Chapter 11
Rolling, Torque, and angular momentum

1 Angular momentum

The instantaneous angular momentum ***L*** of a particle relative to the origin O is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

$$\boldsymbol{p} = \boldsymbol{p}_{//} + \boldsymbol{p}_{\perp}$$

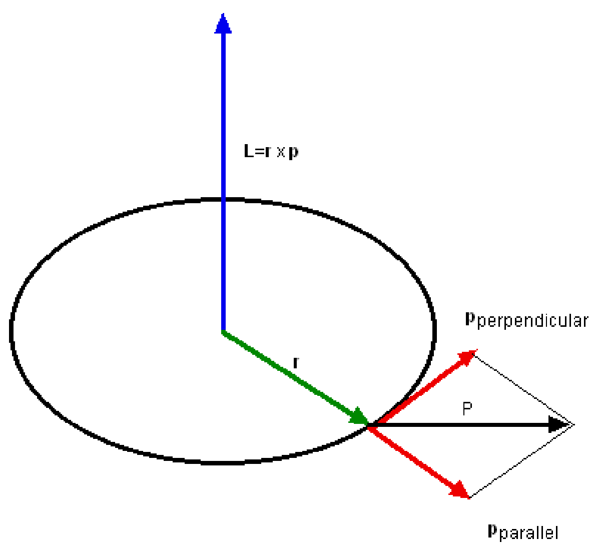
where

$\boldsymbol{p}_{//}$ is the component of ***p*** parallel to the direction of ***r***.

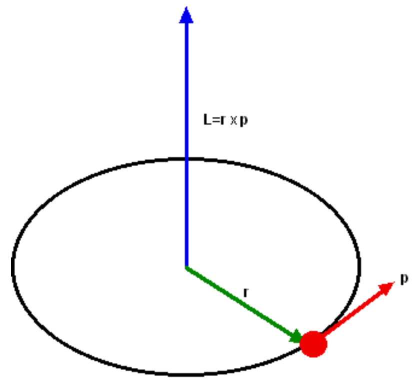
$\boldsymbol{p}_{\perp}$ is the component of ***p*** perpendicular to the direction of ***r***.

Then we have

$$\boldsymbol{L} = \boldsymbol{r} \times (\boldsymbol{p}_{//} + \boldsymbol{p}_{\perp}) = \boldsymbol{r} \times \boldsymbol{p}_{\perp}$$



2 The angular momentum of the circular motion



$$L_z = mvr$$

Note that a particle in uniform circular motion has aconstant angular momentum about an axis through the center of its path. This angular momentum can be rewritten as

$$L_z = mvr = mr^2 \frac{v}{r} = I\omega$$

where I is the moment of inertia and is given by

$$I = mr^2$$

and ω is the angular velocity and is given by

$$\omega = \frac{v}{r}.$$

This form of L_z indicates the close relations such as

$$L \leftrightarrow p, \qquad I \leftrightarrow m, \qquad \omega \leftrightarrow v$$

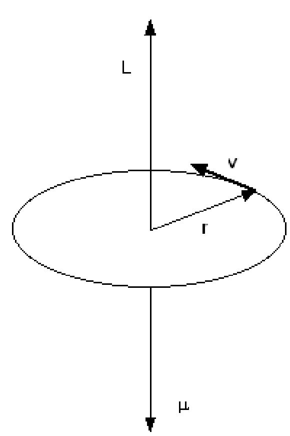


Fig.1 Orbital (circular) motion of electron with mass *m* and a charge –*e*. The direction of orbital angular momentum *L* is perpendicular to the plane of the motion (*x*-*y* plane).

((**Quantum Mechanics**))

The orbital angular momentum of an electron (charge –*e* and mass *m*) *L* is defined by

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m\boldsymbol{v}) \text{, or } L_z = mvr \text{ .} \qquad (1)$$

According to the de Broglie relation

$$p = \frac{h}{\lambda} \qquad (2)$$

where *p* (= *mv*) is the momentum, *h* is the Planck's constant, and λ is the wavelength.

Bohr condition:

$$\frac{2\pi r}{\lambda} = n \qquad \text{or} \qquad r = \frac{n}{2\pi}\lambda$$

The angular momentum L_z is described by

$$L_z = mvr = pr = \frac{h}{\lambda}\frac{n}{2\pi}\lambda = n\frac{h}{2\pi} = n\hbar \qquad (3)$$

The angular momentum is quantized in the units of the Dirac constant ($\hbar = \frac{h}{2\pi}$)

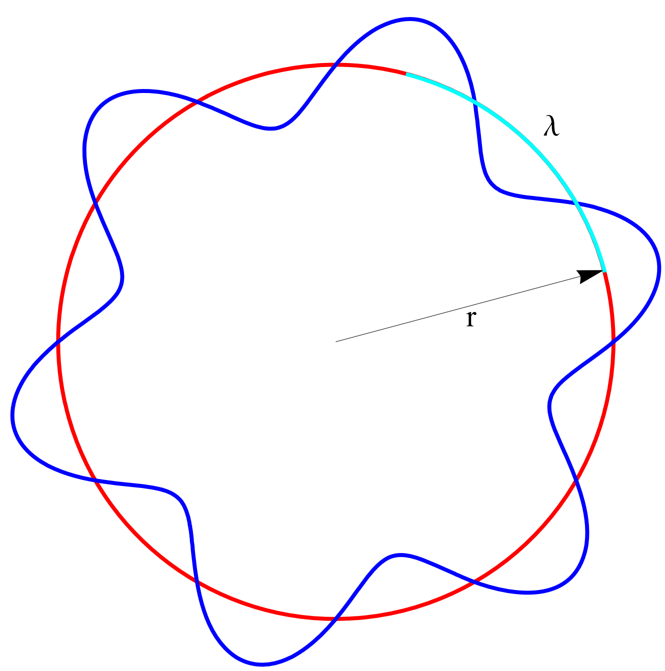


Fig. Acceptable wave on the ring (circular orbit). The circumference should be equal to the integer n (=1, 2, 3,…) times the de Broglie wavelength λ. The picture of fitting the de Broglie waves onto a circle makes clear the reason why the orbital angular momentum is quantized. Only integral numbers of wavelengths can be fitted. Otherwise, there would be destructive interference between waves on successive cycles of the ring.

3. Angular momentum of the system of particles with respect to any reference point

((**Theorem**))

The angular momentum about any point (in our case the origin) is equal to the angular momentum about the center of mass plus the angular momentum due to translation of the center of mass with respect to the point.

$$\boldsymbol{L} = \boldsymbol{R}_{CM} \times M\boldsymbol{V}_{CM} + \boldsymbol{L}_{CM}$$

((**Proof**))

From the definition, the angular momentum $\boldsymbol{L}$ around the origin O is given by

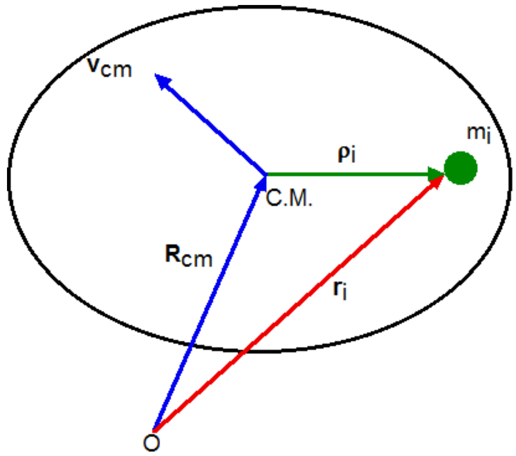
$$\boldsymbol{L} = \sum_{i=1}^{N} \boldsymbol{r}_i \times \boldsymbol{p}_i = \sum_{i=1}^{N} m_i (\boldsymbol{r}_i \times \dot{\boldsymbol{r}}_i)$$

for the N-particles system. The center of mass is given by

$$\boldsymbol{R}_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i \boldsymbol{r}_i$$

$$\boldsymbol{V}_{CM} = \dot{\boldsymbol{R}}_{CM} = \frac{1}{M} \sum_{i=1}^{N} m_i \dot{\boldsymbol{r}}_i$$

where M is the total mass; $M = \sum_{i=1}^{N} m_i$



We assume that

$$\boldsymbol{r}_i = \boldsymbol{R}_{CM} + \boldsymbol{\rho}_i$$
$$\dot{\boldsymbol{r}}_i = \dot{\boldsymbol{R}}CM + \dot{\boldsymbol{\rho}}_i$$

where

$$\sum_{i=1}^{N} m_i \boldsymbol{\rho}_i = 0$$
$$\sum_{i=1}^{N} m_i \dot{\boldsymbol{\rho}}_i = 0$$

