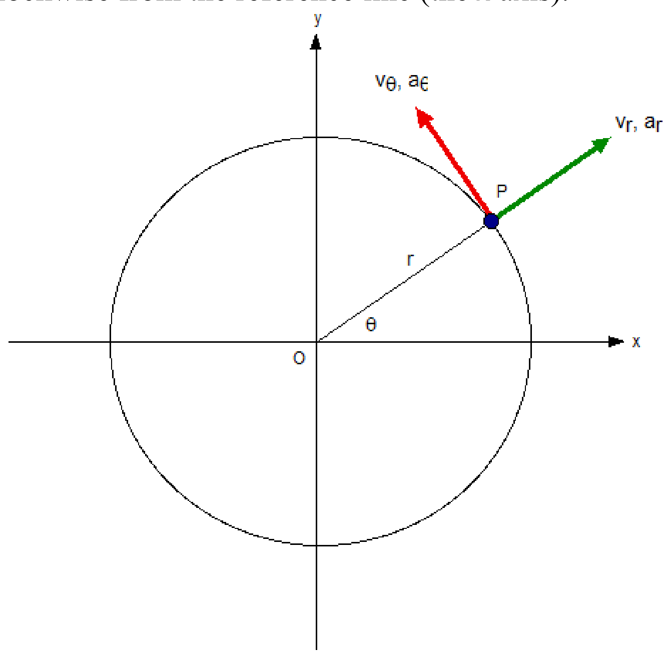
Chapter 10
Rotation

1 Rotation

1.1 Polar coordinates: general case

The point P is located at (r, θ), where r is the distance from the origin and θ is the measured counterclockwise from the reference line (the x axis).



We introduce the unit vectors given by

$$\hat{r} = (\cos\theta, \sin\theta)$$
$$\hat{\theta} = (-\sin\theta, \cos\theta)$$

The position vector (displacement vector) is given by

$$\boldsymbol{r} = (r\cos\theta, r\sin\theta) = r\cos\theta\hat{\boldsymbol{i}} + r\sin\theta\hat{\boldsymbol{j}}$$

The velocity and acceleration are

$$\boldsymbol{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

or

$$v_r = \boldsymbol{v}\cdot\hat{r} = \dot{r}$$
$$v_\theta = \boldsymbol{v}\cdot\hat{\theta} = r\dot{\theta}$$

$$a_r = \boldsymbol{a}\cdot\hat{r} = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = \boldsymbol{a}\cdot\hat{\theta} = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$

((**Note-1**))

$$\boldsymbol{r} = (r\cos\theta, r\sin\theta)$$
$$\boldsymbol{v} = \dot{\boldsymbol{r}} = (\dot{r}\cos\theta - r\dot{\theta}\sin\theta, \dot{r}\sin\theta + r\dot{\theta}\cos\theta) = \dot{r}(\cos\theta,\sin\theta) + r\dot{\theta}(-\sin\theta,\cos\theta) = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

$$\boldsymbol{a} = \ddot{\boldsymbol{r}} = (\ddot{r}\cos\theta - \dot{r}\dot{\theta}\sin\theta - \dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta, \ddot{r}\sin\theta + \dot{r}\dot{\theta}\cos\theta + \dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2 \mathrm{si}$$

or

$$\ddot{\boldsymbol{r}} = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta, \ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)$$

or

$$\ddot{\mathbf{r}} = (\ddot{r}\cos\theta - 2\dot{r}\dot{\theta}\sin\theta - r\ddot{\theta}\sin\theta - r\dot{\theta}^2\cos\theta)\hat{i} + (\ddot{r}\sin\theta + 2\dot{r}\dot{\theta}\cos\theta + r\ddot{\theta}\cos\theta - r\dot{\theta}^2\sin\theta)\hat{j}$$

$$\ddot{\boldsymbol{r}} = (\ddot{r} - r\dot{\theta}^2)(\cos\theta\hat{i} + \sin\theta\hat{j}) + (2\dot{r}\dot{\theta} + r\ddot{\theta})(-\sin\theta\hat{i} + \cos\theta\hat{j}) = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

((**Note-2**))

$$\boldsymbol{e}_r = (\cos\theta, \sin\theta)\,. \qquad \boldsymbol{e}_\theta = (-\sin\theta, \cos\theta)$$

$$\dot{\boldsymbol{e}}_r = \dot{\theta}\,(-\sin\theta,\cos\theta) = \dot{\theta}\,\boldsymbol{e}_\theta\,, \qquad \dot{\boldsymbol{e}}_\theta = -\dot{\theta}\,(\cos\theta,\sin\theta) = -\dot{\theta}\boldsymbol{e}_r$$

$$\boldsymbol{r} = r\,\boldsymbol{e}_\theta$$

The velocity vector:

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\,\boldsymbol{e}_r + r\,\dot{\boldsymbol{e}}_r = \dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\;\boldsymbol{e}_\theta$$

The acceleration vector:

$$\begin{aligned}
\boldsymbol{a} &= \ddot{\boldsymbol{r}} \\
&= \ddot{r}\,\boldsymbol{e}_r + \dot{r}\,\dot{\boldsymbol{e}}_r + \dot{r}\dot{\theta}\;\boldsymbol{e}_\theta + r\ddot{\theta}\;\boldsymbol{e}_\theta + r\dot{\theta}\;\dot{\boldsymbol{e}}_\theta \\
&= \ddot{r}\,\boldsymbol{e}_r + \dot{r}\,\dot{\theta}\,\boldsymbol{e}_\theta + \dot{r}\dot{\theta}\;\boldsymbol{e}_\theta + r\ddot{\theta}\;\boldsymbol{e}_\theta + r\dot{\theta}\;(-\dot{\theta}\boldsymbol{e}_r) \\
&= (\ddot{r} - r\dot{\theta}^2)\boldsymbol{e}_r + (2\dot{r}\dot{\theta} + r\ddot{\theta})\boldsymbol{e}_\theta
\end{aligned}$$

((**Mathematica**))

```
Clear["Global`*"];
R = {r[t] Cos[θ[t]], r[t] Sin[θ[t]]};

V = D[R, t] // FullSimplify

{Cos[θ[t]] r'[t] - r[t] Sin[θ[t]] θ'[t],
 Sin[θ[t]] r'[t] + Cos[θ[t]] r[t] θ'[t]}

A = D[R, {t, 2}] // FullSimplify

{Cos[θ[t]] (-r[t] θ'[t]^2 + r''[t]) -
   Sin[θ[t]] (2 r'[t] θ'[t] + r[t] θ''[t]),
 2 Cos[θ[t]] r'[t] θ'[t] + Sin[θ[t]] (-r[t] θ'[t]^2 + r''[t]) +
   Cos[θ[t]] r[t] θ''[t]}

er = {Cos[θ[t]], Sin[θ[t]]};
eθ = {-Sin[θ[t]], Cos[θ[t]]};

A.er // Simplify

-r[t] θ'[t]^2 + r''[t]

A.eθ // Simplify

2 r'[t] θ'[t] + r[t] θ''[t]

V.er // Simplify

r'[t]

V.eθ // Simplify

r[t] θ'[t]
```

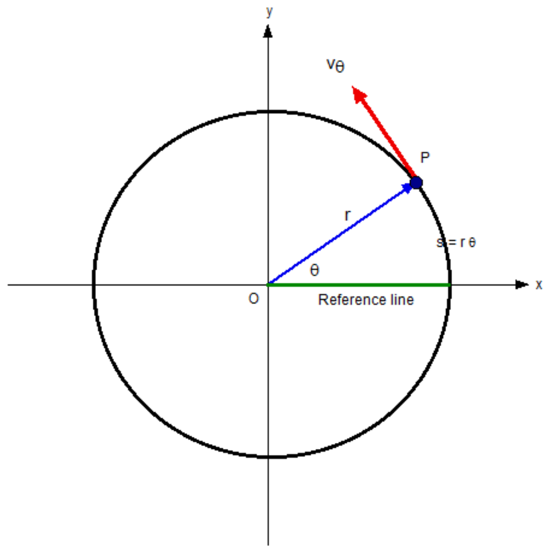
1.2 Circular motion (r = constant)

We consider a circular motion with r = constant. since $\dot{r}=0$ and $\ddot{r}=0$.

$$a_r = -r\dot{\theta}^2 \qquad v_r = 0$$
$$a_\theta = a_t = r\ddot{\theta} \qquad v_\theta = v_t = r\dot{\theta}$$

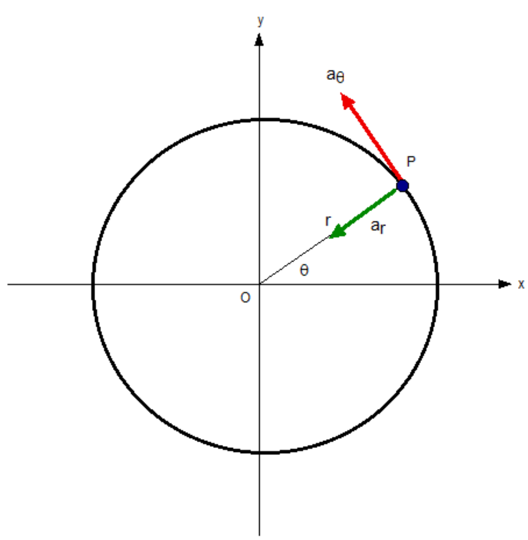
In summary, we have

$$v_\theta = v_t = r\dot{\theta} = v$$



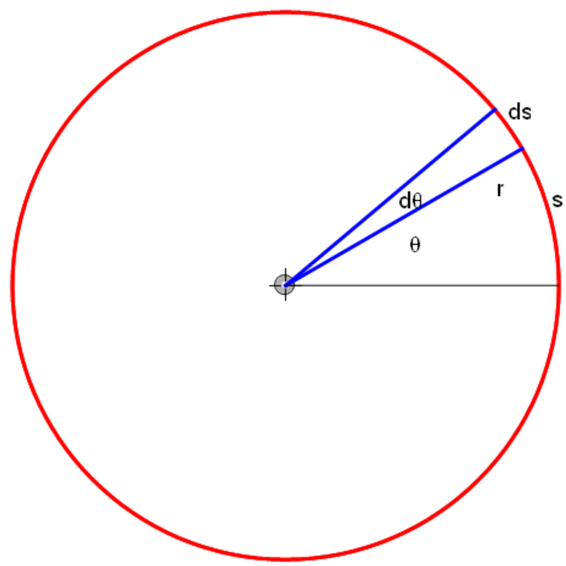
$$a_r = -r\dot{\theta}^2 = -\frac{v^2}{r}$$

$$a_\theta = a_t = r\ddot{\theta} = \frac{dv}{dt}$$



1.3 Angular velocity ω and angular acceleration α

The case when the point P rotates about the origin in a circle of radius r (= constant). As the particle moves, the only coordinates that changes is θ. As the particle moves through θ, it moves through an arc length s.



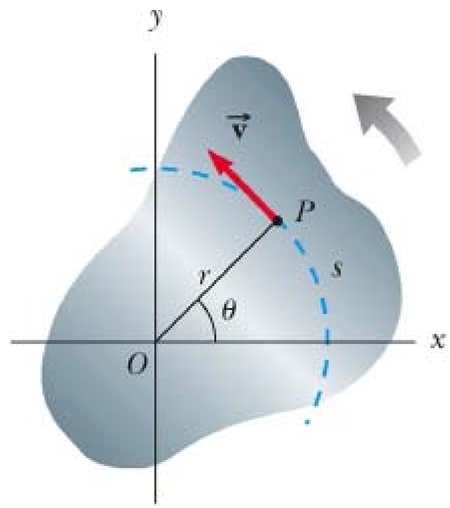
$$ds = rd\theta$$
$$\dot{s} = \frac{ds}{dt} = r\frac{d\theta}{dt} = r\dot{\theta} = v$$
$$\ddot{s} = \frac{d^2s}{dt^2} = r\frac{d^2\theta}{dt^2} = r\ddot{\theta} = \frac{dv}{dt}$$

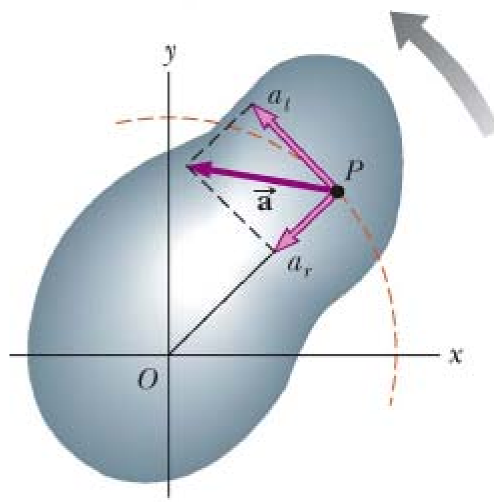
Here we define the angular velocity and angular acceleration as

$\omega = \frac{d\theta}{dt} = \dot{\theta}$	angular velocity	(rad/s)
$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \ddot{\theta}$	angular acceleration	(rad/s^2)

Then we have

$$v_\theta = \frac{ds}{dt} = v = r\omega$$
$$a_\theta = \frac{d^2s}{dt^2} = r\alpha$$
$$a_r = -\frac{v^2}{r} = -\omega^2 r$$





1.4 Summary

Rotational variables

Angular displacement	$\boldsymbol{\theta}$	**displacement**	$\boldsymbol{x}$
Angular velocity	$\boldsymbol{\omega}$	**velocity**	$\boldsymbol{v}$
Angular acceleration	$\boldsymbol{\alpha}$	**acceleration**	$\boldsymbol{a}$

