) and v_{rel} are independent of t. Then we have

$$M\frac{dv}{dt} = Rv_{rel}$$

$$R = -\frac{dM}{dt}$$

From the second equation, M is obtained as

$$M = M_0 - Rt$$

or

$$\frac{M}{M_0} = 1 - \frac{Rt}{M_0}$$

where M_0 is the initial mass at $t = 0$ and $t < M_0/R$. Then we have

$$\frac{dv}{dt} = \frac{Rv_{rel}}{M} = \frac{Rv_{rel}}{M_0 - Rt} = \frac{Rv_{rel}/M_0}{1 - Rt/M_0}$$

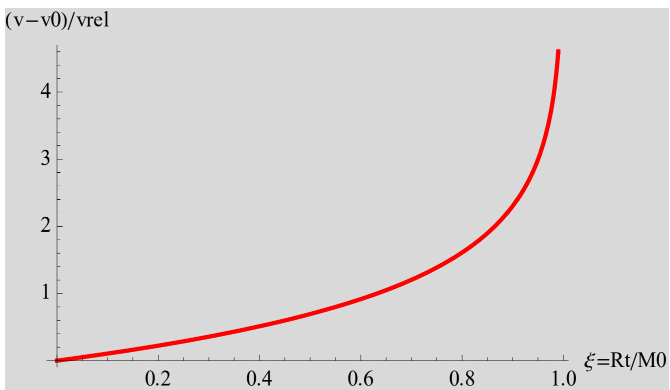
or

$$v - v_0 = \int dv = \int_0^t \frac{Rv_{rel}/M_0}{1 - Rt/M_0} dt = v_{rel} \ln\left(\frac{1}{1 - \frac{R}{M_o} t} \right)$$

where $0 < \frac{Rt}{M_0} < 1$.

(a) Plot of $(v\text{-}v_0)/\mathrm{v}_{\mathrm{rel}}$ vs $\xi = Rt/M_0$.

$$\frac{v - v_0}{v_{rel}} = \ln\left(\frac{1}{1-\xi}\right)$$

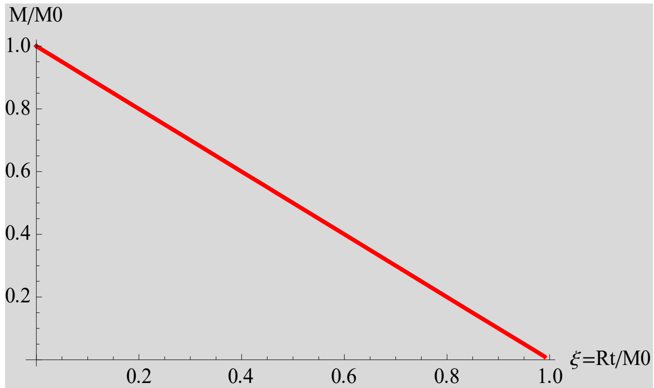


Note that in the limit of $\xi \to 0$ (or $t \to 0$)

$$\frac{v - v_0}{v_{rel}} = \ln\left(\frac{1}{1-\xi}\right) = \xi + \frac{\xi^2}{2} + \frac{\xi^3}{3}$$

(b) Plot of M/M_0 vs $\xi = Rt/M_0$.

$$\frac{M}{M_0} = 1 - \xi$$



In conclusion:

The velocity of the rocket is proportional to the time in early stage. As the mass of rocket decreases in the late stage, the velocity of the rocket rapidly increases.

((**Note**)) The velocity of rocket

10^2 miles/h = 47.7 m/s
10^3 miles/h=0.447 km/s = 447 m/s
10^4 miles/h = 4.4704 km/s

The velocity for the orbit on the earth

$$v_{orbit} = \sqrt{\frac{M_E G}{R_E}} = 7.91 \text{ km/s} = 1.77 \text{ x } 10^4 \text{ miles/h}$$

The escape velocity form the earth:

$$v_{escape} = \sqrt{\frac{2M_E G}{R_E}} = 11.19 \text{ km/s} = 2.50 \text{ x } 10^4 \text{ miles/h.}$$

11.3

Problem 9-78 (10-th edition)

Consider a rocket that is in deep space and at rest relative to an inertial reference frame. The rocket's engine is to be fired for a certain interval. What must be the rocket's *mass ratio* (ratio of initial to final mass) over that interval if the rocket's original speed relative to the inertial frame is to be equal to (a) the exhaust speed (speed of the exhaust products relative to the rocket) and (b) 2.0 times the exhaust speed?

((**Solution**))
Second rocket equation with $v_i = 0$ and $v_f = v$;

$$v_f - v_i = v - 0 = v_{rel}\ln(\frac{M_i}{M_f})$$

$$\frac{M_i}{M_f} = \exp(\frac{v}{v_{rel}})$$

(a) If $v = v_{rel}$, we obtain $\frac{M_i}{M_f} = e^1 = 2.72$

(b) If $v = 2\,v_{rel}$,, we obtain $\frac{M_i}{M_f} = e^2 = 7.39$

11.4

Problem 9-79 (10-th edition)

A rocket that is in deep space and initially at rest relative to an inertial reference frame has a mass of 2.55 x 10^5 kg, of which 1.81 x 10^5 kg is fuel. The rocket engine is then fired for 250 s while fuel is consumed at the rate of 480 kg/s. The speed of the exhaust products relative to the rocket is 3.27 km/s.
(a) What is the rocket's thrust? After the 250 s firing, what are (b) the mass and (c) the speed of the rocket?

((**Solution**))

(a) The thrust of the rocket is given by $T = Rv_{\rm rel}$ where R (=- dM/dt) is the rate of fuel consumption and $v_{\rm rel}$ is the speed of the exhaust gas relative to the rocket. For this problem R = 480 kg/s and $v_{\rm rel} = 3.27 \times 10^3$ m/s, so

$$T = R\,v_{\rm rel} = 1.57 \text{ x } 10^6 \text{ N}$$

(b)
The mass of fuel ejected is given by $M_{\rm fuel} = R\Delta t$, where Δt is the time interval of the burn. Thus, $M_{\rm fuel}$ = (480 kg/s)(250 s) = 1.20×10^5 kg. The mass of the rocket after the burn is

$$M_{\rm f} = M_{\rm i} - M_{\rm fuel} = (2.55 - 1.20) \text{ x } 10^5 \text{ kg} = 1.35 \text{ x } 10^5 \text{ kg.}$$

(c)
Since the initial speed is zero, the final speed is given by the second rocket equation

$$v_f = v_{rel}\ln(\frac{M_i}{M_f}) = 2.08 \text{ x } 10^3 \text{ m/s}$$

11.5

Problem 9-76 (10-th edition)

A 6090 kg space probe moving nose-first toward Jupiter at 105 m/s relative to the Sun fires its rocket engine, ejecting 80.0 kg of exhaust at a speed of 253 m/s relative to the space probe. What is the final velocity of the probe?