from the definition of the center of mass. Then the angular momentum of system with respect to the origin (reference point O) is given by

$$\begin{aligned}\boldsymbol{L} &= \sum_{i=1}^{N} m_i [(\boldsymbol{R}_{CM} + \boldsymbol{\rho}_i) \times (\boldsymbol{V}_{CM} + \dot{\boldsymbol{\rho}}_i)] \\ &= \sum_{i=1}^{N} m_i (\boldsymbol{R}_{CM} \times \boldsymbol{V}_{CM} + \boldsymbol{R}_{CM} \times \dot{\boldsymbol{\rho}}_i + \boldsymbol{\rho}_i \times \boldsymbol{V}_{CM} + \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i) \\ &= \sum_{i=1}^{N} m_i (\boldsymbol{R}_{CM} \times \boldsymbol{V}_{CM} + \boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i) \\ &= M\boldsymbol{R}_{CM} \times \boldsymbol{V}_{CM} + \sum_{i=1}^{N} m_i (\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i)\end{aligned}$$

or

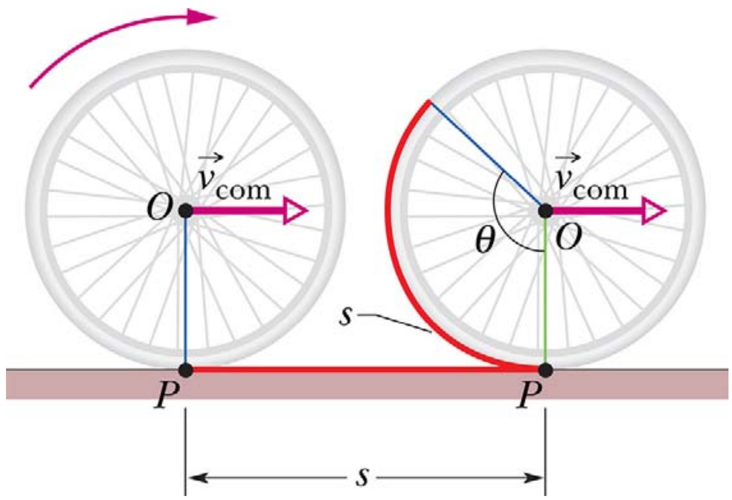
$$\boldsymbol{L} = \boldsymbol{R}_{CM} \times M\boldsymbol{V}_{CM} + \boldsymbol{L}_{CM}$$

where $\boldsymbol{L}_{cm}$ is the angular momentum of the system with respect to the center of mass and is given by

$$\boldsymbol{L}_{CM} = \sum_{i=1}^{N} m_i (\boldsymbol{\rho}_i \times \dot{\boldsymbol{\rho}}_i)$$

((**Example**)) See also the discussion described at Sec.8.

We now consider the angular momentum of the rotating wheel around the point P



The angular momentum around the point P is given by

$$L = L_{CM} + MRV_{CM} = I_{CM}\omega + MR(\omega R) = I_{CM}\omega + MR^2\omega = I_P\omega$$

where

$$I_P = I_{CM} + MR^2$$ (which corresponds to the parallel-axis theorem)

((**Note**))

Since the wheel undergoes one rotation, it takes a period T. The center of mass of the wheel moves the distance $2\pi R$, where R is the radius of the wheel.

$$V_{cm} = \frac{2\pi R}{T} = \frac{2\pi R}{\frac{2\pi}{\omega}} = \omega R$$

4 Newton's second law in angular momentum

4.1 One particle-system

The torque is related to the angular momentum,

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{d\boldsymbol{r}}{dt} \times \boldsymbol{p} + \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \boldsymbol{r} \times \frac{d\boldsymbol{p}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{\tau}$$

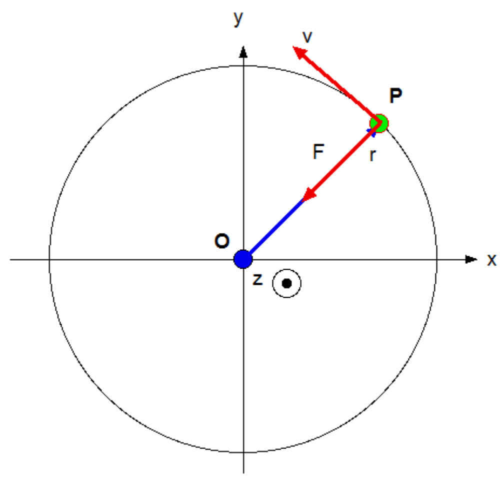
or

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt}$$

The SI units of the angular momentum $\boldsymbol{L}$ is N m s = kg m^2/s.

4.2 Conservation of angular momentum of a particle in the presence of the central force

We consider the case of a single particle uniformly rotating around the origin O. We assume that there is an attractive force between the particle and the center of orbit (central force).



The angular momentum $\boldsymbol{L}$ ($L = mvr$):

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

The vector $\boldsymbol{L}$ is directed along the z axis. The torque around the origin;

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = 0, \qquad \boldsymbol{L} = \text{conserved}$$

since $\boldsymbol{r}$ is antiparallel to $\boldsymbol{F}$ (central field). This means that L is conserved *only* at the origin O.

4.3 Many-particles system

Now we consider the total angular momentum $\boldsymbol{L}$ for the many-particle system,

$$\frac{d}{dt}(\boldsymbol{r}_i \times \boldsymbol{p}_i) = \boldsymbol{v}_i \times \boldsymbol{p}_i + \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i = \boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i$$

$$\boldsymbol{L} = \sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i)$$

$$\begin{aligned}\dot{\boldsymbol{L}} &= \frac{d}{dt}\sum_i (\boldsymbol{r}_i \times \boldsymbol{p}_i) \\ &= \sum_i (\boldsymbol{r}_i \times \dot{\boldsymbol{p}}_i) \\ &= \sum_i \{\boldsymbol{r}_i \times (\boldsymbol{F}_i^{(e)} + \sum_j \boldsymbol{F}_{ji})\}\end{aligned}$$

or

$$\dot{\boldsymbol{L}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(e)} + \sum_{\substack{j \\ i \neq j}} \boldsymbol{r}_i \times \boldsymbol{F}_{ji}$$

The last term on the right in this equation can be considered a sum of the pairs of the form

$$\boldsymbol{r}_i \times \boldsymbol{F}_{ji} + \boldsymbol{r}_j \times \boldsymbol{F}_{ij} = (\boldsymbol{r}_i - \boldsymbol{r}_j) \times \boldsymbol{F}_{ji} = \boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji}$$

using the equality of action and reaction, where $\boldsymbol{r}_{ij} = \boldsymbol{r}_i - \boldsymbol{r}_j$.

((Note))

$$\boldsymbol{r}_{ij} \times \boldsymbol{F}_{ji} = 0$$

for the strong law of action and reaction (the internal forces between two particles in addition to being equal and opposite site, also lie along the line joining the particles).

Then we have

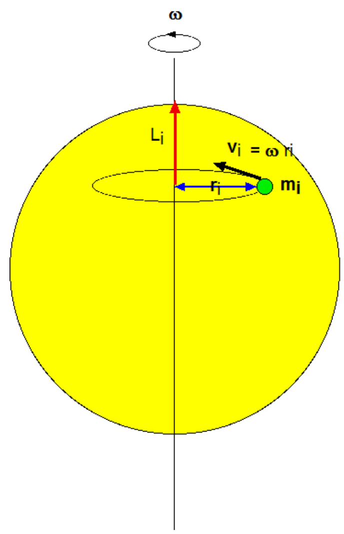
$$\dot{\boldsymbol{L}} = \sum_i \boldsymbol{r}_i \times \boldsymbol{F}_i^{(e)} = \tau^{(e)}$$

The time derivative of the total angular momentum is equal to the moment of the external force about the given point O.