Average angular velocity $\quad \omega_{avg} = \dfrac{\Delta\theta}{\Delta t}$

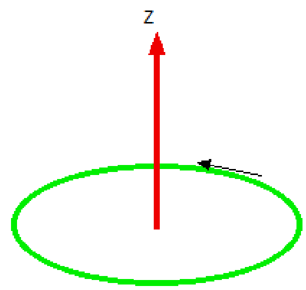
Instantaneous angular velocity $\quad \omega = \dfrac{d\theta}{dt}$

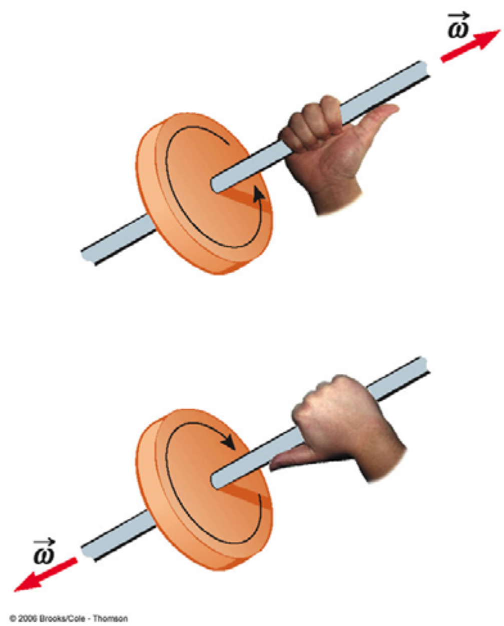
Average angular acceleration $\quad \alpha_{avg} = \dfrac{\Delta\omega}{\Delta t}$

Instantaneous angular acceleration $\quad \alpha = \dfrac{d\omega}{dt} = \dfrac{d^2\theta}{dt^2}$

The direction of $\boldsymbol{\omega}$

The direction of $\boldsymbol{\omega}$ is along the rotation axis.
The sense of $\boldsymbol{\omega}$ is defined by the right-hand rule.
The thumb of the right hand gives the sense of $\boldsymbol{\omega}$.





2 The case of α = constant

$$\alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2} = \text{const}$$

We solve this differential equation with the initial conditions:

$$\dot{\theta}(0) = \omega_0$$
$$\theta(0) = \theta_0$$

The solution is as follows.

$$\omega = \dot{\theta} = \omega_0 + \alpha t$$

$$\theta = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$$

$$\omega^2 - \omega_0{}^2 = 2\alpha(\theta - \theta_0)$$

Problem 10-13 (SP-10 Hint) (10-th edition)**

A flywheel turns through 40 rev as it slows from an angular speed of 1.5 rad/s to a stop. (a) Assuming a constant angular acceleration, find the time for it to come to rest, (b) What is its angular acceleration? (c) How much time is required for it to complete the first 20 of the 40 revolutions?

((Solution)) 1 rev = 2π (rad)

$\omega_i = 1.5 rad/s$

$\omega_f = 0$

$\Delta\theta = 40rev = 40 \times 2\pi = 80\pi$ (rad)

α is constant.

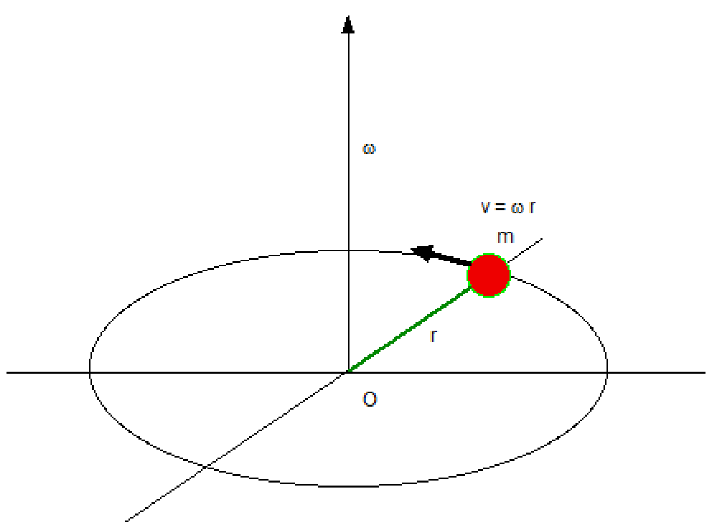
$$\omega_f = \omega_i + \alpha t$$

$$\theta_f = \theta_i + \omega_i t + \frac{1}{2}\alpha t^2$$

$$\omega_f{}^2 - \omega_i{}^2 = 2\alpha(\theta_f - \theta_i)$$

3 Kinetic energy of rotation

We assume that a particle (mass m) rotates around the z axis with an angular velocity ω.

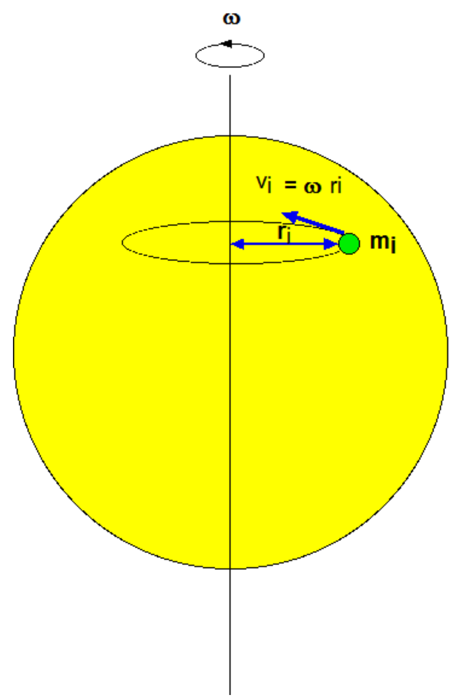


The kinetic energy of the particle is given by

$$K_R = \frac{1}{2}mv^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}mr^2\omega^2$$

where r is distance perpendicular to the axis of rotation.

Now we consider an object as a collection of particles and assume that it rotates about a fixed axis (z-axis) with an angular velocity ω. Figure shows the rotating object and identifies particle element (m_i) on the object located at a distance r_i from the rotation axis.



The total kinetic energy of the rotating object:

$$K_R = \sum_i \frac{1}{2}m_i v_i^2 = \sum_i \frac{1}{2}m_i(\omega r_i)^2 = \frac{1}{2}(\sum_i m_i r_i^2)\omega^2$$

Rotational inertia (moment of inertia) I:

$$I = \sum_i m_i r_i^2$$

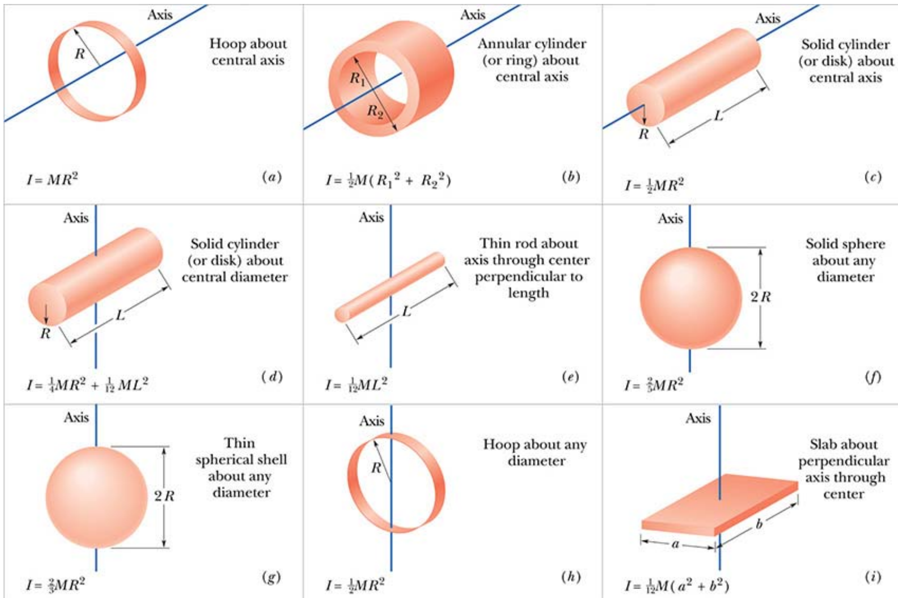
The rotational kinetic energy: K_R

$$K_R = \frac{1}{2}I\omega^2$$

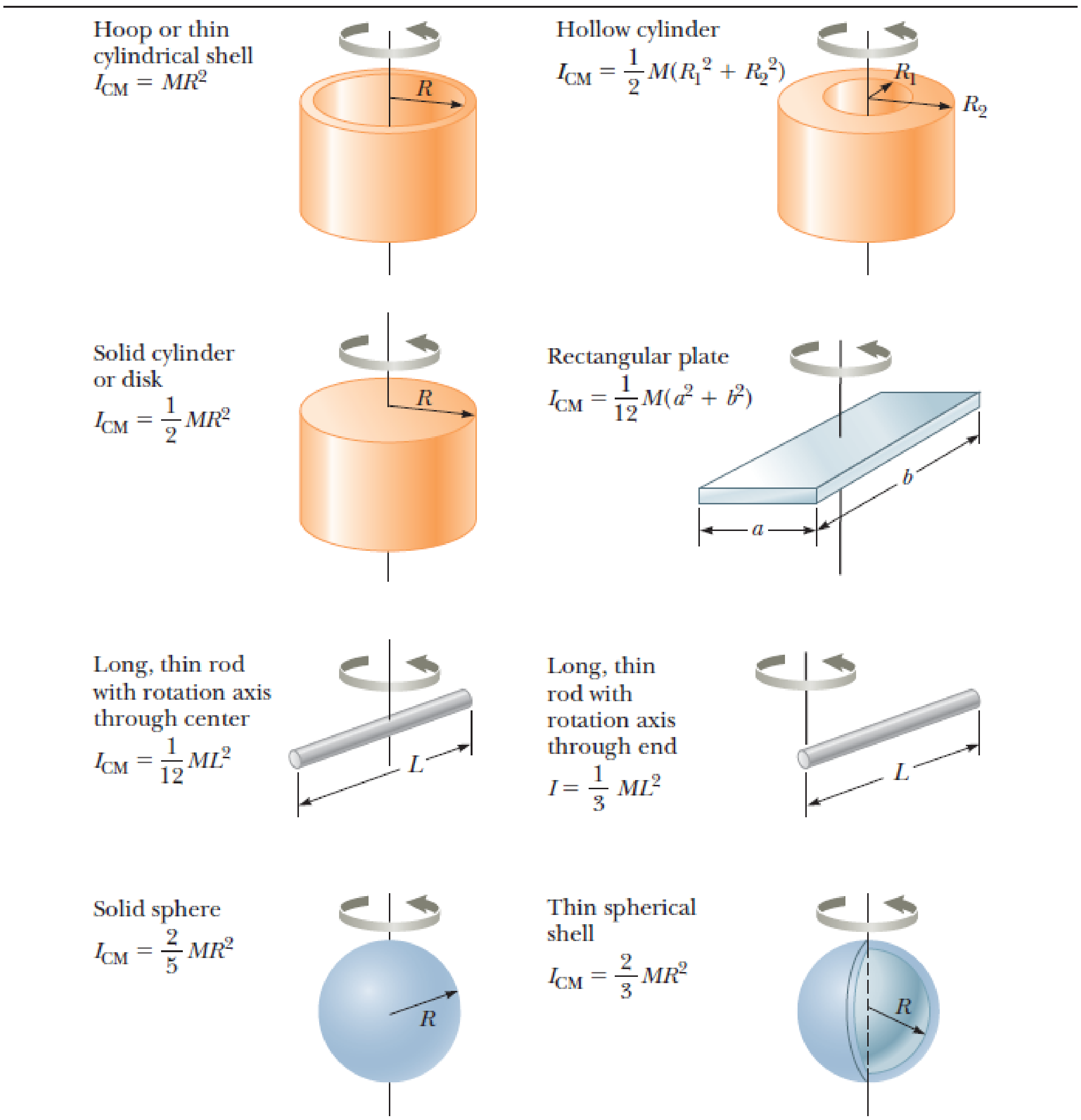
Rotational inertia for the continuous body

$$I = \int r^2 dm$$

(1) From the textbook (Halliday, Resnick, and Walker)



(2) Table from the textbook (Serway and Jewett)

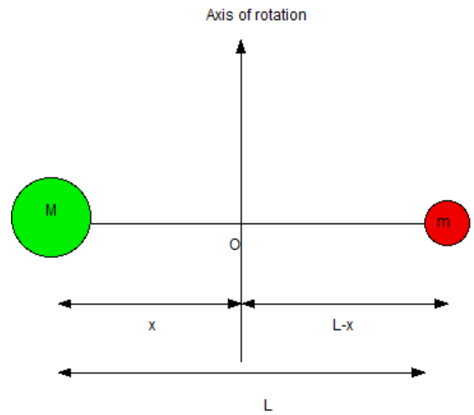


4. Calculation of moment of inertia

4.1 Example

Two balls with masses M and m are connected by a rigid rod of length L and negligible mass as in Figure. For an axis perpendicular to the rod, show that the system has the minimum moment of inertia when the axes passes through the center of mass. Show that this moment of inertia is $I = \mu L^2$, where $\mu = \dfrac{mM}{m+M}$.