((**Solution**))

Second rocket equation;

$$v_f = v_i + v_{rel} \ln(\frac{M_i}{M_f})$$

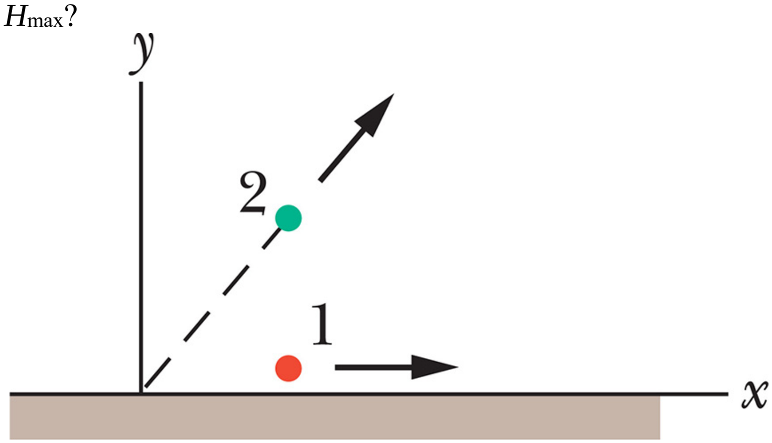
v_f = 105 m/s + (253 m/s) ln(6090 kg/6010 kg) = 108 m/s.

12. Homework and SP-09

12.1

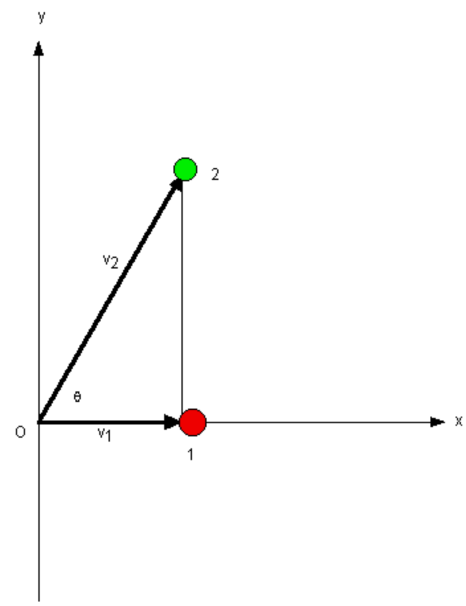
Problem 9-14 (SP-09) (10-th edition)

In Fig., two particles are launched from the origin of the coordinate system at time t = 0. Particle 1 of mass m_1 = 5.00 g is shot directly along the x axis on a frictionless floor, with constant speed 10.0 m/s. Particle of mass m_2 = 3.00 g is shot with a velocity of magnitude 20.0 m/s, at an upward angle such that it always directly above particle 1. (a) What is the maximum height H_{max} reached by the c.o.m. of the two-particle system? In unit-vector notation, what are the (b) velocity and (b) acceleration of the c.o.m. when the c.o.m. reaches H_{max}?



((Solution))

m_1 = 5.00g, m_2 = 3.00 g, v_1 = 10.0 m/s, v_2 = 20 m/s



$$\boldsymbol{v}_1 = (v_1, 0),$$
$$\boldsymbol{v}_2 = (v_2 \cos\theta, v_2 \sin\theta - gt)$$

$\boldsymbol{r}_1 = (x_1, y_1)$ $\boldsymbol{r}_2 = (x_2, y_2)$

$$\boldsymbol{R}_{com} = \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{m_1 + m_2},$$
$$\boldsymbol{V}_{com} = \frac{m_1 \boldsymbol{v}_1 + m_2 \boldsymbol{v}_2}{m_1 + m_2},$$
$$\boldsymbol{A}_{com} = \frac{m_1 \boldsymbol{a}_1 + m_2 \boldsymbol{a}_2}{m_1 + m_2}$$

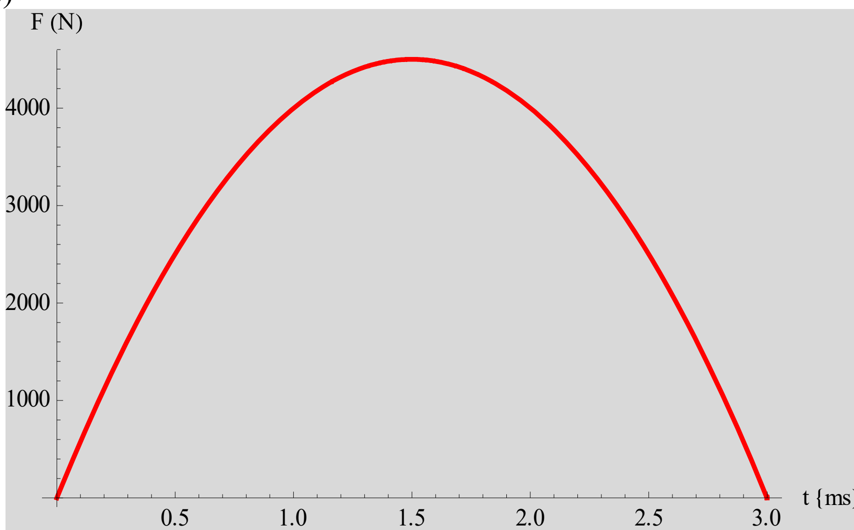
12.2
Problem 9-37 (SP-09) (10-th edition)

A soccer player kicks a soccer ball of mass 0.45 kg that is initially at rest. The player's foot is in contact with the ball for 3.0 x 10^{-3} s, and the force of the kick is given by

$$F(t) = [(6.0 \times 10^6)t - (2.0 \times 10^9)t^2]N$$

for $0 \le t \le 3.0 \times 10^{-3} s$, where t is in seconds. Find the magnitudes of (a) the impulse on the ball due to the kick, (b) the average force on the ball from the player's foot during the period of contact, (c) the maximum force on the ball from the player's foot during the period of contact, and (d) the ball's velocity immediately after it loses contact with player's foot.