5 Angular momentum of a rotating rigid object

5.1 Definition

To find the angular momentum of the entire object, add the angular momentum of all the individual particles

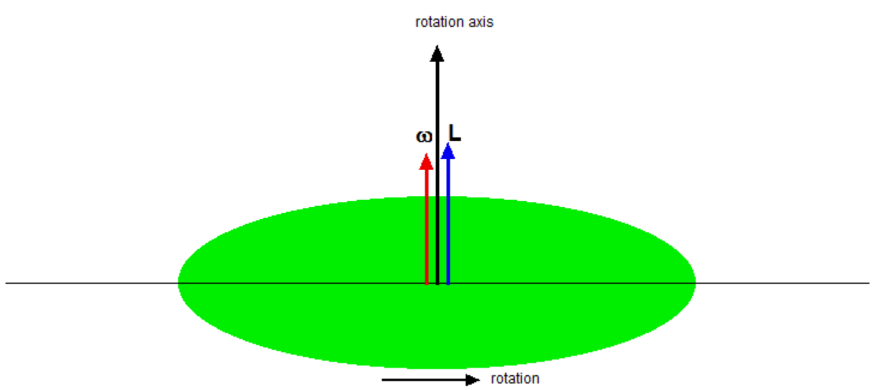


$$L = \sum_i L_i = \sum_i m_i v_i r_i = \sum_i m_i \omega r_i^2 = I\omega$$

or

$$L = I\omega$$

where r_i is the distance between the position of the particle mi and rotational axis. The angular momentum ***L*** is directed along the *z* axis (rotation axis), as is the vector ***ω***.



5.2 Angular momentum conservation

When the net external torque acting on the system is zero, the angular momentum *L* of a system remains constant.

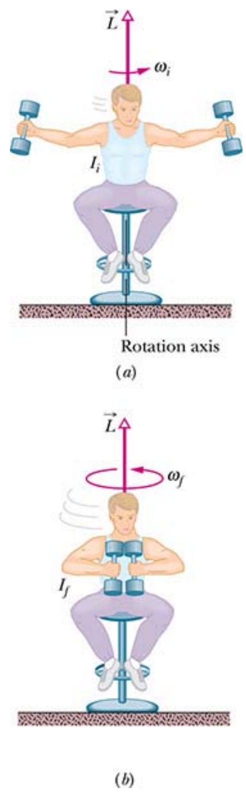
$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = 0$$

or

$$\boldsymbol{L}_f = \boldsymbol{L}_i$$

or

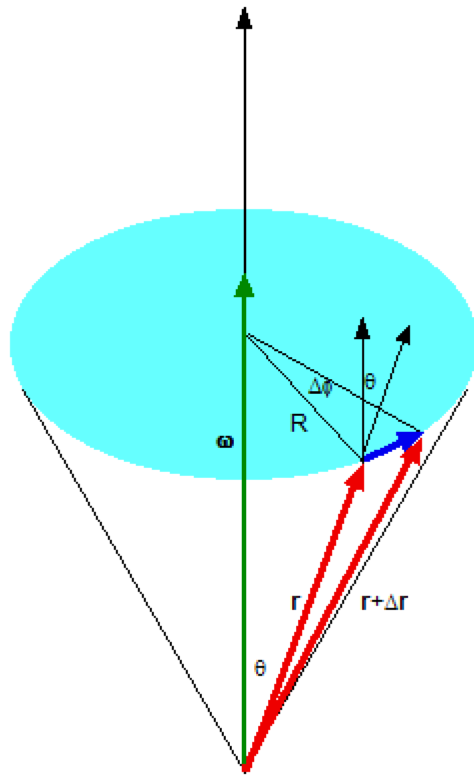
$$I_i \omega_i = I_f \omega_f$$



6. Angular momentum and rotational kinetic energy (general case)

From the Appendix of LN-10, the instantaneous velocity $\boldsymbol{v}$ for a point at position $\boldsymbol{r}$ in the body is given by

$$\boldsymbol{v} = \boldsymbol{\omega} \times \boldsymbol{r}$$



The total angular momentum of the system (with N particles) around the origin is given by

$$\boldsymbol{L} = \sum_{i=1}^{N} [\boldsymbol{r}_i \times m_i (\boldsymbol{\omega} \times \boldsymbol{r}_i)]$$

The rotational kinetic energy is

$$K_R = \sum_{i=1}^{N} \frac{1}{2} m_i (\boldsymbol{\omega} \times \boldsymbol{r}_i) \cdot (\boldsymbol{\omega} \times \boldsymbol{r}_i) = \frac{1}{2} \sum_{i=1}^{N} m_i |\boldsymbol{\omega} \times \boldsymbol{r}_i|^2$$

Here we assume that $\boldsymbol{\omega}$ is given by

$$\boldsymbol{\omega} = (\omega_x, \omega_y, \omega_z)$$

$$\boldsymbol{\omega} \times \boldsymbol{r}_i = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \omega_x & \omega_y & \omega_z \\ x_i & y_i & z_i \end{vmatrix} = (\omega_y z_i - \omega_z y_i, \omega_z x_i - \omega_x z_i, \omega_x y_i - \omega_y x_i)$$

$$\boldsymbol{r}_i \times (\boldsymbol{\omega} \times \boldsymbol{r}_i) = |\boldsymbol{r}_i|^2 \omega - (\boldsymbol{r}_i \cdot \boldsymbol{\omega}) \boldsymbol{r}_i$$

Then we have

$$\boldsymbol{L} = \sum_{i=1}^{N} m_i [|\boldsymbol{r}_i|^2 \boldsymbol{\omega} - (\boldsymbol{r}_i \cdot (\boldsymbol{\omega})\boldsymbol{r}]$$

where

$$\begin{aligned} L_x &= I_{xx}\omega_x + I_{xy}\omega_y + I_{xz}\omega_z \\ L_y &= I_{yx}\omega_x + I_{yy}\omega_y + I_{yz}\omega_z \\ L_z &= I_{zx}\omega_x + I_{zy}\omega_y + I_{zz}\omega_z \end{aligned}$$

where

$$\begin{array}{lll} I_{xx} = \sum_{i=1}^{m} m_i (y_i^2 + z_i^2) & I_{yy} = \sum_{i=1}^{m} m_i (x_i^2 + z_i^2) & I_{zz} = \sum_{i=1}^{m} m_i (x_i^2 + y_i^2) \\ I_{xy} = \sum_{i=1}^{m} m_i (-x_i y_i) & I_{yz} = \sum_{i=1}^{m} m_i (-y_i z_i) & I_{zx} = \sum_{i=1}^{m} m_i (-z_i x_i) \\ I_{xz} = \sum_{i=1}^{m} m_i (-x_i z_i) & I_{yx} = \sum_{i=1}^{m} m_i (-y_i x_i) & I_{zy} = \sum_{i=1}^{m} m_i (-z_i y_i) \end{array}$$

When $\omega_x = \omega_y = 0$, the rotational energy is obtained as

$$K_R = \frac{1}{2}\omega_z^2 \sum_{i=1}^{N} m_i (x_i^2 + y_i^2) = \frac{1}{2} I_{zz}\omega_z^2$$

7. Example

7.1 11-49 Serway

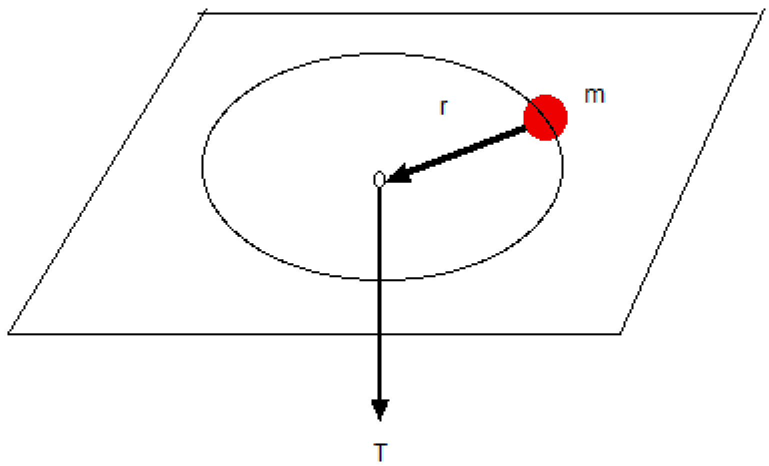
A puck of mass *m* is attached to a cord passing through a small hole in a frictionless, horizontal surface. The puck is initially orbiting with speed v_0 in a circle of radius r_0. The cord is then slowly pulled from below, decreasing the radius of the circle to *r*.

(a) What is the velocity *v* of the puck when the radius is *r*?

(b) Find the tension *T* in the cord as a function of *r*.

(c) How much work *W* is done in moving *m* from r_0 to *r*? (Note: the tension depends on *r*).

(d) Obtain numerical values for *v*, *T*, and *W* when *r* = 0.100 m. *m* = 50.0 g, r_0 = 0.300 m, and v_0 = 1.50 m/s?