((**Solution**))



The moment of inertia is given by

$$I = Mx^2 + m(L - x)^2$$

The derivative of I with respect to x has a local minimum

$$I_{\min} = \frac{mM}{m+M} L^2 = \mu L^2$$

at

$$x = \frac{L}{1+\frac{M}{m}} = \frac{mL}{M+m}$$

((**Mathematica**))

```
Clear["Global`"]; I1 = m (L - x)^2 + M x^2;
rule1 = {x → L y, M → m η};
I2 = I1 / (m L^2) /. rule1 // Expand;

eq1 = Solve[D[I2, y] == 0, y]
```

$$\left\{\left\{y \to \frac{1}{1+\eta}\right\}\right\}$$

```
I22 = I2 /. eq1[[1]] // Simplify
```

$$\frac{\eta}{1+\eta}$$

```
Plot[Evaluate[Table[I2, {η, 0, 1, 0.05}]], {y, 0, 1},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 AxesLabel → {"x/L", "1/(m L^2)"}, Background → LightGray]
```

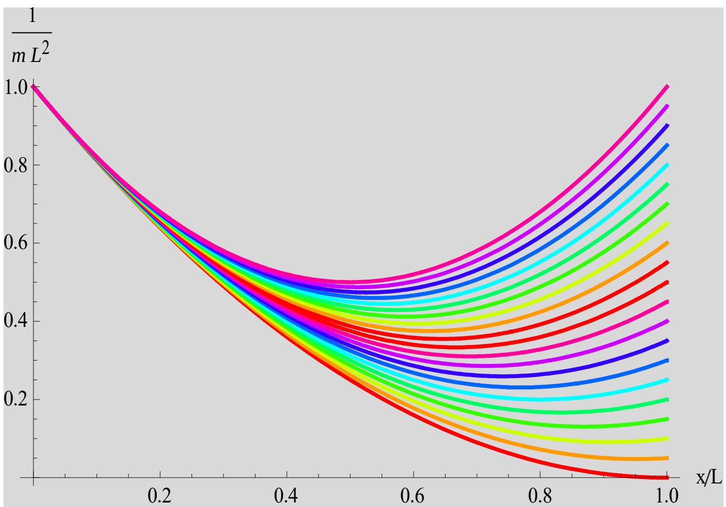
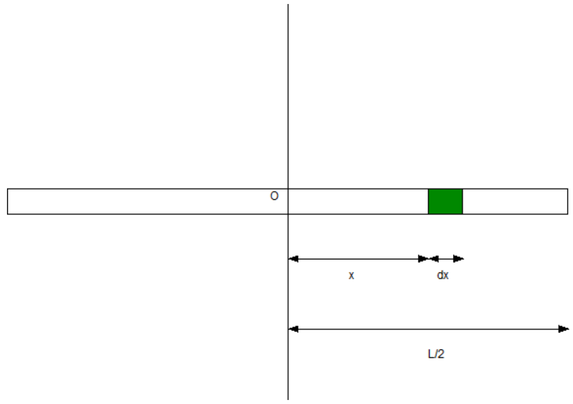


Fig. Plot of $I/(mL^2)$ as a function of x/L, where $\eta = M/m$ is changed between 0 and 1 ($\Delta\eta = 0.05$).

4.2 Thin rod about axis through center perpendicular to length



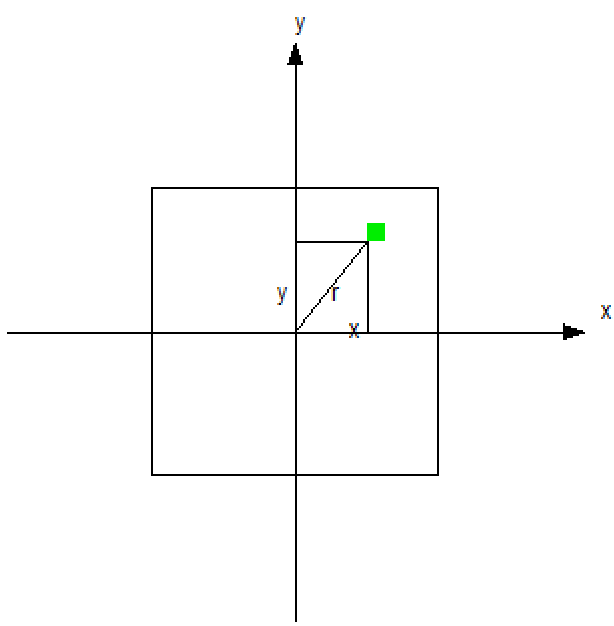
We assume that the total mass of the rod (length L) is M. Then the mass between x and x+dx is

$$dm = \frac{M}{L}dx$$

The moment of inertia for thin rod is

$$I = \int_{-L/2}^{L/2} x^2 dm = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{M}{L}\int_{-L/2}^{L/2} x^2 dx = \frac{2M}{L}\int_{0}^{L/2} x^2 dx = \frac{2M}{L}[\frac{1}{3}x^3]_0^{L/2}$$
$$= \frac{ML^2}{12}$$

4.3 The square about perpendicular axis through center



—

$$\xi = \frac{\int r^2 dm}{\int dm} = \frac{\iint r^2 dxdy}{\iint dxdy} = \frac{\iint (x^2 + y^2)dxdy)}{\iint dxdy} = \frac{4\int_0^{a/2} dx \int_0^{a/2} (x^2 + y^2)dy}{4\int_0^{a/2} dx \int_0^{a/2} dy}$$

or

$$\xi = \frac{\int_0^{a/2} dx[x^2 y + \frac{1}{3}y^3]_0^{a/2}}{(\frac{a}{2})^2} = \frac{\int_0^{a/2} dx(x^2 \frac{a}{2} + \frac{1}{3}\frac{a^3}{8})}{(\frac{a}{2})^2} = \frac{4}{a^2}[\frac{a}{2}\frac{x^3}{3} + \frac{a^3}{24}x]_0^{a/2} = \frac{4}{a^2}(\frac{a^4}{24}) = \frac{1}{6}a^2$$

Therefore the moment of inertia I is given by

$$I = M\xi = \frac{1}{6}Ma^2$$

((**Mathematica**))

```
Clear["Global`"];
Integrate[Integrate[x^2 + y^2, {y, 0, a / 2}],
 {x, 0, a / 2}]
```

$$\frac{a^4}{24}$$

4.4 Solid cylinder

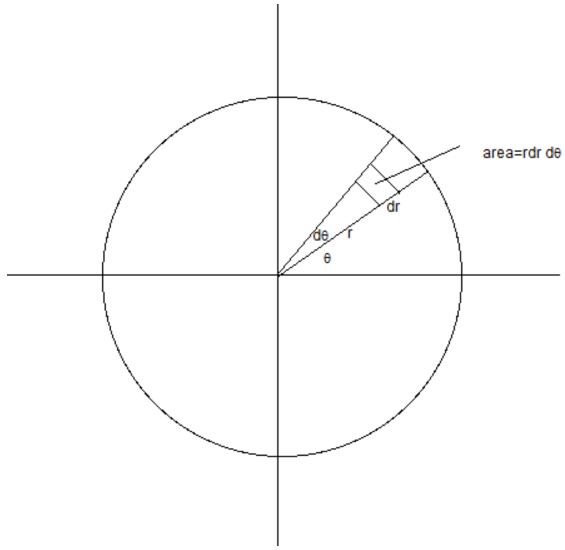
Moment of inertia for the solid cylinder (radius R, height h) (or disk) about central axis:

$$I_z = \frac{1}{2}MR^2$$

The mass: $M = \rho\pi R^2 h$

The moment of Inertia is given by

$$I = \int r^2 dm = \rho h \iint r^2 r dr d\theta = 2\pi\rho h \int_0^R r^3 dr = 2\pi\rho h \frac{R^4}{4} = \frac{\pi h R^4}{2}\frac{M}{\pi h R^2} = \frac{1}{2}MR^2$$



4.5 Annular cylinder

The mass: $M = \rho\pi(R_2{}^2 - R_1{}^2)h$

The moment of inertia is given by

$$I = \int r^2 dm = \rho h \iint r^2 r dr d\theta$$

$$= 2\pi\rho h \int_{R_1}^{R_2} r^3 dr = \frac{2\pi\rho h}{4}[R^4]_{R_1}^{R_2}$$

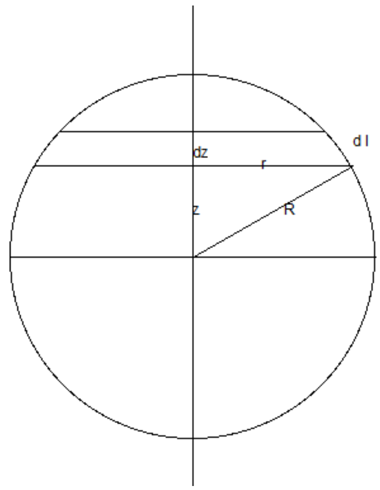
$$= \frac{\pi h(R_2{}^4 - R_1{}^4)}{2} \frac{M}{\pi h(R_2{}^2 - R_1{}^2)}$$

$$= \frac{M}{2}\frac{(R_2{}^4 - R_1{}^4)}{R_2{}^2 - R_1{}^2} = \frac{M}{2}(R_1{}^2 + R_2{}^2)$$

4.6 Solid sphere

(a) Method-1

Moment of inertia for the solid sphere about any diameter: $I_z = \frac{2}{5}MR^2$



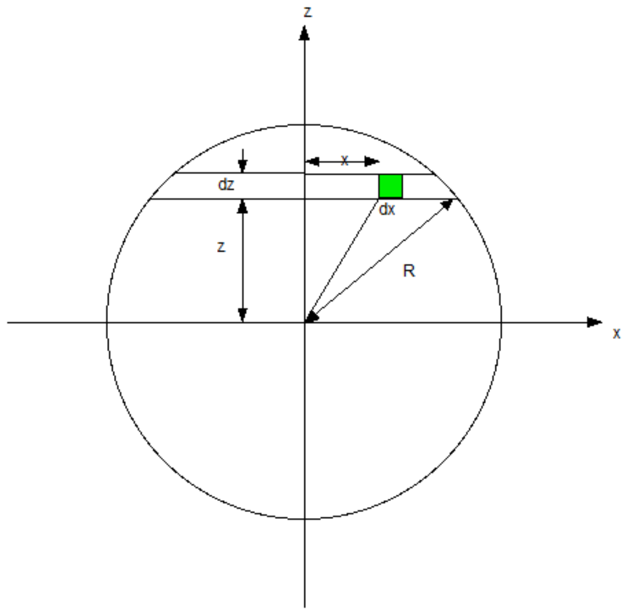
$$dI_z = \frac{1}{2}(dM)r^2 = \frac{1}{2}\rho\pi r^4 dz = \frac{1}{2}\frac{M}{\frac{4\pi}{3}R^3}\pi(R^2 - z^2)^2 dz = \frac{3M}{8R^3}(R^2 - z^2)^2 dz$$

where $dM = \rho\pi r^2 dz$ and $M = \frac{4\pi}{3}R^3\rho$

Then we have

$$I_z = \int_{-R}^{R}\frac{3M}{8R^3}(R^2 - z^2)^2 dz = \frac{3M}{4R^3}\int_0^R (R^2 - z^2)^2 dz = \frac{3M}{4R^3}\frac{8}{15}R^5 = \frac{2}{5}MR^2$$

(b) Method-2: direct calculation



$$\int dm = \frac{4\pi}{3}\rho R^3$$

where ρ is the density of the sphere.

$$\int r^2 dm = \iint \rho x^2 (2\pi x) dx dz = 2\pi\rho \iint x^3 dx dz = 2\pi\rho\xi$$

where

$$\xi = \iint x^3 dx dz = \int_{-R}^{R} dz \int_0^{\sqrt{R^2 - z^2}} x^3 dx = \int_{-R}^{R} dz [\frac{1}{4} x^4]_0^{\sqrt{R^2 - z^2}} = \frac{1}{4}\int_{-R}^{R} dz (R^2 - z^2)^2 = \frac{1}{2}\int_0^R dz (R^2 - z^2)^2$$

or

$$\xi = \frac{1}{2}\int_0^R dz (R^4 - 2R^2 z^2 + z^4) = \frac{1}{2}[R^4 z - 2R^2 \frac{1}{3} z^3 + \frac{1}{5} z^5]_0^R = \frac{4}{15} R^5$$