((**Solution**))



$a = 6.0 \text{ x } 10^6$, $b = 2.0 \text{ x } 10^9$
$t_0 = 3.0 \text{ x } 10^{-3}$ s,
$m = 0.45$ kg

The force is given by

$$F(t) = at - bt^2$$

The impulse J:

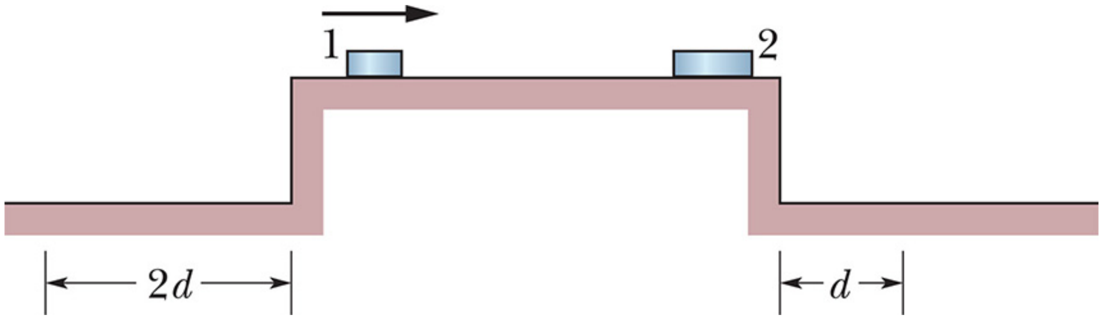
$$J = \int_0^{t_0} F(t)dt$$

The average force F_{av}:

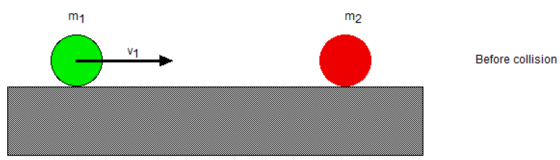
$$F_{av} = \frac{J}{t_0}$$

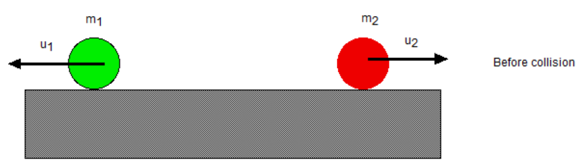
12.3
Problem 9-70* (SP-09) (10-th edition)**

In Fig., puck 1 of mass $m_1 = 0.20$ kg is sent sliding across a frictionless lab bench, to undergo a one-dimensional elastic collision with stationary puck 2. Puck 2 then slides off the bench and lands a distance d from the base of the bench. Puck 1 rebounds from the collision and slides off the opposite edges of the bench, landing a distance $2d$ from the base of the bench. What is the mass of puck2? (Hint: Be careful with signs).

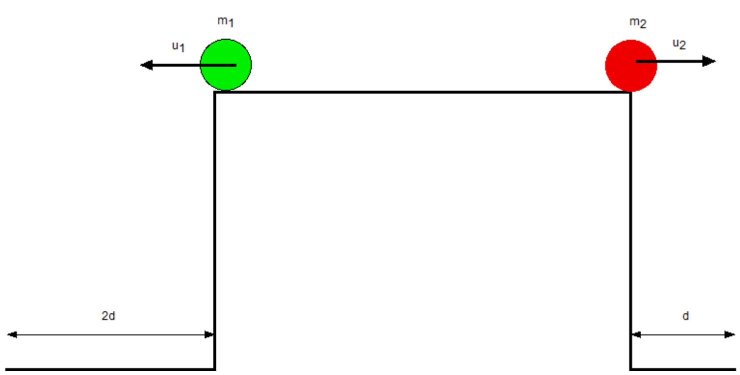


((**Solution**))
$m_1 = 0.2$ kg





Elastic collision



The momentum conservation law:

$$m_1 v_1 + m_2 \cdot 0 = -m_1 u_1 + m_2 u_2 \tag{1}$$

For the mass m_1,

$$x = u_1 t$$

$$y = h - \frac{1}{2} g t^2$$

or

$$y = 0 = h - \frac{1}{2} g t^2 = 0$$

$$t = \sqrt{\frac{2h}{g}}$$

$$x = 2d = u_1 \sqrt{\frac{2h}{g}}$$

$$u_1 = 2d \sqrt{\frac{g}{2h}}$$

For the mass m_2,

$$y = 0 = h - \frac{1}{2} g t^2 = 0$$

$$t = \sqrt{\frac{2h}{g}}$$

$$x = d = u_2 \sqrt{\frac{2h}{g}}$$

$$u_2 = d \sqrt{\frac{g}{2h}}$$

Then we have

$$\frac{u_2}{u_1} = \frac{d\sqrt{\frac{g}{2h}}}{2d\sqrt{\frac{g}{2h}}} = \frac{1}{2} \tag{2}$$

From Eqs.(1) and (2), we have

$$u_1 = \frac{2 m_1 v_1}{m_2 - 2 m_1}$$

$$u_2 = \frac{m_1 v_1}{m_2 - 2 m_1} \tag{3}$$

The energy conservation law:

$$\frac{1}{2}m_1 v_1^2 = \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2$$

The substitution of Eq.(3) into the energy conservation law leads to

$$\frac{m_1(5m_1 - m_2)m_1 v_1}{2m_1 - m_2} = 0$$

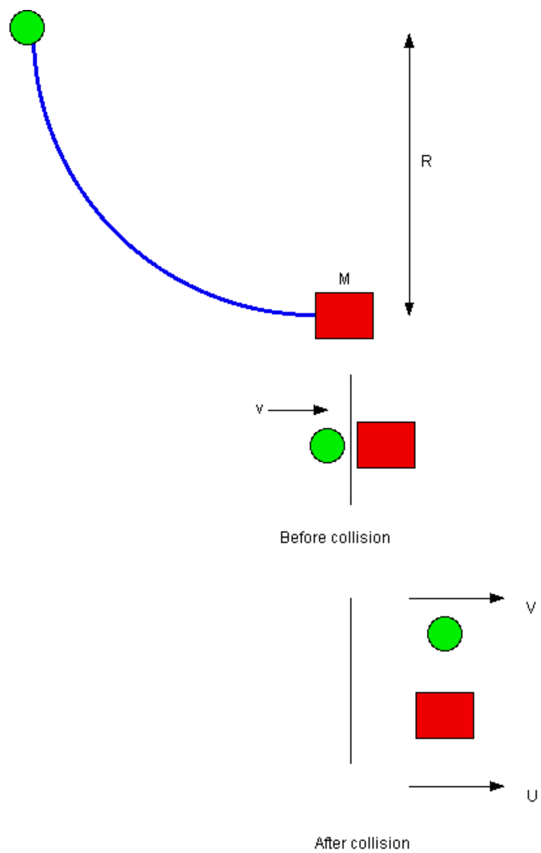
Thus we get

$$m_2 = 5m_1$$

12.4
Problem 9-64 (SP-09) (10-th edition)**

A steel ball of mass 0.500 kg is fastened to a cord that is 70.0 cm long and fixed at the far end. The ball is then released when the cord is horizontal (Fig.) At the bottom of its path, the ball strikes a 2.50 kg steel block initially at rest on a frictionless surface. The collision is elastic, find (a) the speed of the ball and (b) the speed of the block, both just after the collision.

((**Solution**))



$M = 2.50$ kg, $m = 0.5$ kg, $R = 0.7$ m.