((**Solution**))

Angular momentum conservation
No external torque (since the tension vector ***T*** is parallel to ***r***)

$$L_0 = mv_0 r_0 = mvr = L$$

Note:

$$L = I\omega = mr^2\omega = mvr$$

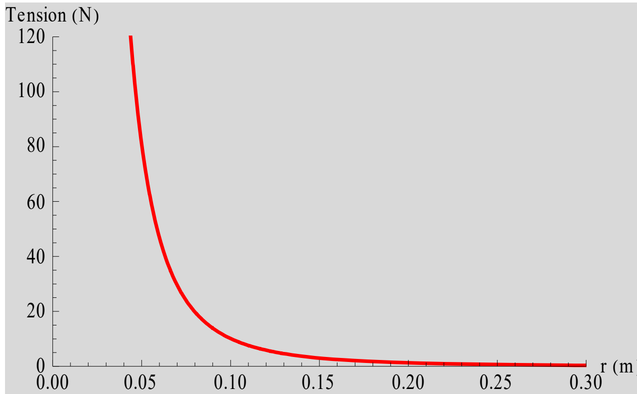
(a) Velocity v

$$v = \frac{mr_0}{mr} v_0 = \frac{r_0}{r} v_0$$

(b) Force T

$$T = m\frac{v^2}{r} = mv_0{}^2 r_0{}^2 \frac{1}{r^3}$$

The tension T is dependent on r.



(c) The work done W is obtained from the work-energy theorem.

$$W = \Delta K = \frac{1}{2}m(v^2 - v_0^2) = \frac{1}{2}m(\frac{r_0^2}{r^2}v_0^2 - v_0^2) = \frac{1}{2}mr_0^2 v_0^2(\frac{1}{r^2} - \frac{1}{r_0^2})$$

Or one can calculate the work done from the integral.

$$dW = \boldsymbol{T} \cdot d\boldsymbol{r} = (\boldsymbol{T} \cdot \hat{r})dr = -mv_0^2 r_0^2 \frac{1}{r^3}dr$$

$$W = \int_{r_0}^{r} (-mv_0^2 r_0^2 \frac{1}{r^3})dr = mv_0^2 r_0^2 \int_{r}^{r_0} \frac{1}{r^3}dr = mv_0^2 r_0^2 [\frac{r^{-3+1}}{-3+1}]_r^{r_0} = \frac{1}{2}mv_0^2 r_0^2 (\frac{1}{r^2} - \frac{1}{r_0^2})$$

(d)

$v = 4.5$ m/s, $T = 10.125$ N, $W = 0.45$ J

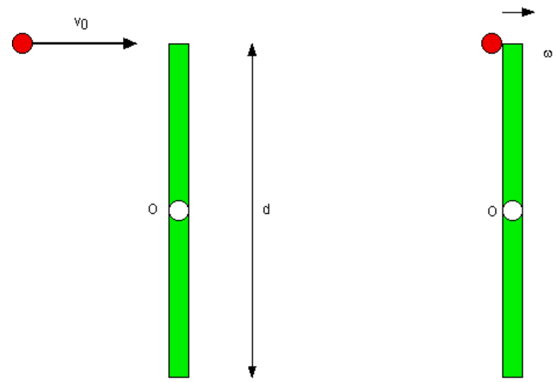
7.2 Example-2

11.50 Serway

A projectile of mass m moves to the right with a speed v_0. The projectile strikes and sticks to the end of a stationary rod of mass M, length d, pivoted about a frictionless axle through its center.

(a) Find the angular velocity of the system right after the collision.

(b) Determine the fractional loss in mechanical energy due to the collision.



The moment of inertia after the projectile strikes and sticks to the end of the rod,

$$I = m(\frac{d}{2})^2 + \frac{1}{12}Md^2 = (\frac{3m+M}{12})d^2$$

The angular momentum conservation law at the collision

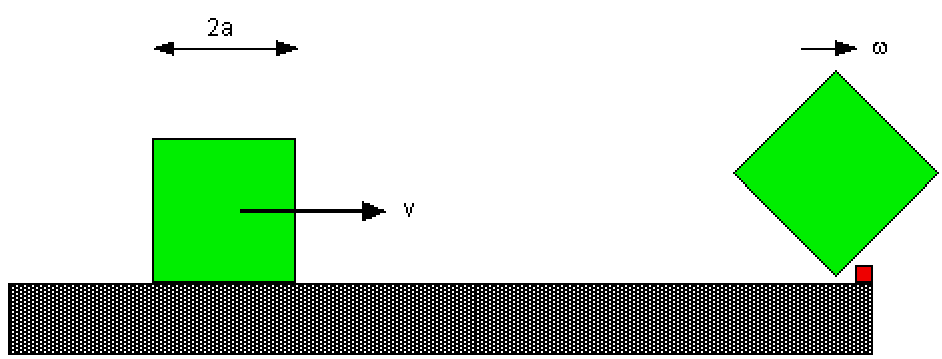
$$mv_0\frac{d}{2} = I\omega = (\frac{3m+M}{12})d^2\omega$$

$$\omega = \frac{m}{3m+M}\frac{6v_0}{d}$$

Ratio ξ of the mechanical energy before and after the collision

$$\xi = \frac{\frac{1}{2}I\omega^2}{\frac{1}{2}mv_0^{\ 2}} = \frac{3m}{3m+M}$$

7.3 Example-3
Serway 11-55
A solid cube of side 2*a* and mass *M* is sliding on a frictionless surface with uniform velocity *v* (=*v*0). It hits a small obstacle at the end of the table, which causes the cube to tilt as in Figure. Find the minimum value of *v* such that the cube falls off the table. Note that the moment of inertia of the cube about an axis along one of its edges is $8Ma^2/3$.



$$I_{CM} = \frac{2}{3}Ma^2$$ for cubic with edge (2*a* x 2*a* x 2*a*).

$$I = \frac{2}{3}Ma^2 + M(\sqrt{2}a)^2 = \frac{8}{3}Ma^2 \qquad \text{(parallel-axis theorem)}$$

Angular-momentum conservation law

$$L_i = Mv_0 a = L_f = I\omega_0 \qquad \text{or} \qquad \omega_0 = \frac{Mv_0 a}{I} = \frac{Mv_0 a}{\frac{8}{3}Ma^2} = \frac{3}{8}\frac{v_0}{a}$$

Energy conservation law

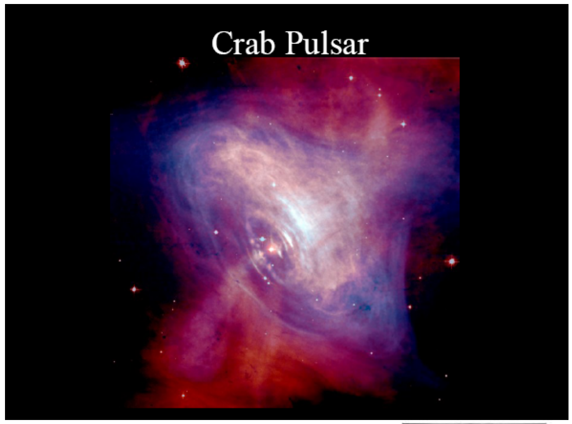
$$\Delta K + \Delta U = 0$$
$$\Delta K = \frac{1}{2}I(\omega^2 - \omega_0^{\ 2})$$
$$\Delta U = Mg(\sqrt{2}-1)a$$

Then we have

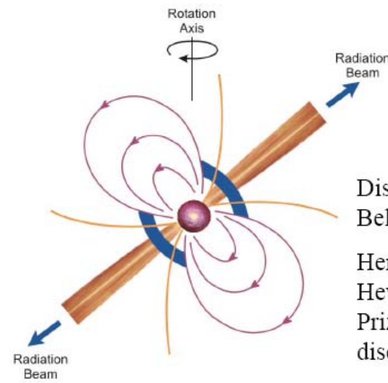
$$\omega^2 = \omega_0^{\ 2} - \frac{2Mg(\sqrt{2}-1)a}{I} \geq 0$$
$$\omega_0^{\ 2} = \frac{9v_0^{\ 2}}{64a^2} \geq \frac{2Mg(\sqrt{2}-1)a}{\frac{8}{3}Ma^2} = \frac{3g(\sqrt{2}-1)}{4a}$$
$$v \geq \sqrt{\frac{16\cdot(\sqrt{2}-1)}{3}ga} = 1.49\sqrt{ga}$$

8. Neutron star (pulsar)

For Crab pulsar: ν = 30/s (T = 33 ms), M = 1.4 solar masses, R = 12 km.