Then we have

$$\frac{I}{M} = \frac{\int r^2 dm}{\int dm} = \frac{2\pi\rho \frac{4}{15} R^5}{\frac{4\pi}{3}\rho R^3} = \frac{2}{5} R^2$$

or

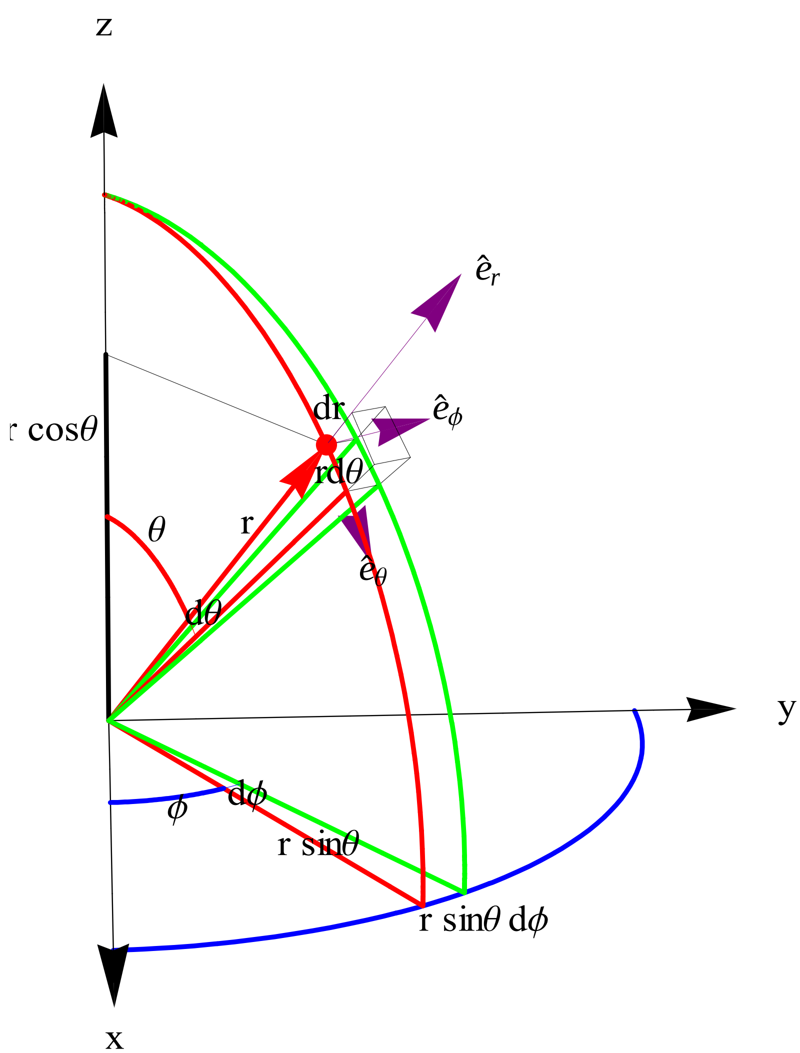
$$I = \frac{2}{5} M R^2$$

((**Mathematica**))

```
Clear["Global`*"];
Integrate[Integrate[x^3, {x, 0, Sqrt[R^2 - z^2]}],
  {z, -R, R}]
```

$$\frac{4 R^5}{15}$$

4.8 Sphere (using the spherical co-ordinate)



The moment of inertia I for the sphere can be calculated as

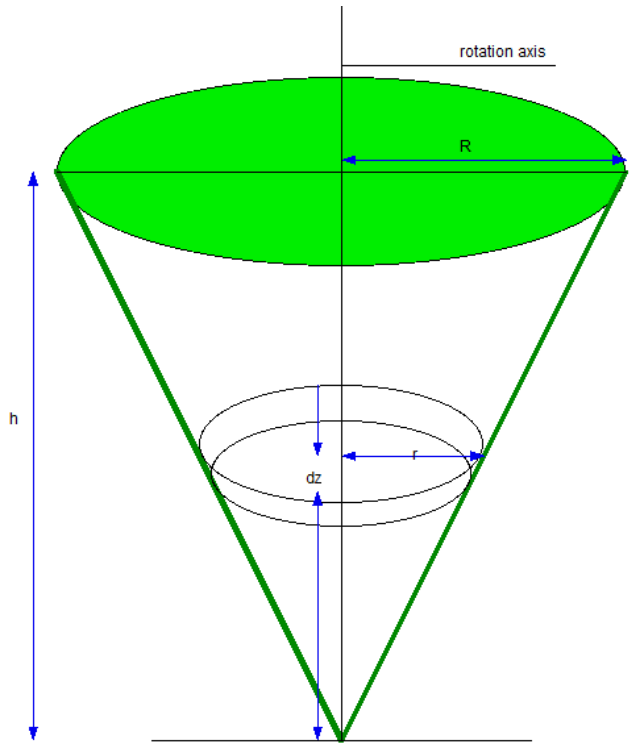
$$
\begin{aligned}
\frac{I}{M} &= \frac{\iiint \rho r^2 \sin^2\theta (r^2 \sin\theta) dr d\theta d\phi}{\frac{4\pi}{3}\rho R^3} \\
&= \frac{3}{4\pi R^3}\int_0^R r^4 dr \int_0^\pi \sin^3\theta d\theta \int_0^{2\pi} d\phi \\
&= \frac{3}{4\pi R^3}\frac{R^5}{5}\frac{8\pi}{3} \\
&= \frac{2}{5}R^2
\end{aligned}
$$

with the use of the spherical co-ordinate.

$$\int_0^{\pi} \sin^3 \theta d\theta = \frac{4}{3}$$

4.8 Solid cone

Show that the moment of inertia of a uniform solid cone about an axis through its center is given by $I = (3/10)MR^2$. The cone has mass M and altitude h. The radius of its circular base is R.



The density ρ of the cone is

$$\rho = \frac{M}{V} = \frac{M}{\frac{1}{3}\pi R^2 h} = \frac{3M}{\pi R^2 h}$$

From the geometry, we have

$$\frac{z}{h} = \frac{r}{R} \qquad \text{or} \qquad r = \frac{R}{h} z$$

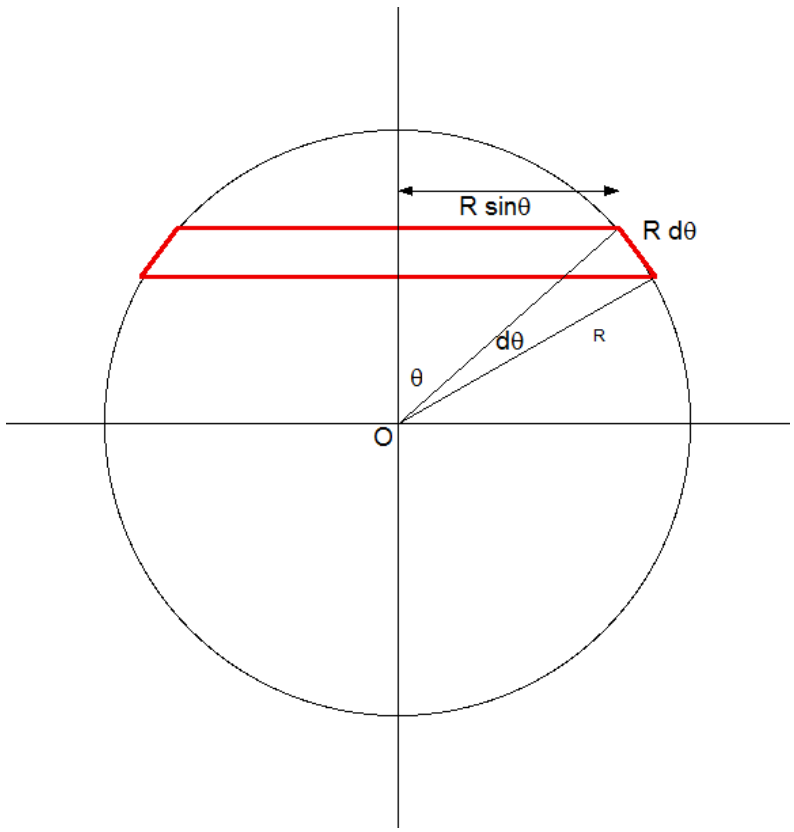
The moment of inertia;

$$dI = \frac{1}{2}(dm)r^2 = \frac{1}{2}(\rho\pi r^2 dz)r^2 = \frac{1}{2}\frac{3M}{\pi R^2 h}\pi r^4 dz = \frac{3}{2}\frac{M}{R^2 h}r^4 dz = \frac{3}{2}\frac{M}{R^2 h}\frac{R^4}{h^4}z^4 dz$$
$$= \frac{3}{2}\frac{MR^2}{h^5}z^4 dz$$

or

$$I = \frac{3}{2}\frac{MR^2}{h^5}\int_0^h z^4 dz = \frac{3}{2}\frac{MR^2}{h^5}\frac{h^5}{5} = \frac{3}{10}MR^2$$

4.9 Moment of inertia for spherical shell



The mass per unit area σ is given by

$$\sigma = \frac{M}{4\pi R^2}$$

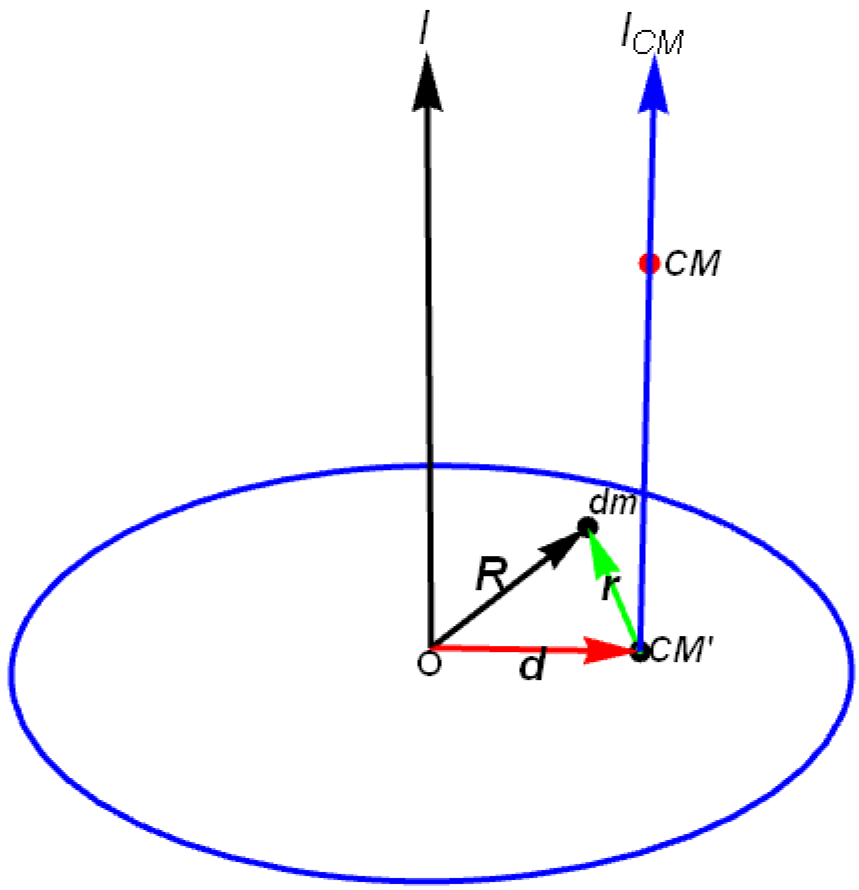
The moment of inertia for the spherical shell is

$$I = \int_0^\pi \sigma(R\sin\theta)^2(2\pi R\sin\theta)Rd\theta = \frac{M}{4\pi R^2}2\pi R^4\int_0^\pi \sin^3\theta d\theta = \frac{MR^2}{2}\frac{4}{3} = \frac{2}{3}MR^2$$

5 Parallel-axis theorem

The moment of inertia about any axis is equal to the moment of inertia about a parallel axis through the center of mass, plus the mass of the body times the square of the distance between the two axes.

$$I = I_{CM} + Md^2$$



We consider the rotation axis passing through the origin, and the axis passing through the center of mass. This axis is parallel to the rotation axis. We consider a plane which is perpendicular the rotation axis. The projection of the center of mass on this plane is denoted by CM' (red dot). In the above figure, the vectors $\boldsymbol{R}$, $\boldsymbol{r}$, and $\boldsymbol{d}$ are on this plane.

$$\boldsymbol{R} = \boldsymbol{r} + \boldsymbol{d}$$

The moment of inertia around the origin is

$$I = \int \boldsymbol{R}^2 dm = \int [(\boldsymbol{r}^2 + \boldsymbol{d}^2 + 2\boldsymbol{r}\cdot\boldsymbol{d})dm = \int (\boldsymbol{r}^2 + \boldsymbol{d}^2)dm + 2\boldsymbol{d}\cdot\int \boldsymbol{r}dm = \int (\boldsymbol{r}^2 + \boldsymbol{d}^2)dm$$

Note that

$$\int \boldsymbol{r}dm = 0$$

From the definition of ***d***,

$$\boldsymbol{d} = \frac{\int \boldsymbol{R}dm}{\int dm}$$

$$\int \boldsymbol{R}dm = \int \boldsymbol{d}dm$$

$$\int (\boldsymbol{R} - \boldsymbol{d})dm = \int \boldsymbol{r}dm = 0$$

Then we have

$$I = \int (\boldsymbol{r}^2 + \boldsymbol{d}^2)dm = I_{CM} + Md^2$$

where I_{CM} is the moment of inertia around the center of the mass.

((Note)) **Perpendicular axis theorem**

The perpendicular axis theorem for planar objects can be demonstrated by looking at the contribution to the three axis moments of inertia from an arbitrary mass element. From the point mass moment, the contributions to each of the axis moments of inertia are

$$I_x = \int y^2 dm$$

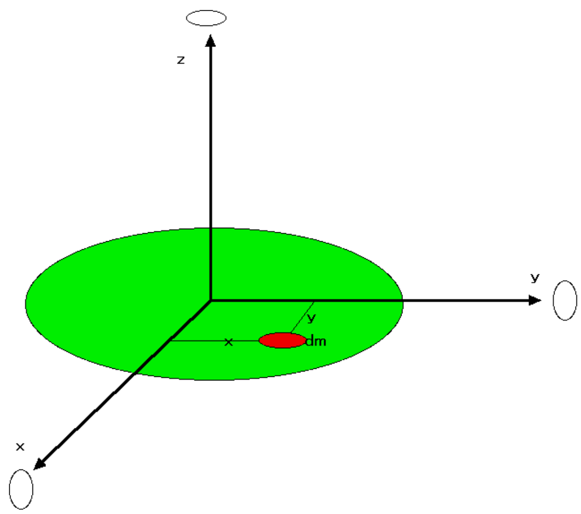
$$I_y = \int x^2 dm$$

$$I_z = \int (x^2 + y^2)dm$$

Then we have

$$I_z = I_x + I_y$$

for planar object.