Energy conservation law

$$mgR = \frac{1}{2}mv^2 \qquad \text{or} \qquad v = \sqrt{2gR} = 3.70m/s$$

Momentum conservation law

$$mv = MU + mV$$

The collision is elastic.

$$\frac{1}{2}mv^2 = \frac{1}{2}MU^2 + \frac{1}{2}mV^2$$

$$V = \left(\frac{m-M}{m+M}\right)v = -2.47m/s$$

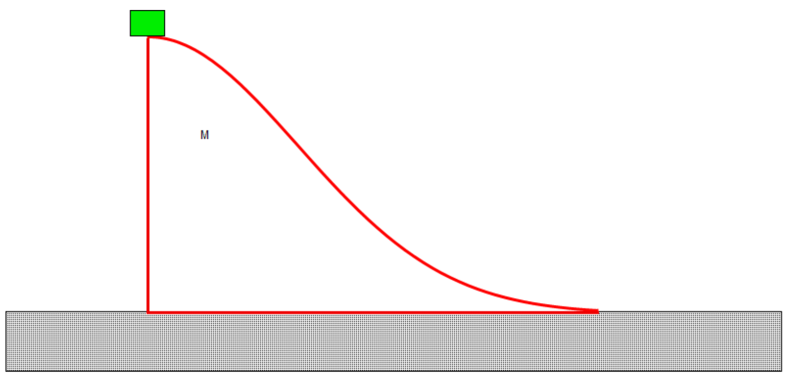
$$U = \frac{2mv}{m+M} = 1.23m/s$$

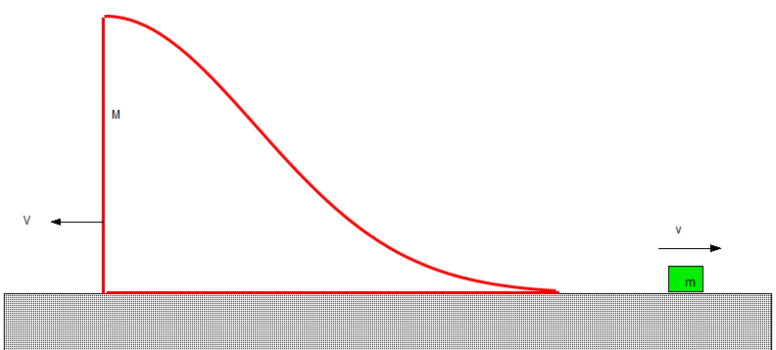
13. Problems from Sewway

13.1 Serway Problem 8-51

A small block of mass m = 0.500 kg is released from rest at the top of a curve-shaped frictionless wedge of mass M = 3.00 kg, which sits on a frictionless, horizontal surface. When the block leaves the wedge, the velocity is measured to be v = 4.00 m/s to the right. (a) What is the velocity of the wedge after the block reaches the horizontal surface? (b) What is the height h of the wedge?

((**Solution**))





((Solution))
M = 3.00 kg
m = 0.500 kg
v = 4.00 m/s

(a)
The initial momentum of the system is zero, which remains constant through the motion (the momentum conservation). When the block of mass m leaves the wedge, we have

$$mv - MV = 0$$

or

$V = 0.667$ m/s

(b) Energy conservation

$E_i = E_f$

where

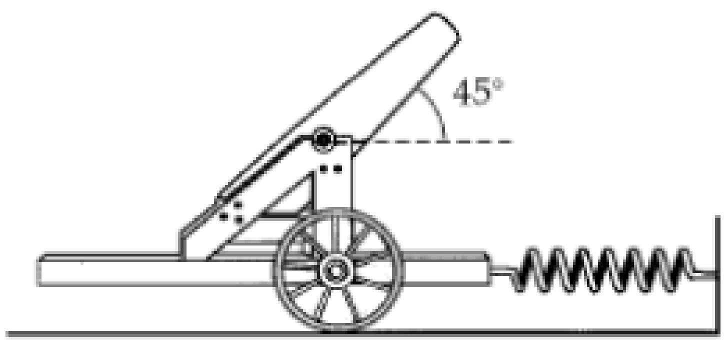
$$E_i = mgh$$
$$E_f = \frac{1}{2}mv^2 + \frac{1}{2}MV^2$$

Then we have $h = 0.952$ m.

13.2 Serway Problem 8-58

A cannon is rigidly attached to a carriage, which can move along horizontal rails but is connected to a post by a large spring, initially stretched and with force constant k = 2.00 x 10^4 N/m. The cannon fires a 200-kg projectile at a velocity of 125 m/s directed 45.0° above the horizontal.

(a) Assuming that the mass of the cannon and its carriage is 5000 kg, find the recoil speed of the cannon.
(b) Determine the maximum extension of the string.
(c) Find the maximum force the spring exerts on the carriage.
(d) Consider the system consisting of the cannon, carriage, and projectile. Is the momentum of this system conserved during the firing? Why or why not?



((**Solution**))
k = 2.00 x 10^4 N/m
m_p = 200 kg; mass for the projectile
v_p = 125 m/s; velocity for the projectile
θ = 45.0°.
M = 5000 kg for the cannon and carriage

(a) Use conservation of the horizontal component of momentum for the system of the shell, the cannon, and the carriage, from just before to just after the cannon firing.

$$m_p v_p \cos 45° + M v_{recoil} = 0$$

or

v_{recoil} = -3.54 m/s

(b) Use conservation of energy for the system of the cannon, the carriage, and the spring from right after the cannon is fired to the instant when the cannon comes to rest.

$E_f = E_i$.

where

$$E_i = \frac{1}{2} m v_{recoil}{}^2$$
$$E_f = \frac{1}{2} k x_{\max}{}^2$$

Then we have

$$x_{\max} = \sqrt{\frac{m}{k}} \left| v_{recoil} \right| = 1.77m$$

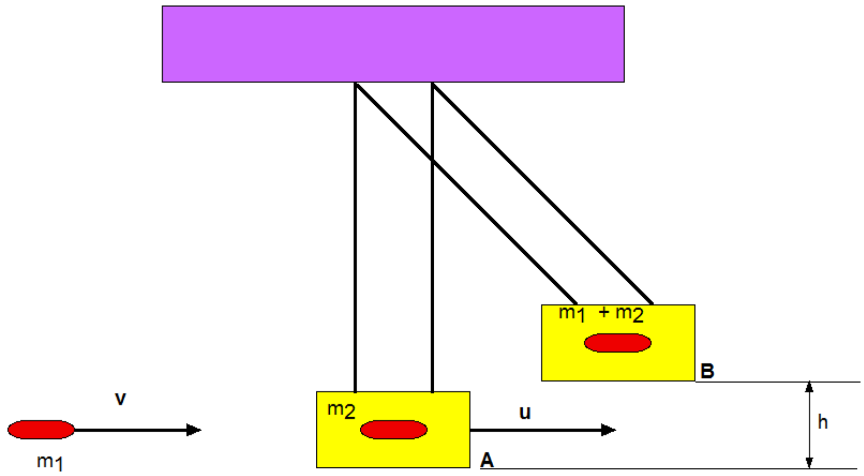
(c) $F_{\max} = k x_{\max} = 3.54 \times 10^4 N$

(d)
No. The rail exerts a vertical external force (the normal force) on the cannon and prevents it from recoiling vertically. Momentum is not conserved in the vertical direction. The spring does not have time to stretch during the cannon firing. Thus, no external horizontal force is exerted on the system (cannon, carriage, and shell) from just before to just after firing. Momentum of this system is conserved in the horizontal direction during this interval.