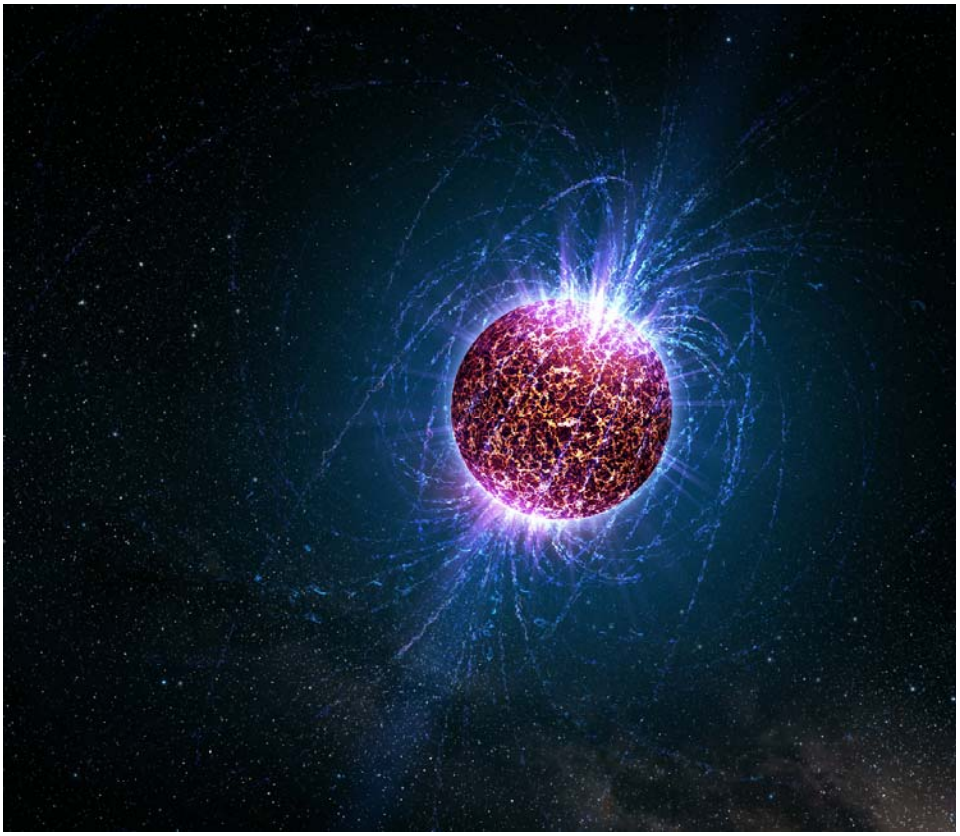


Pulsars



Discovered by Jocelyn Bell in 1967.

Her advisor, Anthony Hewish, won the Nobel Prize in Physics for the discovery in 1974.



For a rotating object to remain bound, the gravitational force at the surface must exceed the centripetal acceleration:

$$m\frac{GM}{r^2} > mr\omega^2 \Rightarrow \frac{GM}{r^3} > \omega^2 = \frac{4\pi^2}{T^2} \Rightarrow \frac{G\rho}{r^3}\frac{4\pi}{3}r^3 > \frac{4\pi^2}{T^2} \Rightarrow \rho > \frac{3\pi}{T^2 G}$$

For the Crab pulsar, $T = 33$ ms so the density must be greater than 1.3×10^{11} g/cm^3 = $1.3\text{x}10^{14}$ kg/m^3. This exceeds the maximum possible density for a white dwarf.

Spinning up of neutron star

Angular momentum of sphere

$$L = I\omega = \frac{2}{5}MR^2\omega$$

where M is mass, R is radius, ω is the angular velocity.

If the Sun ($T = 25$ days, radius 7×10^8 m, mass $1.988\text{x}10^{30}$ kg) were to collapse to a neutron star with a radius of 16 km, how fast would it be spinning?

$$R_i^{\ 2}\omega_i = R_f^{\ 2}\omega_f \,, \qquad \text{or} \qquad \frac{\omega_f}{\omega_i} = \frac{R_i^{\ 2}}{R_f^{\ 2}} = \left(\frac{7\times 10^8}{16\times 10^3}\right)^2 = \frac{49\times 10^{10}}{256} = 2\times 10^9$$

In other words, the star is rotating 2 billion times faster after the collapse than it was before.

$$\frac{T_f}{T_i} = \frac{1}{2\times 10^9}\,, \quad T_f = \frac{25\times(24\times 3600)}{2\times 10^9} = 0.001\text{sec}$$

((**Crab pulsar**))

R (radius) = 12 km = 1.2 x 10^3 m
$T = 2\pi/\omega$ = 33.5028583 m sec.

Although the pulsar were first noted for their regular periods, careful timing measurements by radio telescope soon revealed that the pulsar is gradually slowing down. Its period increases by 36.4 ns each day. Even such the slightest slowdown corresponds to a tremendous loss of energy.

The moment of inertia I:

$$I = \frac{2}{5}MR^2 = 1.377\times 10^{38} \text{ kg m}^2$$

The angular frequency ω:

$$\omega = \frac{2\pi}{T} = 187.54 \text{ rad/s}$$

The frequency f:

f = 29.85 Hz

The rotational kinetic energy:

$$K_{rot} = \frac{1}{2}I\omega^2 = \frac{1}{2}\frac{2}{5}MR^2\omega^2 = 2.42 \text{ x } 10^{42} \text{ J}$$

The loss of energy per day:

$$P = -\frac{4\pi^2}{T^3}I\frac{\Delta T}{\Delta t} = 6.09 \text{ x } 10^{31} \text{ W},$$

where ΔT = 36.4 ns per Δt = 1 day = 24 x 3600 s.

((Mathematica))

Crab pulsar : kinetic rotational energy

```
rule1 = {M → 2.39 × 10^30, R → 12 × 10^3,
   T → 33.5028583 × 10^-3, ΔT → 36.4 × 10^-9,
   Δt → 24 3600}
```

$\{M \to 2.39 \times 10^{30},\ R \to 12000,\ T \to 0.0335029,\ \Delta T \to 3.64 \times 10^{-8},\ \Delta t \to 86400\}$

Moment of inertia (kg m^2)

$$I1 = \frac{2}{5} M R^2 \ /.\ rule1$$

1.37664×10^{38}

Angular frequency (rad/s)

$$\omega = \frac{2\pi}{T};$$

ω /. rule1

187.542

Frequency (Hz)

$$f = \frac{\omega}{2\pi} \ /.\ rule1$$

29.8482

Rotational kinetic energy (J)

$$Krot = \frac{1}{2} I1\ \omega^2 \ /.\ rule1$$

2.42095×10^{42}

Energy loss per day

$$P1 = \frac{-4\pi^2 I1}{T^3} \frac{\Delta T}{\Delta t} \ /.\ rule1$$

-6.08867×10^{31}

9. Pure rolling motion

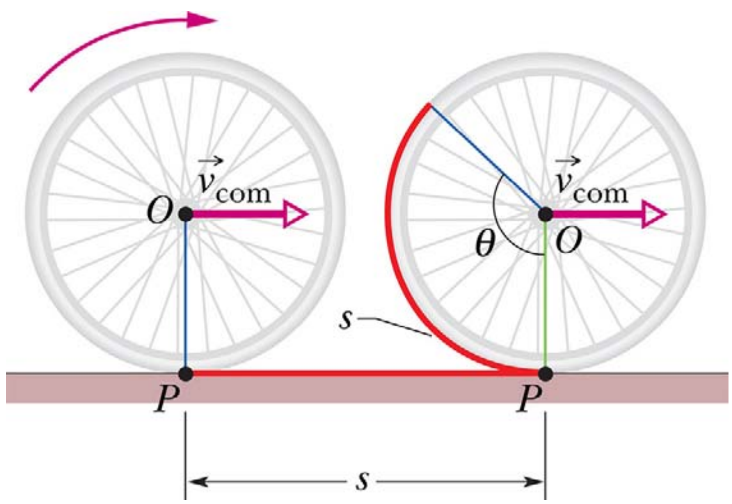
The surfaces must exert friction forces on each other.

Otherwise, the object would slide rather than roll. In such a case, there is a simple relationship between its rotational and translational motions.