((**Example**))

(a) Circular disk (mass M and radius R)

$$I_x = I_y = \frac{1}{4}MR^2$$

$$I_z = \frac{1}{2}MR^2$$

(b) Circular hoop (mass M and radius R)

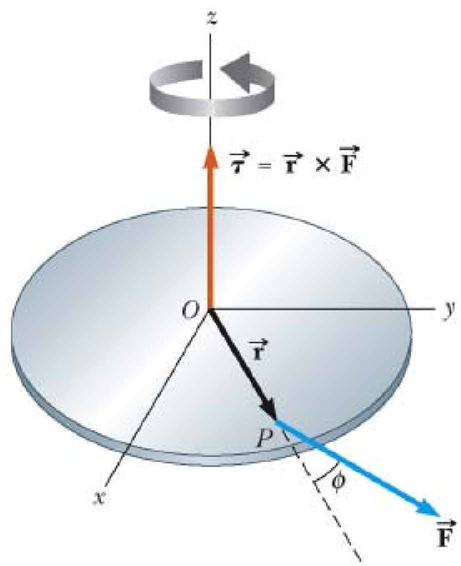
$$I_x = I_y = \frac{1}{2}MR^2$$

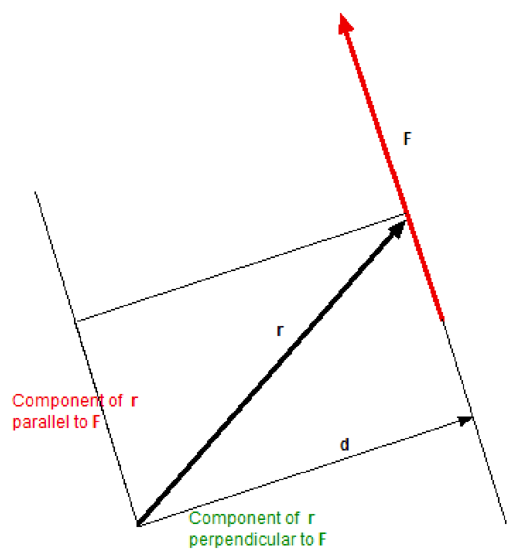
$$I_z = MR^2$$

6 Torque

The torque $\boldsymbol{\tau}$ is defined by

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$$





In this figure

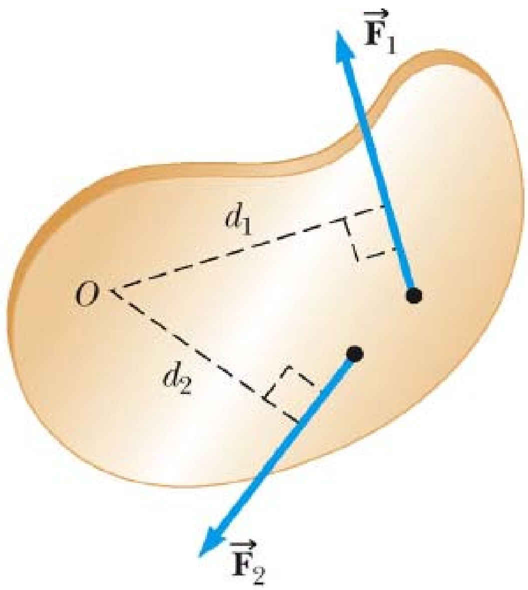
$$\tau = dF$$

The moment arm of ***F***:
d*:* the component of ***r*** perpendicular to the force ***F***. It is called

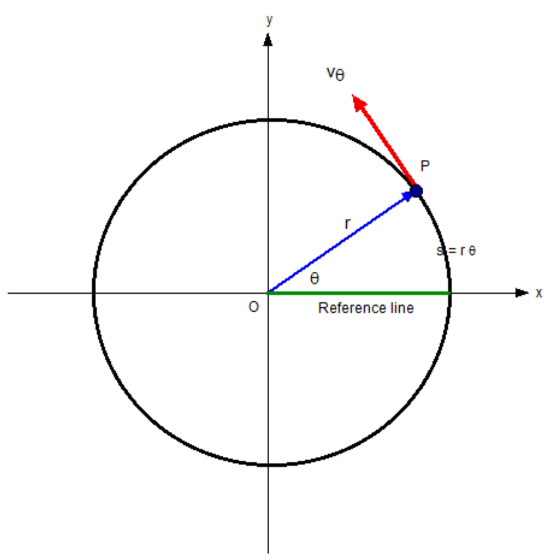
The line of action of ***F***:
the component of ***r*** parallel to the force ***F***.

The torque is a vector.
The direction of the torque vector is perpendicular to the plane formed of ***r*** and ***F***.

With the choice that counterclockwise torques are positive and clockwise torques are negative.



7 Newton's second law of rotation



When r = constant,

$$v_\theta = r\dot{\theta} = v$$

$$a_r = -r\dot{\theta}^2$$

$$a_\theta = a_t = r\ddot{\theta}$$

Torque

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = (r\hat{r}) \times (F_r\hat{r} + F_\theta\hat{\theta}) = rF_\theta\hat{z} = rma_\theta\hat{z}$$

Since

$$a_\theta = r\ddot{\theta}$$

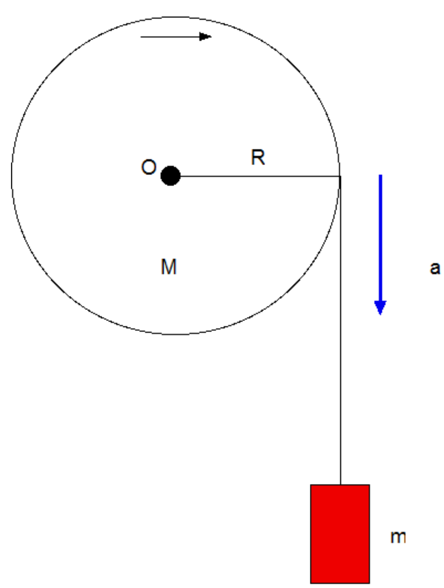
we have

$$\tau_z = mr^2\ddot{\theta} = I\ddot{\theta} = I\alpha$$

8. Example

8.1 Example-1: Angular acceleration of a solid cylinder subject

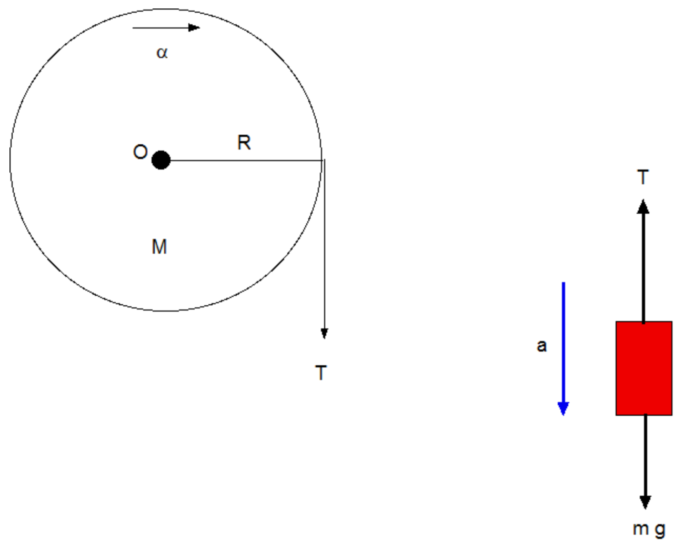
We consider a solid cylinder, free to rotate about a fixed axis coinciding with its geometric axis, subject to an applied torque. The torque is provided by a mass hanging from a string wrapped around the cylinder.



The moment of inertia for the cylinder is

$$I = \frac{1}{2}MR^2$$

The free-body diagram



$$I\alpha = RT$$

$$a = R\alpha$$

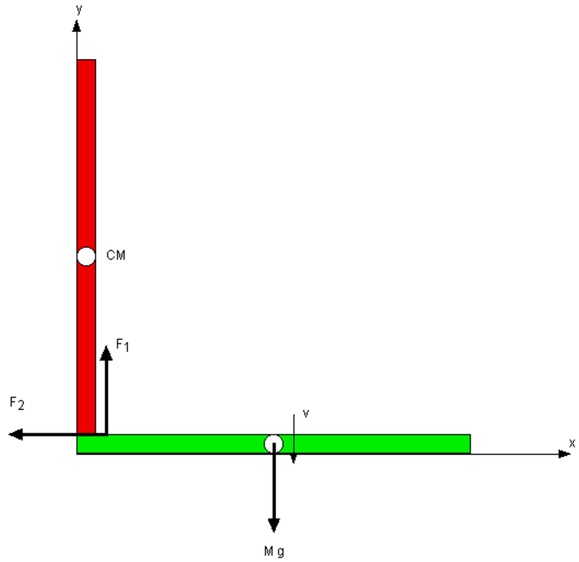
$$mg - T = ma$$

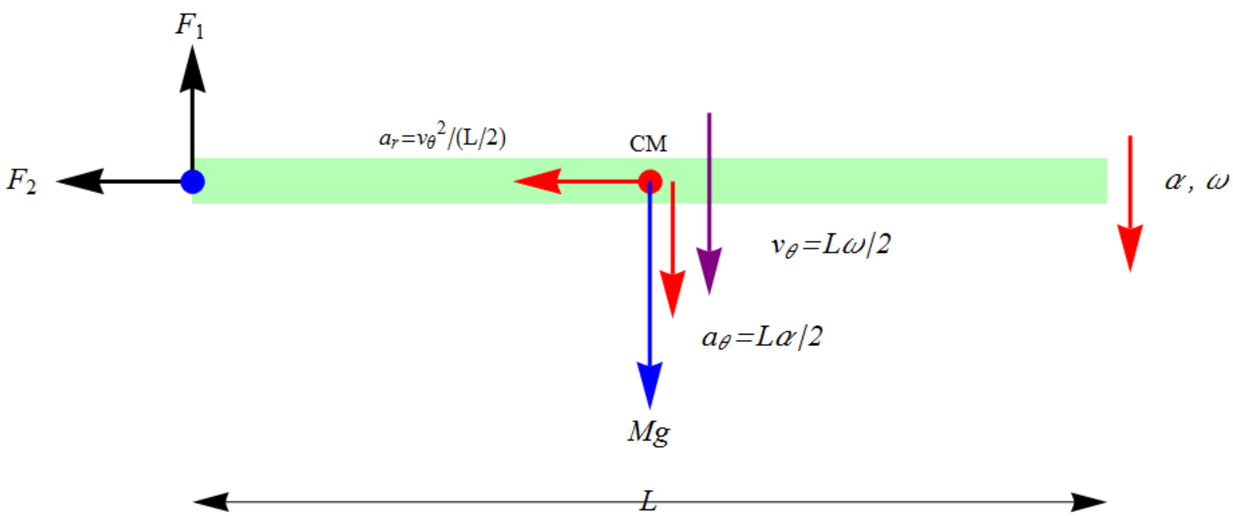
From these equations, we have

$$a = \frac{mg}{m + \frac{M}{2}}, \qquad \alpha = \frac{m}{m + \frac{M}{2}}\frac{g}{R}$$

8.2 ((Example 2))
Serway 10-61

A long uniform rod of length L and mass M is pivoted about a horizontal frictionless pin through one end. The rod is released from rest in a vertical position. At the instant the rod is horizontal, find (a) its angular velocity, (b) the magnitude of its angular acceleration of its center of mass, (c) the x and y components of the acceleration of its center of mass, and (d) the components of the reaction force at the point.





((**Solution**))
The moment of inertia of the rod around the pin,

$$I = \frac{1}{3}ML^2 = \frac{1}{12}ML^2 + M(\frac{L}{2})^2$$

from the parallel axis theorem.