Appendix

A.1 Ballistic pendulum

The ballistic pendulum is an apparatus used to measure the speed of a fast-moving projectile, such as a bullet. A bullet of mass m_1 is fired into a large block of wood of mass m_2 suspended from some light wires. The bullet embeds in the block, and the entire system swings through a height h. The speed of the bullet can be determined from the measurement of h.



(a) **Momentum conservation law on the collision**

$$m_1 v + m_2 .0 = (m_1 + m_2)u$$

or

$$u = \frac{m_1 v}{m_1 + m_2}, \qquad (1)$$

where v is the velocity of bullet before the collision and u is the velocity of bullet and block.

(b) **Energy conservation law**:

The kinetic energy at the point A is given by

$$K_A = \frac{1}{2}(m_1 + m_2)u^2 .$$

The potential energy at the point C is given by

$$U_B = (m_1 + m_2)gh$$

Then we have

$$\frac{1}{2}(m_1 + m_2)u^2 = (m_1 + m_2)gh\,, \qquad (2)$$

where h is the height between the point A and the point B. From Eqs.(1) and (2), we have

$$v = \left(\frac{m_1 + m_2}{m_1}\right)\sqrt{2gh}\,.$$

A.2 A variable-mass drop

We consider a raindrop falling through a cloud of small water droplets Some of these small droplets adhere to the raindrop, thereby increasing its mass as it falls. The force of the rain drop is

$$F_{ext} = \frac{dp}{dt} = m\frac{dv}{dt} + v\frac{dm}{dt}\,.$$

Suppose the mass of the raindrops depends on the distance x that it has fallen. Then $m = kx$, where k is constant, and dm/dt = kv. This gives

$$mg = m\frac{dv}{dt} + v(kv)$$

or

$$xg = x\frac{dv}{dt} + v^2$$

since $F_{\text{ext}} = mg$.

Reference:

K.S. Krane, Am. J. Phys. **49**, 113 (1981).

We assume that $v = dx/dt = v_0$ and $x = x_0$ at $t = 0$. We solve this problem (nonlinear differential equation) using the Mathematica (NDSolve) numericallty.

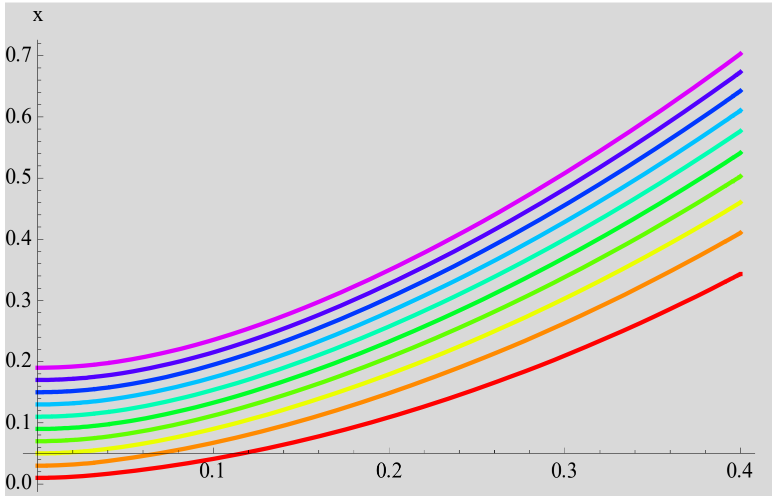


Fig. x vs t, where $v_0 = 0$ and $x_0 = 0.01 - 0.20$ ($\Delta x_0 = 0.02$)

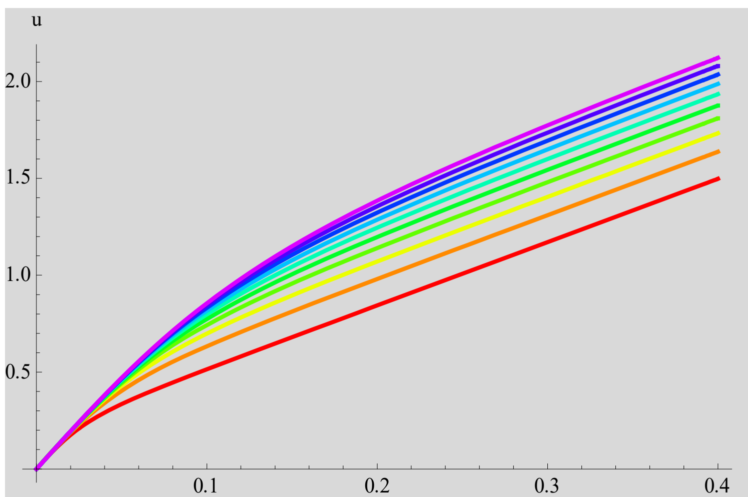


Fig. dx/dt vs t, where $v_0 = 0$ and $x_0 = 0.01 - 0.20$ ($\Delta x_0 = 0.02$).

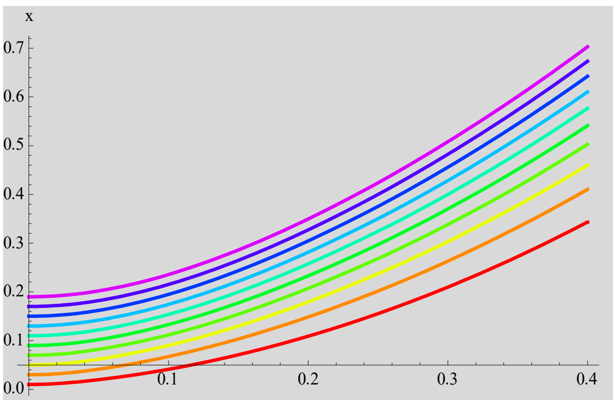
Raindrop

```
Clear["Global`*"]

g = 9.8;

timex[x0_, u0_, tmax_, opts__] :=
 Module[{numso1, numgraph},
  numso1 = NDSolve[{x[t] u'[t] + u[t]^2 == g x[t], x'[t] == u[t], x[0] == x0, u[0] == u0},
    {x[t], u[t]}, {t, 0, tmax}];
  numgraph = Plot[Evaluate[x[t] /. numso1[[1]]], {t, 0, tmax}, opts, DisplayFunction → Identity]]

timelistx =
 timex[#, 0, 0.4,  PlotStyle → {Hue[4.5 (# - 0.01)], Thick}, AxesLabel → {"t", "x"},
    Background → LightGray, PlotRange → All,  DisplayFunction → Identity] & /@
  Range[0.01,  0.2, 0.02]; Show[timelistx, DisplayFunction → $DisplayFunction]
```