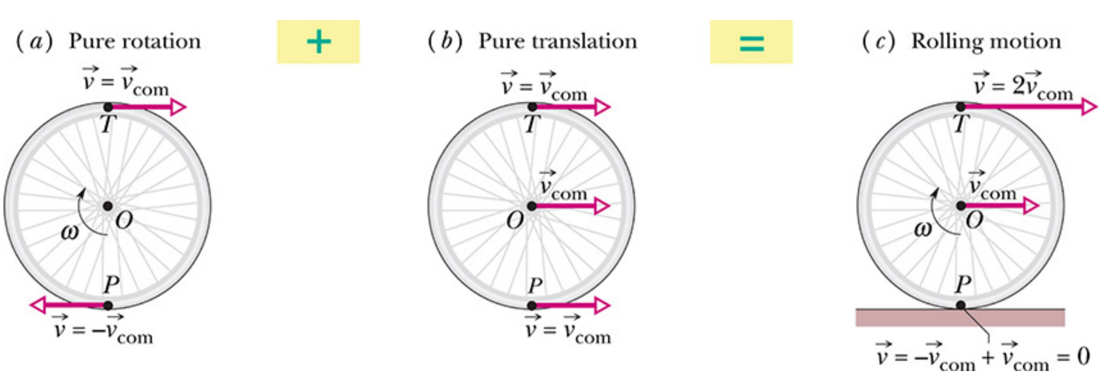
$$s = R\theta$$

$$v_{cm} = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega$$

$$a_{cm} = \frac{d^2 s}{dt^2} = R\frac{d\omega}{dt} = R\alpha$$

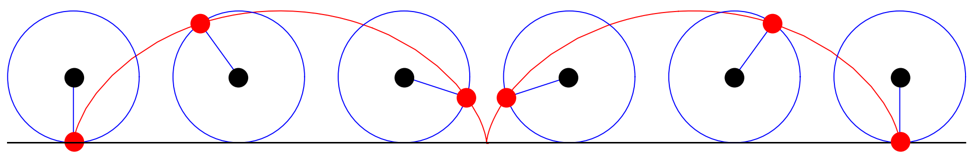


The rolling motion of a wheel is a combination of purely translational and purely rotational motions. For the purely rotational motion (as if the rotation axis through the center were stationary), every point on the wheel rotates about the center with angular velocity ω. For the purely translational motion (as if the wheel did not rotate at all), every point on the wheel moves to the right with speed v_{cm}.



Cycloid

The cycloid is the locus of a point on the rim of a circle of radius rolling along a straight line. It was studied and named by Galileo in 1599. Galileo attempted to find the area by weighing pieces of metal cut into the shape of the cycloid. Torricelli, Fermat, and Descartes all found the area. The cycloid was also studied by Roberval in 1634, Wren in 1658, Huygens in 1673, and Johann Bernoulli in 1696. Roberval and Wren found the arc length (MacTutor Archive). Gear teeth were also made out of cycloids, as first proposed by Desargues in the 1630s (Cundy and Rollett 1989).



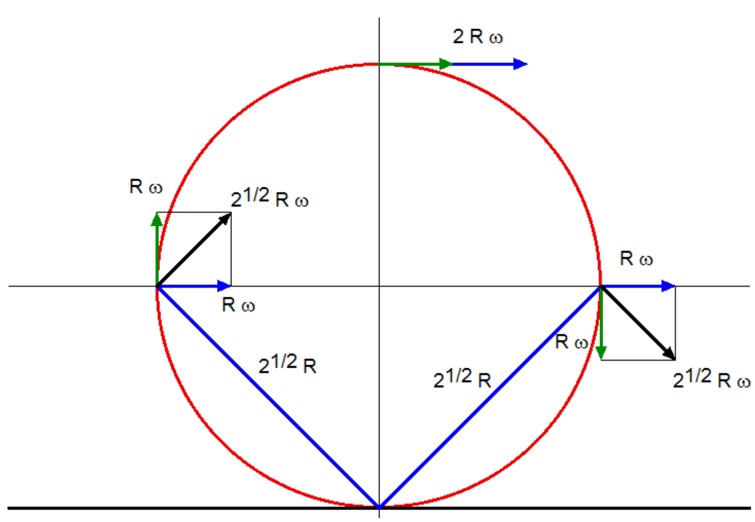
$$T = \frac{2\pi}{\omega}$$

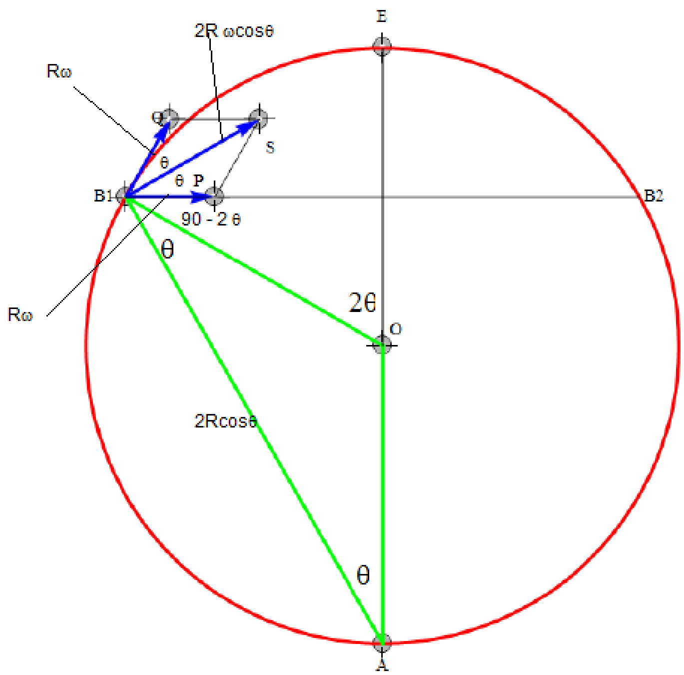
$$2\pi R = v_{CM} T = 2\pi \frac{v_{CM}}{\omega}$$

$$v_{CM} = \omega R$$

10 Rolling as pure rotation

The combination of pure rotation and pure translation yields the actual rolling motion of the wheel. Note that in this combination of motions, the portion of the wheel at the bottom (at point P) is stationary and the portion at the top (at point T) is moving at speed $2v_{cm}$, faster than any other portion of the wheel.





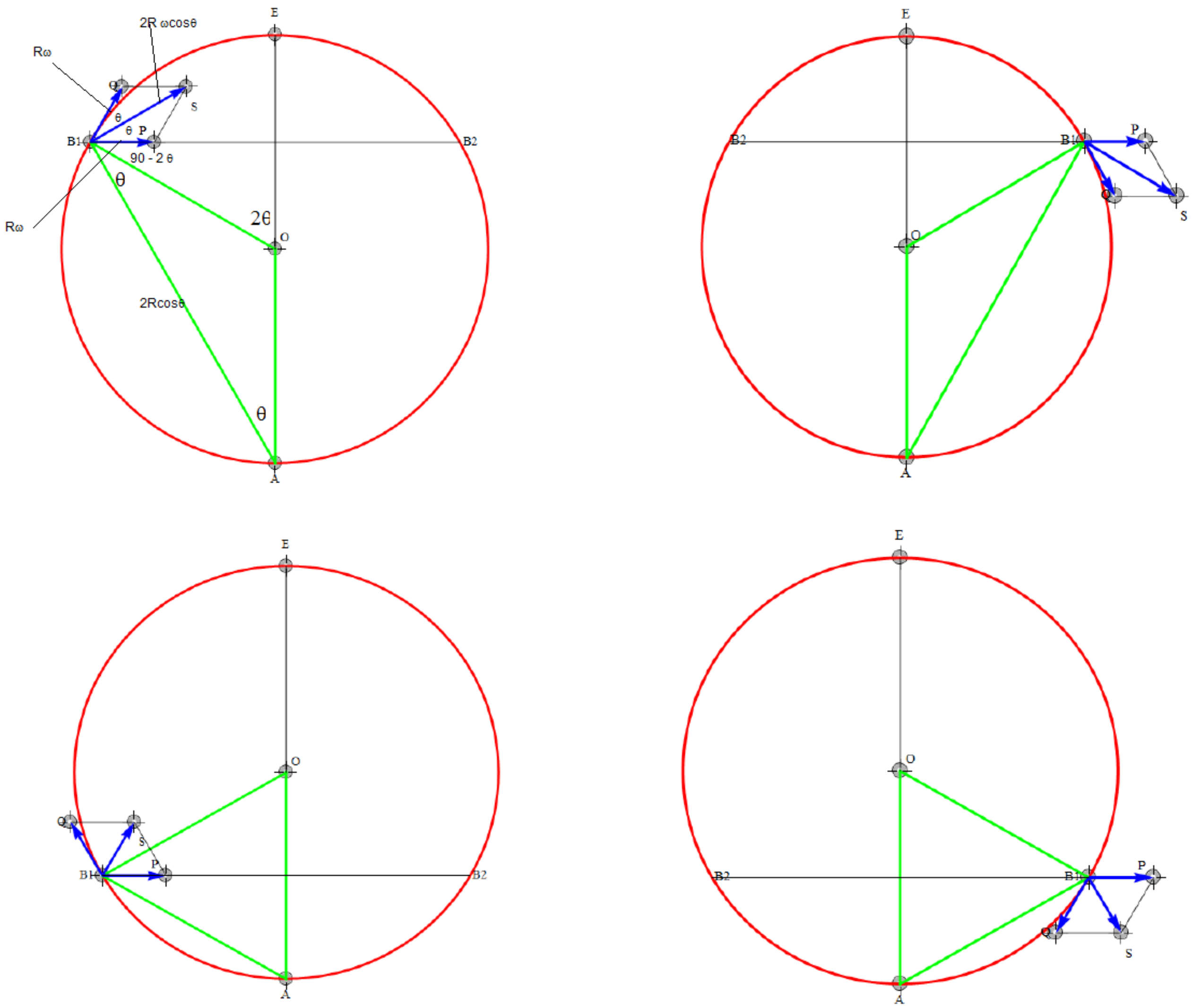
Using the above figure we now discuss the motion of the wheel geometrically. The line B_1B_2 (a horizontal line) is perpendicular to the line AE. The line B_1Q is the tangential line at the point B_1; $\angle OB_1Q = \dfrac{\pi}{2}$. For convenience we assume that $\angle B_1AO = \theta$. The distance $\overline{AB_1}$ is $2R\cos\theta$. Then we have