The energy conservation law:

$$E = K_R + U$$

where K_R is the rotational energy and U is the potential energy

$$E_i = Mg\frac{L}{2}$$

$$E_f = \frac{1}{2}I\omega^2$$

or

$$E_f = E_i \qquad \omega^2 = \frac{3g}{L}$$

(a) The angular velocity ω: $\omega = \sqrt{\frac{3g}{L}}$

(b) The angular acceleration α $\alpha = \frac{3g}{2L}$

$$\tau = I\alpha = Mg\frac{L}{2}$$

$$\alpha = \frac{Mg\frac{L}{2}}{\frac{1}{3}ML^2} = \frac{3g}{2L}$$

(c) Acceleration of the center of mass along the (-y) direction.

$$a_t = \frac{L}{2}\alpha = \frac{3g}{4}$$

(d) Centripetal acceleration of the center of mass:

$$a_r = \omega^2\frac{L}{2} = \frac{3g}{L}\frac{L}{2} = \frac{3g}{2}$$

(e) The component of the reaction force along the (-y axis)

$$Mg - F_1 = Ma_t = \frac{3Mg}{4}$$

$$F_1 = \frac{Mg}{4}$$

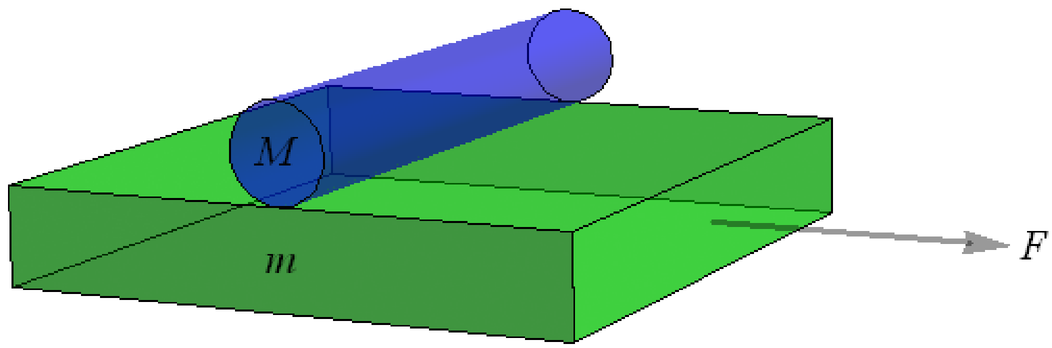
(f) The component of the reaction force toward the center

$$F_2 = Ma_r = \frac{3Mg}{2}$$

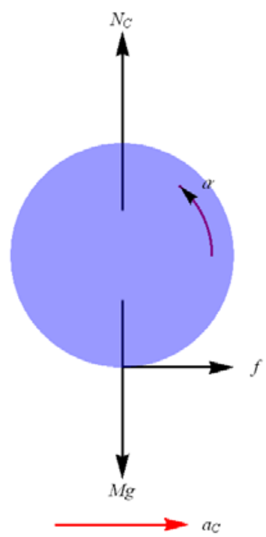
8.3 Example

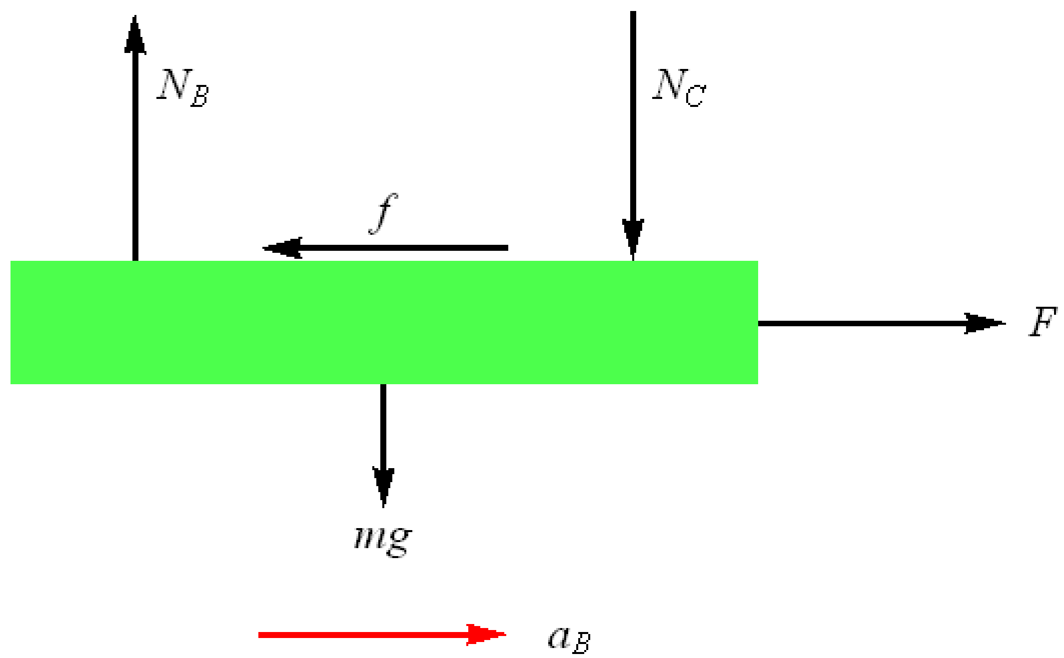
Example from Tipler and Mosca Chapter 9 problems 9-96 and 9-97

A uniform cylinder of mass M and radius R is at rest on a block of mass m, which in turn rests on a horizontal, frictionless table. If a horizontal force F is applied to the block, it accelerates and the cylinder rolls without slipping. (a) Find the acceleration of the block. (b) Find the angular acceleration of the cylinder. Is the cylinder rotating clockwise or counterclockwise? (c) What is the cylinder's linear acceleration relative to the table? Let the direction of *F* be the positive direction. (d) What is the linear acceleration relative to the block



((**Solution**))
We draw the two free body diagrams.





((**Solution**))

The moment of inertia for the cylinder is

$$I = \frac{1}{2}MR^2 \qquad \text{for cylinder}$$

We note that

$$x_{cG} = x_{cB} + x_{BG} = -R\theta + x_{BG}$$

where G denotes the ground. This means that

$$a_C = -R\alpha + a_B$$

For the cylinder, we have

$$N_C = Mg$$

$$f = Ma_C$$

$$I\alpha = fR \qquad \text{or} \qquad f = \frac{1}{2}MR\alpha$$

For the block, we have

$$N_B - N_C = mg$$

$$F - f = ma_B$$

Then we have

$$N_C = Mg\ ,\ N_B = (M + m)g\ ,$$

$$a_B = 3a_C = \frac{3F}{M + 3m},$$

$$a_C = \frac{F}{M + 3m},$$

$$\alpha = \frac{1}{R}\frac{2F}{M + 3m}$$

$$a_C - a_B = -R\alpha = -\frac{2F}{M + 3m}$$

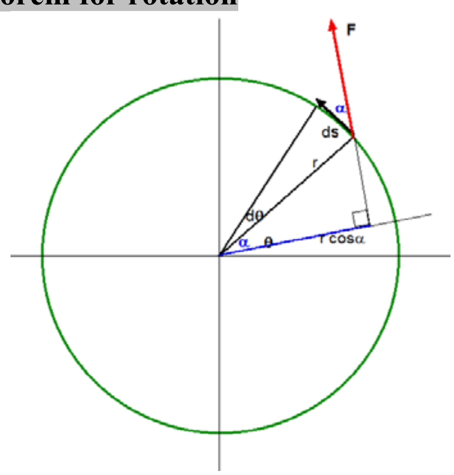
(a) $a_B = \dfrac{3F}{M + 3m}$

(b) $\alpha = \dfrac{1}{R}\dfrac{2F}{M + 3m}$. α is in the counterclockwise direction.

(c) $a_C = \dfrac{F}{M + 3m}$.

(d) $a_C - a_B = -R\alpha = -\dfrac{2F}{M + 3m}$; the acceleration of the cylinder relative to the block

9 Work - energy theorem for rotation



$$dW = \mathbf{F} \cdot d\mathbf{s} = Fds\cos\alpha = Frd\theta\cos\alpha = F(r\cos\alpha)d\theta = \tau d\theta$$

where

$$\tau = F(r\cos\alpha)$$

The work done by the rotation

$$dW = \tau d\theta = I\alpha d\theta = I\frac{d\omega}{dt}d\theta = I\frac{d\omega}{d\theta}\frac{d\theta}{dt}d\theta = I\omega\frac{d\omega}{d\theta}d\theta = I\omega d\omega$$

with

$$\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$$

Work-rotational energy theorem

$$W = \int \tau d\theta = \int I\omega d\omega = \frac{1}{2}I\omega_f^{\ 2} - \frac{1}{2}I\omega_i^{\ 2} = \Delta K_R$$

This equation has exactly the same mathematical form as the work-energy theorem for translation. If an object is both rotating and translating, we have

$$W = \Delta K_T + \Delta K_R$$

10 Power in rotational motion

The power delivered to a rotating object is

$$P = \frac{dW}{dt} = \tau\frac{d\theta}{dt} = \tau\omega$$

This expression is analogous to $P = Fv$ in the case of translation motion.

11 Analogies between translational and rotational motion

θ	x
ω	v
α	a
$K_R = \frac{1}{2}I\omega^2$	$K_T = \frac{1}{2}mv^2$
I	m
$\tau = I\alpha$	$F = ma$

τ	F
Power= $\tau\omega$	Power = Fv
Work = $\tau\, d\theta$	Work = $F\, dx$

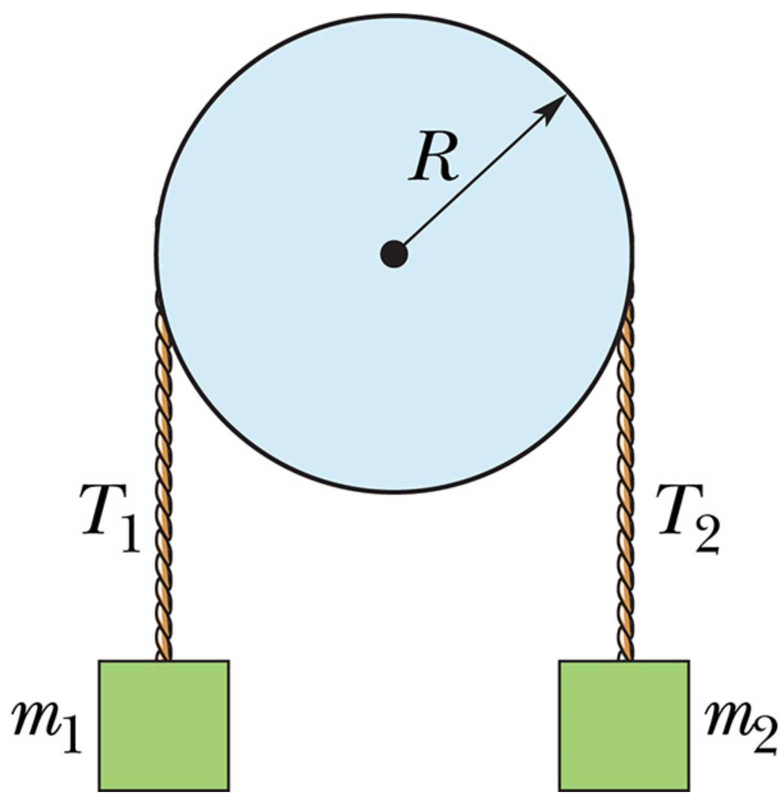
Momentum $p = mv$ Angular momentum $L = I\omega$

12. Homework (Hint) and SP problems

12.1

Problem 10-51 (SP-10) (10-th edition)

In Fig., block 1 has mass m_1 = 460 g, block has mass m_2 = 500 g, and the pulley, which is mounted on a horizontal axle with negligible friction, has radius R = 5.00 cm. When released from rest, block 2 falls 75.0 cm in 5.00 s without the cord slipping on the pulley. (a) What is the magnitude of the acceleration of the blocks? What are (b) tension T_1 and (c) tension T_2? (d) What is the magnitude of the pulley's angular acceleration? (e) What is its rotational inertia?

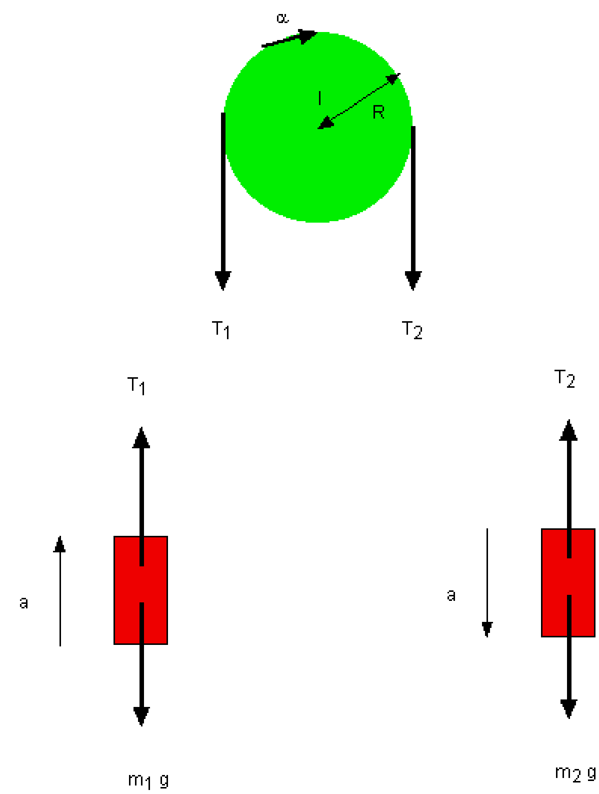


((**Solution**))

$m_1 = 0.46$ kg, $m_2 = 0.5$ kg, $R = 0.05$ m,
$h = 0.75$ m, $t = 5$ sec

Free-body diagram

12.2



$$T_1 - m_1 g = m_1 a$$

$$I\alpha = I\frac{a}{R} = R(T_2 - T_1)$$

,

$$m_2 g - T_2 = m_2 a$$

$$a = R\alpha$$

Determination of a:

$$a = \frac{2h}{t^2} = 0.06 m/s^2$$

$$T_1 = m_1(a+g) = 4.54N$$

$$T_2 = m_2(-a+g) = 4.87N$$

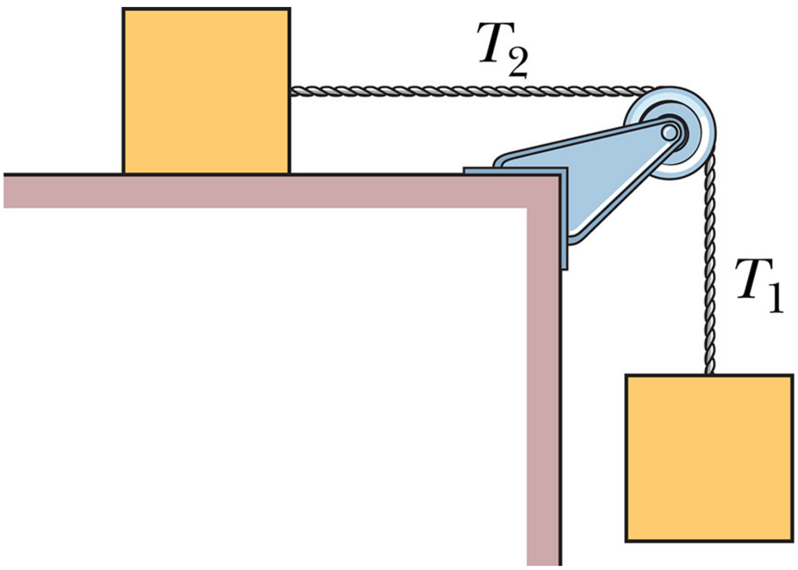
$$I = -\frac{[g(m_1 - m_2) + a(m_1 + m_2)]R^2}{a} = 0.0139 kgm^2$$

$$\alpha = \frac{a}{R} = 1.2 rad/s^2$$

12.2

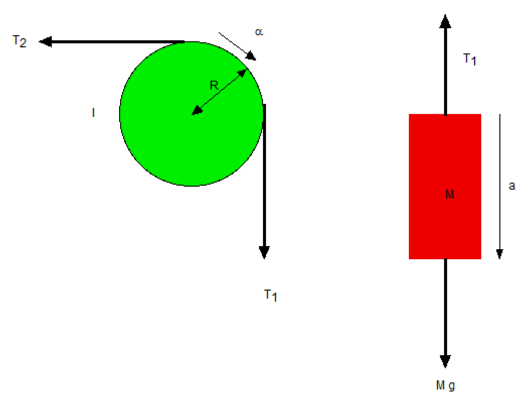
Problem 10-71 (HW-10, Hint) (10-th edition)

In Fig., two 6.20 kg blocks are connected by a massless string over a pulley of radius 2.40 cm and rotational inertia 7.40 x 10^{-4} kg m^2. The string does not slip on the pulley; it is not known whether there is friction between the table and the sliding block; the puley's axis is frictionless. When this system is released from rest, the pulley turns through 1.30 rad in 91.0 ms and the acceleration of the blocks is constant. What are (a) the magnitude of the pulley's angular acceleration, (b) the magnitude of either block's acceleration, (c) string tension T_1, and (d) string tension T_2?