B. Center of mass frame and laboratory frame for elastic collision

B.1 Momentum conservation and the energy conservation

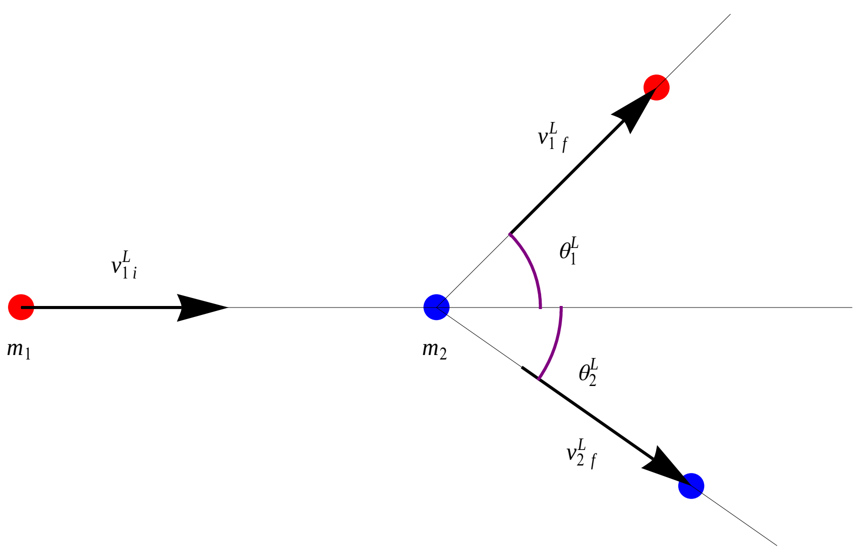


Fig. Laboratory frame. The initial momentum is $m_i \boldsymbol{v}_{1i}^L$. The scattering angle of the particle 1 with mass m_1 is θ_1^L.

In the laboratory frame, the velocities of the particles 1 and 2 before and after collision, are defined by

$$\boldsymbol{v}_{1i}^L, \quad \boldsymbol{v}_{2i}^L, \quad \boldsymbol{v}_{1f}^L, \quad \boldsymbol{v}_{2f}^L,$$

where

$$\boldsymbol{v}_{2i}^L = 0$$

since the particle with mass m_2 is at rest before the collision. The center of mass velocity is given by

$$\boldsymbol{v}_{CM} = \frac{m_1}{m_1 + m_2} \boldsymbol{v}_{1i}^L$$

In the center of mass frame

$$\boldsymbol{v}_{1i}^{CM}, \quad \boldsymbol{v}_{2i}^{CM}, \quad \boldsymbol{v}_{1f}^{CM}, \quad \boldsymbol{v}_{2f}^{CM},$$

We have the relation

$$\boldsymbol{v}_{1i}^L = \boldsymbol{v}_{1i}^{CM} + \boldsymbol{v}_{CM}, \qquad \boldsymbol{v}_{2i}^L = \boldsymbol{v}_{2i}^{CM} + \boldsymbol{v}_{CM}, \tag{1a}$$

$$\boldsymbol{v}_{1f}^L = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}, \qquad \boldsymbol{v}_{2f}^L = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM}, \tag{1b}$$

where

$$\boldsymbol{v}_{1i}^{CM} = \boldsymbol{v}_{1i}^L - \boldsymbol{v}_{CM} = \frac{m_2}{m_1 + m_2} \boldsymbol{v}_{1i}^L, \qquad \boldsymbol{v}_{2i}^{CM} = \boldsymbol{v}_{2i}^L - \boldsymbol{v}_{CM} = -\frac{m_1}{m_1 + m_2} \boldsymbol{v}_{1i}^L.$$

(i) The momentum conservation in the laboratory frame;

$$m_1 \boldsymbol{v}_{1i}^L + m_2 \boldsymbol{v}_{2i}^L = m_1 \boldsymbol{v}_{1f}^L + m_2 \boldsymbol{v}_{2f}^L.$$

which means that the velocity of the center of mass remains unchanged before and after the collision. Using Eqs.(1a) and (1b), we get

$$m_1 \boldsymbol{v}_{1i}^L = m_1 (\boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}) + m_2 (\boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM})$$

or

$$m_1 \boldsymbol{v}_{1f}^{CM} + m_2 \boldsymbol{v}_{2f}^{CM} = m_1 \boldsymbol{v}_{1i}^{L} - (m_1 + m_2)\boldsymbol{v}_{CM} = 0$$

or

$$m_1 \boldsymbol{v}_{1f}^{CM} + m_2 \boldsymbol{v}_{2f}^{CM} = 0, \qquad \text{or} \qquad \boldsymbol{v}_{2f}^{CM} = -\frac{m_1}{m_2}\boldsymbol{v}_{1f}^{CM} \qquad (2)$$

(ii) The energy conservation law;

$$\frac{1}{2}m_1\left(v_{1i}^{L}\right)^2 + \frac{1}{2}m_2\left(v_{2i}^{L}\right)^2 = \frac{1}{2}m_1\left(v_{1f}^{L}\right)^2 + \frac{1}{2}m_2\left(v_{2f}^{L}\right)^2$$

Using Eqs.(1a) and (1b), we get

$$\frac{1}{2}m_1\left(\boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}\right)^2 + \frac{1}{2}m_2(\boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM})^2 = \frac{1}{2}m_1\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}m_2(\boldsymbol{v}_{2f}^{CM})^2 + (m_1\boldsymbol{v}_{1f}^{CM} + m_2\boldsymbol{v}_{1f}^{CM})\cdot\boldsymbol{v}_{CM} + \frac{1}{2}(m_1 + m_2)(\boldsymbol{v}_{CM})^2 = \frac{1}{2}m_1\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}m_2(\boldsymbol{v}_{2f}^{CM})^2 = \frac{1}{2}m_1\left(\boldsymbol{v}_{1i}^{L}\right)^2 - \frac{1}{2}(m_1 + m_2)(\boldsymbol{v}_{CM})^2 = \frac{1}{2}\mu\left(\boldsymbol{v}_{1i}^{L}\right)^2 \qquad (3)$$

where μ is the reduced mass and is given by

$$\mu = \frac{m_1 m_2}{m_1 + m_2}.$$

Using Eqs.(2) and (3),

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2 + \frac{1}{2}m_2(-\frac{m_1}{m_2}\boldsymbol{v}_{1f}^{CM})^2 = \frac{1}{2}\mu\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$$\frac{1}{2}m_1(\boldsymbol{v}_{1f}^{CM})^2+\frac{1}{2}\frac{m_1{}^2}{m_2}(\boldsymbol{v}_{1f}^{CM})^2=\frac{1}{2}\frac{m_1m_2}{m_1+m_2}\left(\boldsymbol{v}_{1i}^{L}\right)^2$$

or

$$v_{1f}^{CM}=\frac{m_2v_{1i}^L}{m_1+m_2}, \qquad \text{and} \qquad v_{2f}^{CM}=\frac{m_1v_{1i}^L}{m_1+m_2}$$

and

$$v_{1f}^{CM}+v_{2f}^{CM}=\frac{(m_1+m_2)v_{1i}^L}{m_1+m_2}=v_{1i}^L$$

in magnitudes.