$$\angle B_1OE = 2\theta$$

$$\angle OB_1B_2 = \frac{\pi}{2} - 2\theta$$

$$\angle B_2B_1Q = 2\theta$$

The vector $\overrightarrow{B_1Q}$ (blue line) is the velocity of the rotation motion. The vector $\overrightarrow{B_1P}$ is the velocity of the translational motion of the center of mass. $B_1Q = B_1P = R\omega$. The vector $\overrightarrow{B_1S}$ $(= \overrightarrow{B_1Q} + \overrightarrow{B_1P})$ is the resultant velocity; $\angle QB_1S = \angle PB_1S = \theta$ The magnitude of the resultant velocity is $2R\omega\cos\theta\ (= \overline{AB_1}\ \omega)$ The line B_1S is perpendicular to the line AB_1, since $\angle AB_1S = \theta + (\pi/2 - 2\theta) + \theta = \pi/2$.



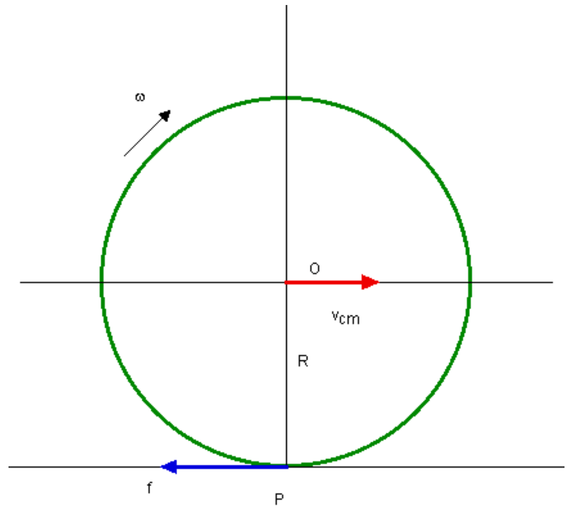
These figures suggest another way to look at the rolling motion of a wheel—namely, as pure rotation about an axis that always extends through the point where the wheel contacts the street as the wheel moves. We consider the rolling motion to be pure rotation about an axis passing through point P and perpendicular to the plane of the figure. The vectors in the Figure then represent the instantaneous velocities of points on the rolling wheel.

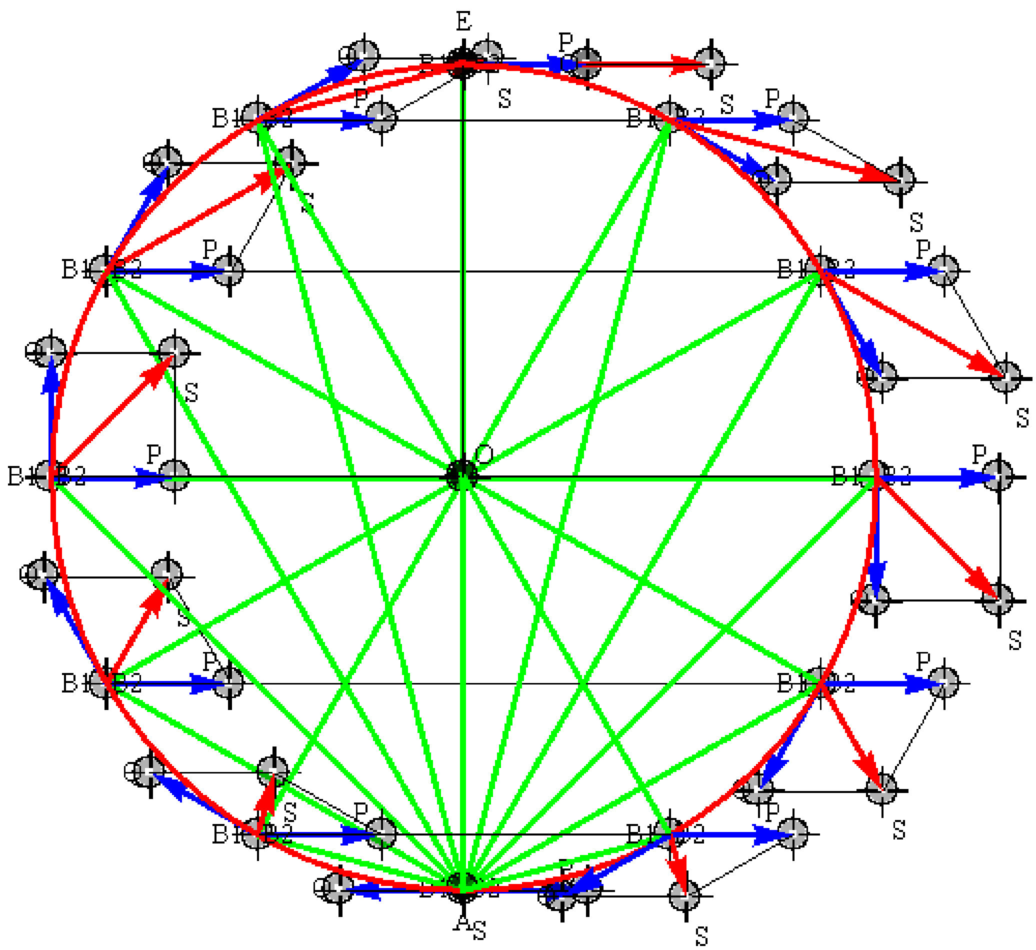
If we review the rolling as pure rotation about an axis through P, we have

$$K = \frac{1}{2} I_P \omega^2 = \frac{1}{2}(I_{cm} + MR^2)\omega^2 = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} MR^2\omega^2 = \frac{1}{2} I_{cm}\omega^2 + \frac{1}{2} M v_{cm}{}^2$$

The kinetic energy consists of the rotational energy and the translational energy.

((Force of rolling))





In summary, we realize that all parts of the wheel instantaneously rotate around the point P at the angular velocity *ω*. The magnitude of the velocity is equal to the distance (between the point P and each point of the wheel) times *ω*. Then the total kinetic energy is the rotational energy around the point P with the moment of inertia I_p. Using the parallel axis theorem, the moment of inertia I_p is expressed by

$$I_p = I_{CM} + MR^2$$

The kinetic energy is given by

$$K = \frac{1}{2} I_p \omega^2 = \frac{1}{2}(I_{CM} + MR^2)\omega^2 = \frac{1}{2} I_{CM}\omega^2 + \frac{1}{2} M v_{CM}{}^2$$

where

$$v_{CM} = R\omega$$

This means that the total kinetic energy is the sum of the rotational energy around the center of mass and the kinetic energy arising from the translation of the center of mass.