((**Solution**))

M = 6.2 kg, R = 0.024 m, I = 7.40 x 10^{-4} kg m^2,
θ = 1.30 rad in t = 91 ms.



Determination of the angular acceleration α and the acceleration a

$$\theta = \frac{1}{2}\alpha t^2$$

$$a = R\alpha = \frac{2\theta R}{t^2}$$

Equations of motion:

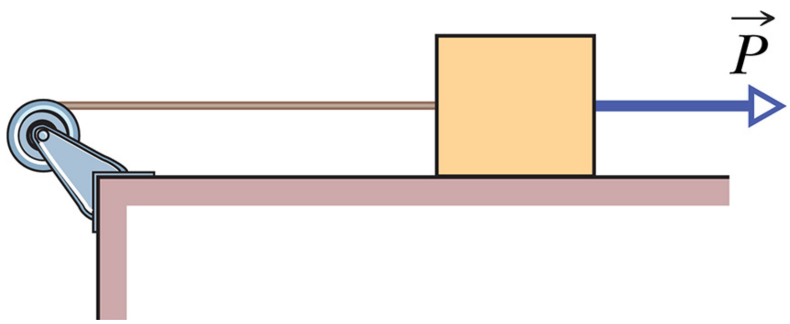
$$(T_1 - T_2)R = I\alpha = I\frac{a}{R} \qquad \text{or} \qquad T_1 - T_2 = I\frac{a}{R^2}$$

$$Mg - T_1 = Ma$$

12.3

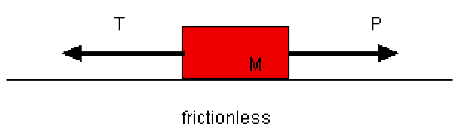
Problem 10-93 (10-th edition)

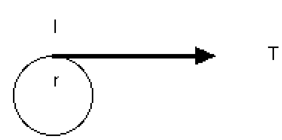
A wheel of radius 0.20 m is mounted on a frictionless horizontal axis. The rotational inertia of the wheel about the axis is 0.50 kg m^2. A massless cord wrapped around the wheel is attached to a 2.0 kg block that slides on a horizontal frictionless surface. If a horizontal force of magnitude P = 3.0 N is applied to the block as shown in Fig., what is the magnitude of the angular acceleration of the wheel? Assume the cord does not slip on the wheel.



((**Solution**))

m = 0.05 kg, I = 0.05 kg m^2, r = 0.2 m, M = 2.0 kg, P = 3N.





Newton’s second law

$$\sum F_x = P - T = Ma$$

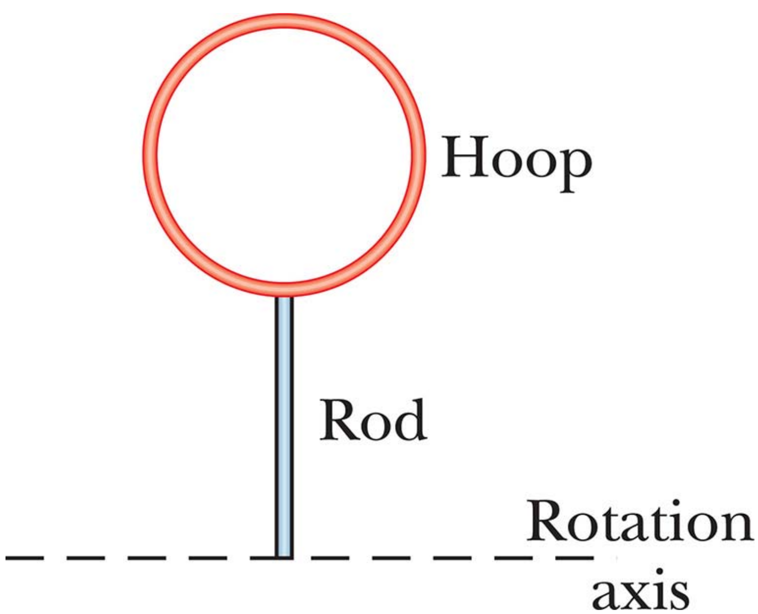
$$I\alpha = rT$$
$$a = r\alpha$$

From these equations,

$$a = \frac{Tr^2}{I} = \frac{r^2 P}{Mr^2 + I}$$
$$T = \frac{P}{1 + \dfrac{Mr^2}{I}}$$
$$\alpha = \frac{a}{r} = 4.6 rad / s^2$$

12.4
Problem 10-67* (SP-10) (10-th edition)**

Figure shows a rigid assembly of a thin hoop (of mass *m* and radius *R* = 0.150 m) and a thin radial rod (of mass *m* and length *L* = 2.00 *R*). The assembly is upright, but if we give it a slight nudge, it will rotate around a horizontal axis in the plane of the rod and hoop, through a lower end of the rod. Assuming that the energy given to the assembly in such a nudge is negligible, what would be the assembly’s angular speed about the rotation axis when it passes through the upside-down (inverted) orientation?



((**My solution**))
R = 0.150 m, $L = 2R$

$$E_f = \frac{1}{2} I\omega^2 - mg(3R + R)$$

$$E_i = mg(3R + R)$$

$$I = [m(3R)^2 + \frac{1}{2} mR^2] + [mR^2 + \frac{1}{12} m(2R)^2] = 10.83 mR^2$$

The energy conservation law

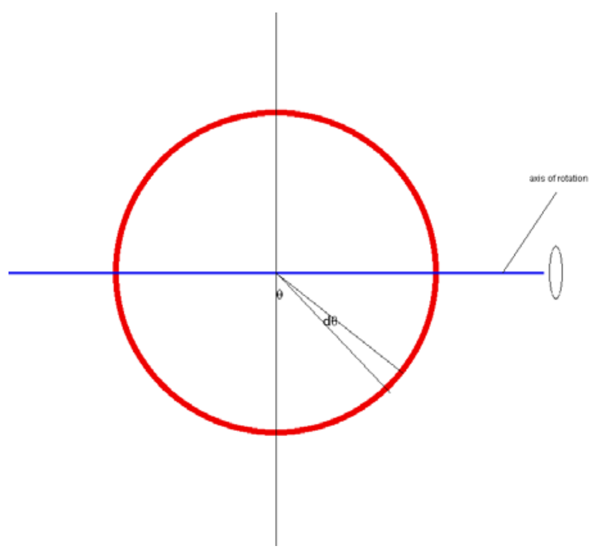
$$\frac{1}{2} I\omega^2 - mg(3R + R) = mg(3R + R)$$

or

$$\frac{1}{2} I\omega^2 = mg(8R)$$

$$\omega = \sqrt{\frac{mg(16R)}{I}} = 9.8 rad/s$$

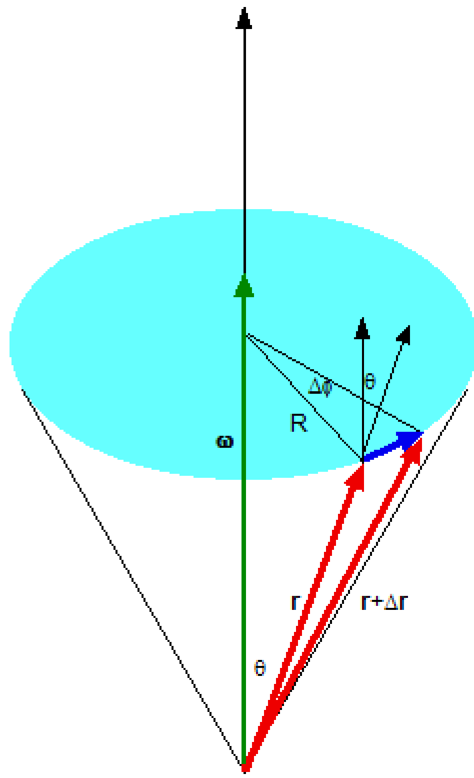
((Note)) The moment of inertia for the circular hoop



$$I = 2\int_0^{\pi} (R\cos\theta)^2 \frac{M}{2\pi R}(Rd\theta) = \frac{1}{2} MR^2$$

Appendix A

The expression of the velocity of the particle rotating around the axis



From the above geometry, we have

$$\Delta r = R\Delta\phi$$
$$R = r\sin\theta$$

Then

$$\frac{\Delta r}{\Delta t} = R\frac{\Delta\phi}{\Delta t} = \omega r\sin\theta$$

where

$$\omega = \frac{d\phi}{dt}$$

Taking into account of the direction of the velocity, we get

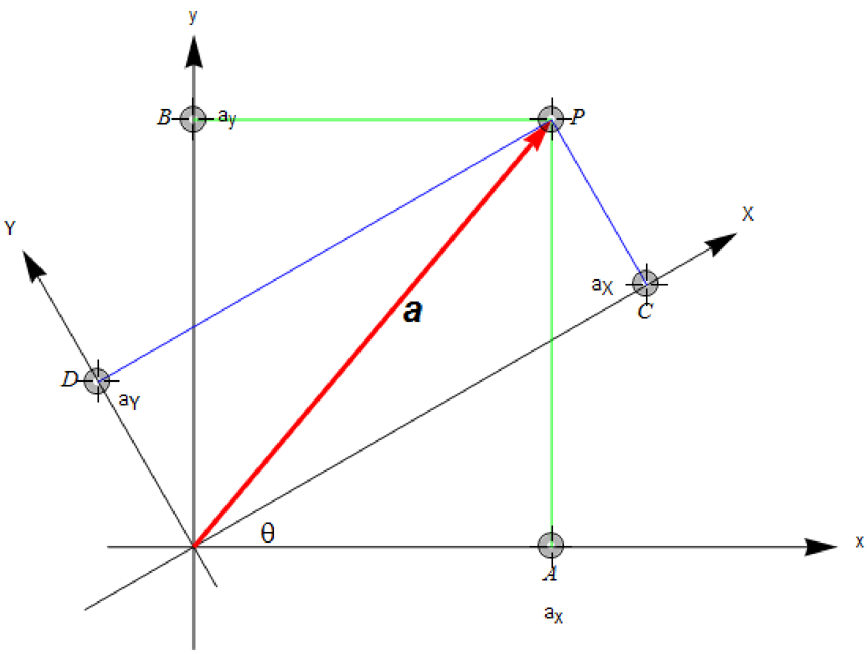
$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$

B. **((Richard Feynman, Rotation in space, Feynman Lectures on Physics))**

Is a torque in three dimension a vector?

B1. Definition of vector

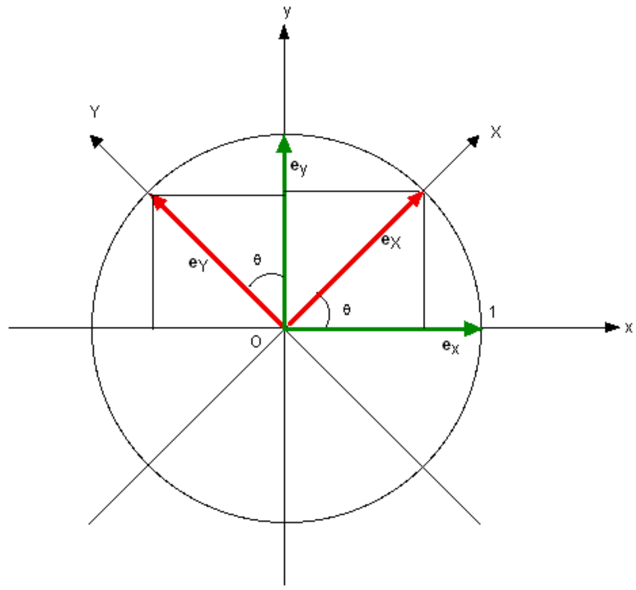
We now consider the one coordinate system is rotated by a fixed angle θ, such that the axis z- and z'- are the same.