In conclusion we have

$$\boldsymbol{v}_{2f}^{CM}=-\frac{m_1}{m_2}\boldsymbol{v}_{1f}^{CM}\ .\ \boldsymbol{v}_{2i}^{CM}=-\frac{m_1}{m_1+m_2}\boldsymbol{v}_{1i}^{L}\ .$$

with

$$\left|\boldsymbol{v}_{1f}^{CM}\right|=\left|\boldsymbol{v}_{1i}^{CM}\right|=\frac{m_2v_{1i}^L}{m_1+m_2}\ .\ \left|\boldsymbol{v}_{2f}^{CM}\right|=\left|\boldsymbol{v}_{2i}^{CM}\right|=\frac{m_1v_{1i}^L}{m_1+m_2}\ .$$

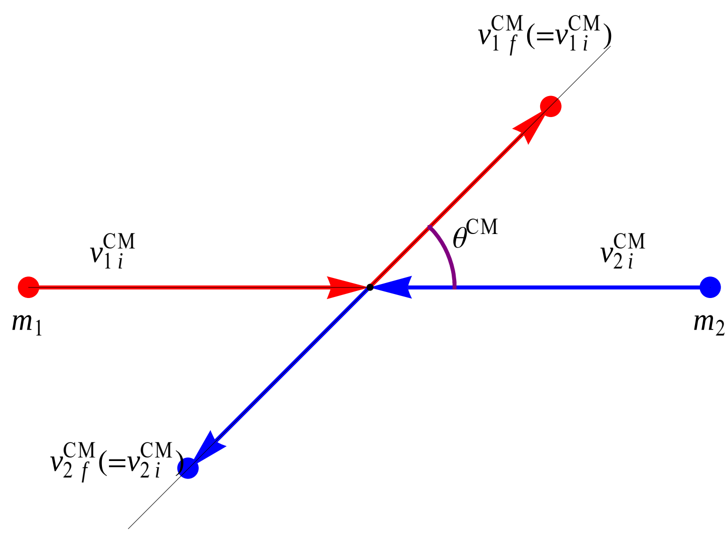


Fig. The center of mass frame. $v_{1i}^{CM} = v_{1f}^{CM} = \dfrac{m_2 v_{1i}^L}{m_1 + m_2}$,

$v_{2i}^{CM} = v_{2f}^{CM} = \dfrac{m_1 v_{1i}^L}{m_1 + m_2} = v_{CM}$. $v_{1f}^{CM} + v_{2f}^{CM} = v_{1i}^{CM} + v_{2i}^{CM} = v_{1i}^L$. The scattering angle of the particle 1 with mass m_1 is θ^{CM}.

Note that

$$\begin{aligned}
\left|\boldsymbol{v}_{2f}^L - \boldsymbol{v}_{1f}^L\right| &= \left|(\boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM}) - (\boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM})\right| \\
&= \left|\boldsymbol{v}_{2f}^{CM} - \boldsymbol{v}_{1f}^{CM}\right| \\
&= \left|\frac{m_1 v_{1i}^L}{m_1 + m_2} + \frac{m_2 v_{1i}^L}{m_1 + m_2}\right| = v_{1i}^L
\end{aligned}$$

$$\begin{aligned}
\left|\boldsymbol{v}_{2i}^L - \boldsymbol{v}_{1i}^L\right| &= \left|(\boldsymbol{v}_{2i}^{CM} + \boldsymbol{v}_{CM}) - (\boldsymbol{v}_{1i}^{CM} + \boldsymbol{v}_{CM})\right| \\
&= \left|\boldsymbol{v}_{2i}^{CM} - \boldsymbol{v}_{1i}^{CM}\right| \\
&= \left|-\frac{m_1}{m_1 + m_2}\boldsymbol{v}_{1i}^L - \frac{m_2}{m_1 + m_2}\boldsymbol{v}_{1i}^L\right| \\
&= v_{1i}^L
\end{aligned}$$

That is, the rate at which two objects approach each other before an elastic collision is the same at the rate at which they separate afterward. We can use this result to identify elastic collisions in any inertial reference frame. Namely, the relative velocity of two objects at a given time, that is, the difference in the velocity vectors of the objects must be the same in all inertial reference frames.

B.2 Scattering angles θ^{CM}, θ_1^L and θ_2^L

The y and z components of v_{1f}^{CM} and v_{2f}^{CM};

$$\left(\boldsymbol{v}_{1f}^{CM}\right)_y = v_{1f}^{CM} \sin\theta^{CM} = \frac{m_2 v_{1i}^L}{m_1 + m_2} \sin\theta^{CM}$$

$$\left(\boldsymbol{v}_{1f}^{CM}\right)_z = v_{1f}^{CM} \cos\theta^{CM} = \frac{m_2 v_{1i}^L}{m_1 + m_2} \cos\theta^{CM}$$

$$\left(\boldsymbol{v}_{2f}^{CM}\right)_y = -v_{2f}^{CM} \sin\theta^{CM} = -\frac{m_1 v_{1i}^L}{m_1 + m_2} \sin\theta^{CM}$$

$$\left(\boldsymbol{v}_{2f}^{CM}\right)_z = -v_{2f}^{CM}\cos\theta^{CM} = -\frac{m_1 v_{1i}^L}{m_1+m_2}\cos\theta^{CM}$$

where the z axis is in the horizontal direction and the y axis is in the vertical direction.