We assume that the vector $\boldsymbol{a}$ is given by

$$\boldsymbol{a} = a_x \hat{e}_x + a_y \hat{e}_y = a_X \hat{e}_X + a_Y \hat{e}_Y$$

$$\hat{e}_X = (\cos\theta)\hat{e}_x + (\sin\theta)\hat{e}_y$$
$$\hat{e}_Y = (-\sin\theta)\hat{e}_x + (\cos\theta)\hat{e}_y$$



or

$$\hat{e}_x = (\cos\theta)\hat{e}_X + (-\sin\theta)\hat{e}_Y$$
$$\hat{e}_y = (\sin\theta)\hat{e}_X + (\cos\theta)\hat{e}_Y$$

Then we have

$$\begin{aligned} a_X\hat{e}_X + a_Y\hat{e}_Y &= a_x\hat{e}_x + a_y\hat{e}_y \\ &= a_x(\cos\theta\hat{e}_X - \sin\theta\hat{e}_Y) + a_y(\sin\theta\hat{e}_X + \cos\theta\hat{e}_Y) \end{aligned}$$

or

$$\begin{aligned} a_X\hat{e}_X + a_Y\hat{e}_Y &= \\ &= (a_x\cos\theta + a_y\sin\theta)\hat{e}_X + (-a_x\sin\theta + a_y\cos\theta)\hat{e}_Y \end{aligned}$$

or

$$\begin{aligned} a_X &= a_x\cos\theta + a_y\sin\theta \\ a_Y &= -a_x\sin\theta + a_y\cos\theta \end{aligned}$$

or

$$\begin{pmatrix} a_X \\ a_Y \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} a_x \\ a_y \end{pmatrix}$$

or

$$\begin{pmatrix} a_y \\ a_x \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a_X \\ a_Y \end{pmatrix}$$

In the three dimensional notation, we have

$$\begin{pmatrix} a_X \\ a_Y \\ a_Z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a_x \\ a_y \\ a_z \end{pmatrix}$$

In general, any vector transforms into the new system in the same way as do a_x, a_y, and a_z. For convenience, we use the notation

$$a_X = a_{x'}$$
$$a_Y = a_{y'}$$
$$a_Z = a_{z'}$$

B2. Torque $\boldsymbol{\tau}$

Torque is defined by

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix} = \tau_x \boldsymbol{i} + \tau_y \boldsymbol{j} + \tau_z \boldsymbol{k}$$

where

$$\tau_x = yF_z - zF_y$$
$$\tau_y = zF_x - xF_z$$
$$\tau_z = xF_y - yF_x$$

Now we consider how the torque transforms under the rotation of the coordinate by the angle θ around the z axis. The torque in the new coordinate is

$$\boldsymbol{\tau}' = \boldsymbol{r}' \times \boldsymbol{F}' = \begin{vmatrix} \boldsymbol{i}' & \boldsymbol{j}' & \boldsymbol{k}' \\ x' & y' & z' \\ F_{x'} & F_{y'} & F_{z'} \end{vmatrix} = \tau_{x'} \boldsymbol{i}' + \tau_{y'} \boldsymbol{j}' + \tau_{z'} \boldsymbol{k}'$$

$$\tau_{x'} = y' F_{z'} - z' F_{y'}$$
$$\tau_{y'} = z' F_{x'} - x' F_{z'}$$
$$\tau_{z'} = x' F_{y'} - y' F_{x'}$$

Using

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \begin{pmatrix} F_{x'} \\ F_{y'} \\ F_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} F_x \\ F_y \\ F_z \end{pmatrix}$$

We can show that

$$\tau_{x'} = y' F_{z'} - z' F_{y'} = \tau_x \cos\theta + \tau_y \sin\theta$$
$$\tau_{y'} = z' F_{x'} - x' F_{z'} = -\tau_x \sin\theta + \tau_y \cos\theta$$
$$\tau_{z'} = x' F_{y'} - y' F_{x'} = \tau_z$$

or

$$\begin{pmatrix} \tau_{x'} \\ \tau_{y'} \\ \tau_{z'} \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \tau_x \\ \tau_y \\ \tau_{z'} \end{pmatrix}$$

This means that the torque is a vector.

((**Mathematica**))

```
A = {{Cos[θ], Sin[θ], 0},
  {-Sin[θ], Cos[θ], 0}, {0, 0, 1}}

{{Cos[θ], Sin[θ], 0},
 {-Sin[θ], Cos[θ], 0}, {0, 0, 1}}

A // MatrixForm
```

$$\begin{pmatrix} \mathrm{Cos}[\theta] & \mathrm{Sin}[\theta] & 0 \\ -\mathrm{Sin}[\theta] & \mathrm{Cos}[\theta] & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

```
F = {Fx, Fy, Fz}

{Fx, Fy, Fz}

R = {x, y, z}

{x, y, z}

Fn = A.F

{Fx Cos[θ] + Fy Sin[θ],
 Fy Cos[θ] - Fx Sin[θ], Fz}

Rn = A.R

{x Cos[θ] + y Sin[θ], y Cos[θ] - x Sin[θ], z}

τ = Cross[R, F]

{Fz y - Fy z, -Fz x + Fx z, Fy x - Fx y}

τn = Cross[Rn, Fn] // Simplify

{(Fz y - Fy z) Cos[θ] + (-Fz x + Fx z) Sin[θ],
 (-Fz x + Fx z) Cos[θ] + (-Fz y + Fy z) Sin[θ],
 Fy x - Fx y}
```

APPENDIX C

The kinetic rotation energy of the earth

The kinetic rotation energy for the earth is given by

$$K = \frac{1}{2} I \omega^2$$

where I is the moment of inertia of the earth and w is the angular velocity,

$$I = \frac{2}{5} M_E R_E{}^2, \qquad T = \frac{2\pi}{\omega} = \frac{2\pi R_E}{v}, \qquad v = \omega R_E$$

M_E is the mass of the earth, R_E is the radius and T is the period (24 hours). The rotation energy K can be rewritten as

$$K = \frac{1}{2}\frac{2}{5} M_E R_E{}^2 \omega^2 = \frac{1}{5} M_E v^2 = \frac{1}{5} M_E \left(\frac{2\pi R_E}{T}\right)^2$$

The value of K can be estimated as

$K = 2.56537 \times 10^{29}$ J.

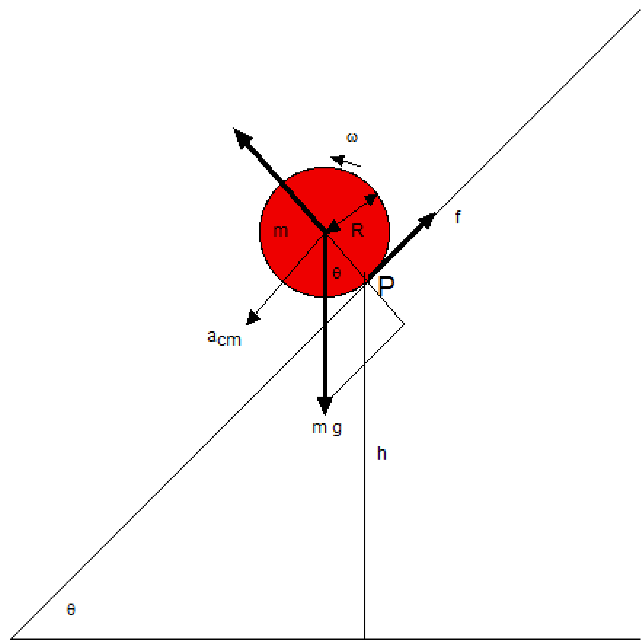
((**Mathematica**))

```
Clear["Global`*"];

rule1 = {ME → 5.9736 × 10^24, RE → 6.372 10^6 , hour → 3600};

K = 2/5 1/2 ME (2 π RE / T1)^2 /. {T1 → 24 hour} //. rule1

2.56537 × 10^29
```

APPENDIX D: How to determine the moment of inertia (experimentally)

Walter Lewin;
https://www.youtube.com/watch?v=cB8GNQuyMPc
The concept of moment of inertia is demonstrated by rolling a series of cylinders down an inclined plane.

Suppose that either hollow cylinder or solid cylinder roll down on the incline with the angle θ. There is no slipping on the surface of the incline. First we calculate the acceleration using the Newton's second law for the translation and rotation.



The Torque around the central axis is

$$\tau = fR = I_{CM}\alpha$$

where the force f is considered. The Newton's second law for the movement for the center of mass,

$$N - Mg\cos\theta = 0$$

and

$$Ma = Mg\sin\theta - f$$

where M is the mass of the cylinder and θ is the angle of incline. The condition for no slipping is given by

$$a = a_{cm} = R\alpha$$

Then we get

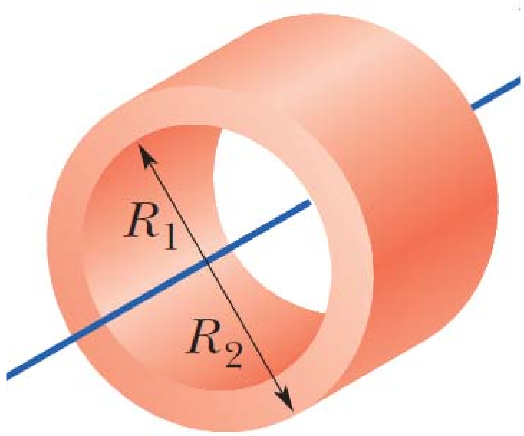
$$f = \frac{I_{CM}\alpha}{R} = \frac{I_{CM}}{R^2}a$$

The acceleration a is obtained by

$$a = \frac{g \sin\theta}{1 + \frac{I_{CM}}{MR^2}}$$

We consider an annular cylinder (hollow cylinder) about the central axis. The inner radius is R_1 and the outer radius is R_2. The moment of inertia around the central axis is given by

$$I_{CM} = \frac{M}{2}(R_1^{\ 2} + R_2^{\ 2})$$



Noting that $R = R_2$, we have the expression for the acceleration as

$$a = \frac{g \sin\theta}{1 + \frac{I_{CM}}{MR_2^{\ 2}}} = \frac{2}{3 + \frac{R_1^2}{R_2^{\ 2}}} g \sin\theta = \frac{2}{3 + x^2} g \sin\theta$$

Thus the acceleration a depends only on the ratio $x = R_1 / R_2$ (≤ 1). It does not depend on the mass (M), the length of cylinder (L), and the kind of materials for the hollow cylinder (such as copper, aluminum, wood, and so on). In other words, a is a universal function of the ratio x.

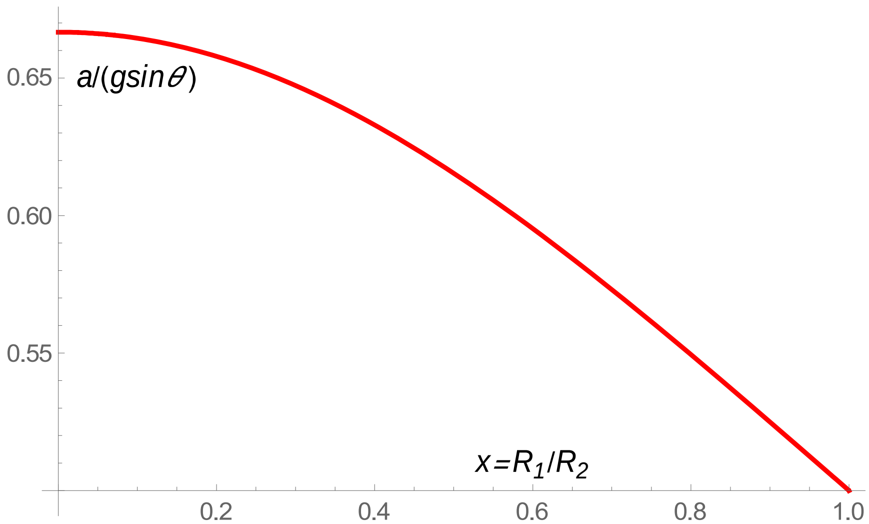
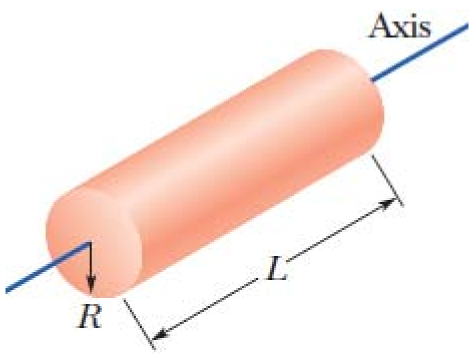


Fig. $a/(g\sin\theta)$ vs the rario $x = R_1 / R_2$ for the hollow cylinder.

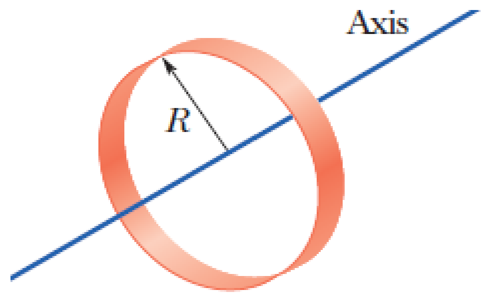
(a) x =0, (R_1=0): solid cylinder (or disk) about the central axis

$$\boxed{a = \frac{2}{3} g \sin\theta}$$



(b) x = 1 (R_1 = R_2): hoop about the central axis

$$\boxed{a = \frac{1}{2} g \sin\theta}$$



Then the acceleration a for the solid cylinder is larger than that for the hoop.