Using Eq.(1b), we have

$$\boldsymbol{v}_{1f}^L = \boldsymbol{v}_{1f}^{CM} + \boldsymbol{v}_{CM}, \qquad \boldsymbol{v}_{2f}^L = \boldsymbol{v}_{2f}^{CM} + \boldsymbol{v}_{CM},$$

or

$$\left(\boldsymbol{v}_{1f}^L\right)_y = \left(\boldsymbol{v}_{1f}^{CM}\right)_y + \left(\boldsymbol{v}_{CM}\right)_y = v_{1f}^{CM}\sin\theta^{CM} = \frac{m_2 v_{1i}^L}{m_1+m_2}\sin\theta^{CM}$$

$$\begin{aligned}\left(\boldsymbol{v}_{1f}^L\right)_z &= \left(\boldsymbol{v}_{1f}^{CM}\right)_z + \left(\boldsymbol{v}_{CM}\right)_z \\ &= v_{1f}^{CM}\cos\theta^{CM} + v_{CM} \\ &= \frac{m_2 v_{1i}^L}{m_1+m_2}\cos\theta^{CM} + \frac{m_1 v_{1i}^L}{m_1+m_2} \\ &= \frac{(m_1 + m_2\cos\theta^{CM})}{m_1+m_2} v_{1i}^L\end{aligned}$$

$$\left(\boldsymbol{v}_{2f}^L\right)_y = \left(\boldsymbol{v}_{2f}^{CM}\right)_y + \left(\boldsymbol{v}_{CM}\right)_y = -\frac{m_1 v_{1i}^L}{m_1+m_2}\sin\theta^{CM}$$

$$\begin{aligned}\left(\boldsymbol{v}_{2f}^L\right)_z &= \left(\boldsymbol{v}_{2f}^{CM}\right)_z + \left(\boldsymbol{v}_{CM}\right)_z \\ &= -\frac{m_1 v_{1i}^L}{m_1+m_2}\cos\theta^{CM} + \frac{m_1 v_{1i}^L}{m_1+m_2} \\ &= \frac{(m_1 - m_1\cos\theta^{CM})}{m_1+m_2} v_{1i}^L\end{aligned}$$

where

$$\boldsymbol{v}_{CM} = \frac{m_1}{m_1+m_2}\boldsymbol{v}_{1i}^L .$$

The scattering angle in the laboratory scheme is obtained as

$$\tan\theta_1^L = \frac{(\boldsymbol{v}_{1f}^L)_y}{(\boldsymbol{v}_{1f}^L)_z} = \frac{m_2 \sin\theta^{CM}}{m_1 + m_2\cos\theta^{CM}}$$

$$\tan\theta_2^L = \frac{(\boldsymbol{v}_{2f}^L)_y}{(\boldsymbol{v}_{2f}^L)_z} = \frac{m_1 \sin\theta^{CM}}{m_1 - m_1\cos\theta^{CM}} = \frac{\sin\theta^{CM}}{1-\cos\theta^{CM}} = \cot\left(\frac{\theta^{CM}}{2}\right)$$

When $m_1 = m_2$, we have

$$\tan\theta_1^L = \frac{\sin\theta^{CM}}{1+\cos\theta^{CM}} = \frac{2\sin\left(\frac{\theta^{CM}}{2}\right)\cos\left(\frac{\theta^{CM}}{2}\right)}{2\cos^2\left(\frac{\theta^{CM}}{2}\right)} = \tan\left(\frac{\theta^{CM}}{2}\right).$$

Then we have

$$\theta_1^L + \theta_2^L = \frac{\pi}{2}$$

since

$$\tan\theta_1^L \tan\theta_2^L = 1 .$$

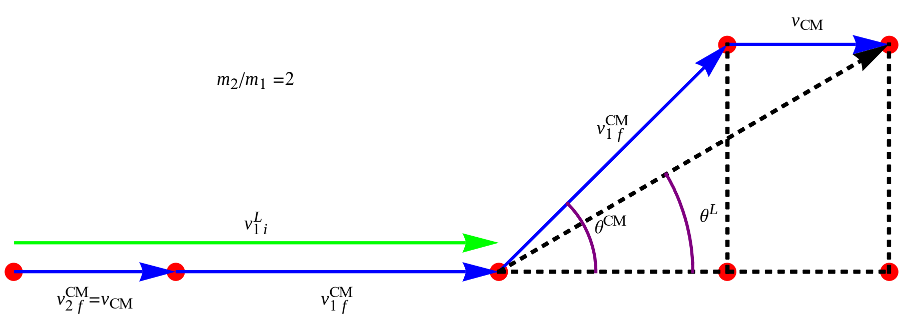


Fig. The relation between scattering angle in the laboratory frame and in the center of mass frame. $v_{1i}^{CM} = v_{1f}^{CM} = \dfrac{m_2 v_{1i}^L}{m_1+m_2}$, $v_{2i}^{CM} = v_{2f}^{CM} = \dfrac{m_1 v_{1i}^L}{m_1+m_2} = v_{CM}$. $v_{1i}^L = v_{1f}^{CM} + v_{2f}^{CM}$. $v_{1i}^L = v_{1i}^{CM} + v_{2i}^{CM}$. In this figure, we assume that $m_2/m_1 = 2$.

APPENDIX C Energy of a system of many particles

The kinetic energy of many particles is given by

$$K = \frac{1}{2}\sum_i m_i(\boldsymbol{v}_i \cdot \boldsymbol{v}_i)$$

The velocity in the laboratory frame $\boldsymbol{v}_i$ is given by

$$\boldsymbol{v}_i = \boldsymbol{v}_{CM} + \boldsymbol{v}_i^{CM}$$

where $\boldsymbol{v}_{CM}$ is the velocity of the center of mass and $\boldsymbol{v}_i^{CM}$ is the velocity in the center of mass frame. Then K can be rewritten as

$$\begin{aligned}
K &= \frac{1}{2}\sum_i m_i(\boldsymbol{v}_{CM} + \boldsymbol{v}_i^{CM}) \cdot (\boldsymbol{v}_{CM} + \boldsymbol{v}_i^{CM}) \\
&= \frac{1}{2}\sum_i m_i[\boldsymbol{v}_{CM}{}^2 + 2\boldsymbol{v}_i^{CM} \cdot \boldsymbol{v}_{CM} + \boldsymbol{v}_i^{CM\,2}) \\
&= \frac{1}{2}M\boldsymbol{v}_{CM}{}^2 + \frac{1}{2}\sum_i m_i(\boldsymbol{v}_i^{CM})^2 + (\sum_i m_i\boldsymbol{v}_i^{CM}) \cdot \boldsymbol{v}_{CM} \\
&= \frac{1}{2}M\boldsymbol{v}_{CM}{}^2 + \frac{1}{2}\sum_i m_i(\boldsymbol{v}_i^{CM})^2 \\
&= K_{rel} + K_{CM}
\end{aligned}$$

where

$$\sum_i m_i\boldsymbol{v}_i^{CM} = \sum_i m_i(\boldsymbol{v}_i - \boldsymbol{v}_{CM}) = M\boldsymbol{v}_{CM} - M\boldsymbol{v}_{CM} = 0$$

Then K consists of the kinetic energy of the center of mass (K_{CM}) and the kinetic energy of the objects as viewed in the center of mass frame (K_{rel}).