As is described previously, the angular momentum around the point P is given by

$$L_P = L_{CM} + MRV_{CM} = I_{CM}\omega + MR(\omega R) = I_{CM}\omega + MR^2\omega = I_P\omega$$

where

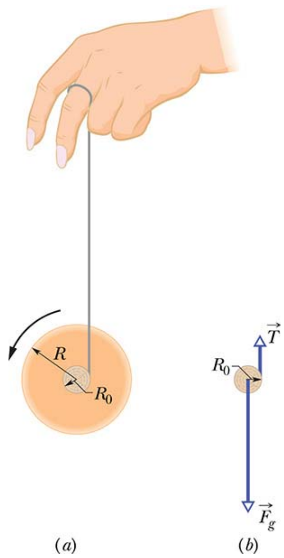
$$I_P = I_{CM} + MR^2 \qquad \text{(which corresponds to the parallel-axis theorem)}$$

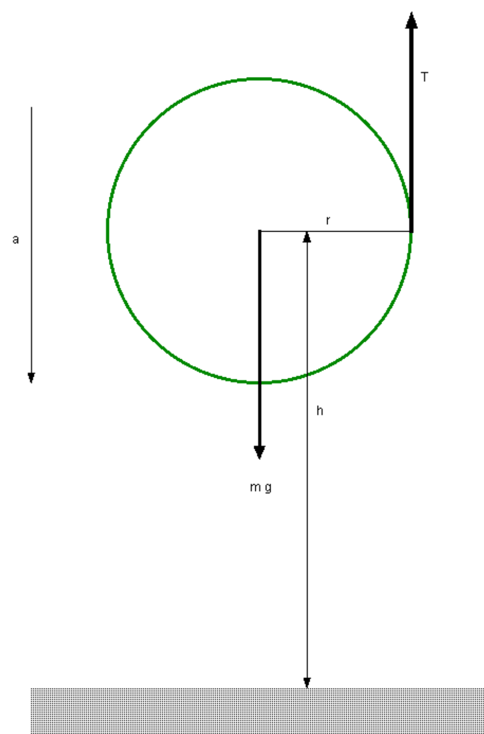
11. Typical Example

11.1 Yo-yo

Problem 11-17 (HW-11) (10-th edition)

A yo-yo has a rotational inertia of 950 g cm^2 and a mass of 120 g. Its axle radius is 3.2 mm, and its string is 120 cm long. The yo-yo rolls from rest down to the end of the string. (a) What is the magnitude of its linear acceleration? (b) How long does it take to reach the end of the string? As it reaches the end of the string, what are its (c) linear speed, (d) translational kinetic energy, (e) rotational kinetic energy, and (f) angular speed?





$$Tr = I\alpha$$
$$mg - T = ma$$
$$a = r\alpha$$

or

$$T = \frac{I}{r^2} a$$

$$mg - T = ma$$

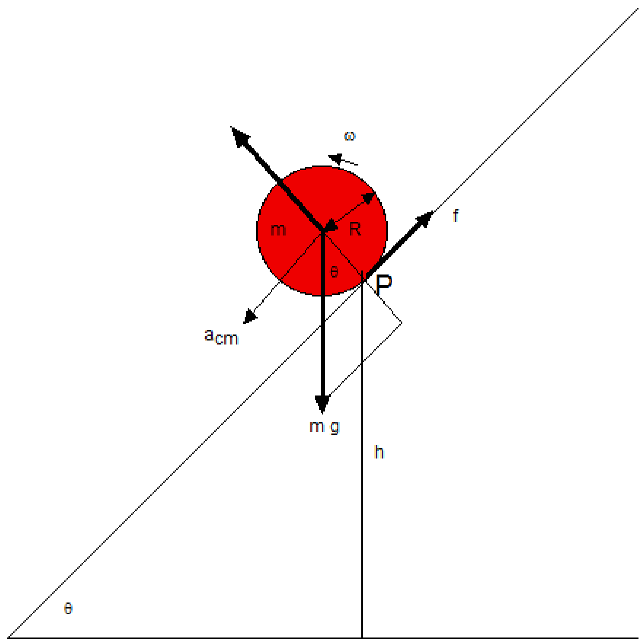
or

$$a = \frac{g}{1 + \frac{I}{mr^2}}$$

11.2 A ball rolling down on a ramp (energy conservation law)

The rotational friction f produces the net torque required for rotation. Suppose that the ball does not slip. For sphere, the moment of inertia is given by

$$I_{cm} = \frac{2}{5} mR^2 .$$



$$ma_{cm} = mg\sin\theta - f$$
$$I_{cm}\alpha = fR$$
$$a_{cm} = R\alpha$$

$N = mg\cos\theta$ (the normal force)

or

$$ma_{cm} = mg\sin\theta - f$$

$$\frac{I_{cm}a_{cm}}{R^2} = f$$

or

$$a_{cm} = \frac{mg\sin\theta}{m + \frac{I_{cm}}{R^2}} = \frac{g\sin\theta}{1 + \frac{I_{cm}}{mR^2}} = \frac{5}{7}g\sin\theta$$

$$f = \frac{I_{cm}}{R^2}\frac{mg\sin\theta}{m + \frac{I_{cm}}{R^2}} = \frac{I_{cm}}{mR^2 + I_{cm}}mg\sin\theta = \frac{2}{7}mg\sin\theta$$

The rotational friction f is related to the normal force as

$$\frac{f}{N} = \frac{2}{7}\tan\theta$$

which is different from the kinetic friction ($f_k = \mu_k N$, independent of the angle θ).

Kinetic-energy theorem

$$v^2 - 0^2 = 2a_{cm}d = 2a_{cm}\frac{h}{\sin\theta} = 2\frac{g\sin\theta}{1 + \frac{I_{cm}}{mR^2}}\frac{h}{\sin\theta} = \frac{2gh}{1 + \frac{I_{cm}}{mR^2}}$$

$$m(v^2 - 0^2) = \frac{2mgh}{1 + \frac{I_{cm}}{mR^2}}$$

This means that the energy conservation is valid in the case of rotating without slipping (in spite of the existence of the rotational friction force which is essential for the rotation)

Energy conservation:

$$\frac{1}{2}mv^2 + \frac{1}{2}I_{cm}\omega^2 = mgh$$

((**Another method**))

The angular momentum at the point P is given by

$$L_P = I_{cm}\omega + mR^2\omega = (I_{cm} + mR^2)\omega = I_P\omega$$

The torque around the point P is

$$\tau_P = mg(R\sin\theta)$$

From the Newton's second law for rotation, we get

$$\tau_P = \frac{dL_P}{dt} = I_P \frac{d\omega}{dt} = I_P \alpha$$

or

$$\alpha = \frac{\tau_P}{I_P} = \frac{mgR\sin\theta}{I_{cm} + mR^2} = \frac{gR\sin\theta}{\frac{I_{cm}}{m} + R^2} = \frac{1}{R}\frac{g\sin\theta}{1 + \frac{I_{cm}}{mR^2}}$$

$$a = R\alpha = \frac{g\sin\theta}{1 + \frac{I_{cm}}{mR^2}}$$

((Note))
Thick walled cylindrical tube with open ends with outside radius R and inter radius r

The moment of inertia is

$$I_{cm} = \frac{1}{2}M(R^2 + r^2)$$

Thus we have

$$\frac{I_{cm}}{MR^2} = \frac{1}{2}(1 + \frac{r^2}{R^2})$$

which is dependent only on the ratio of $r/R\ (\leq 1)$.

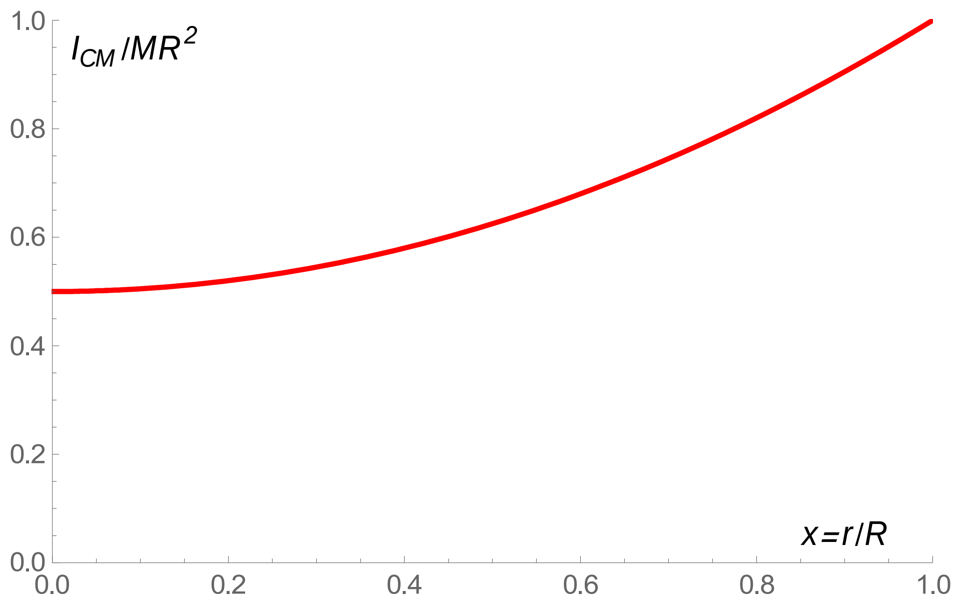


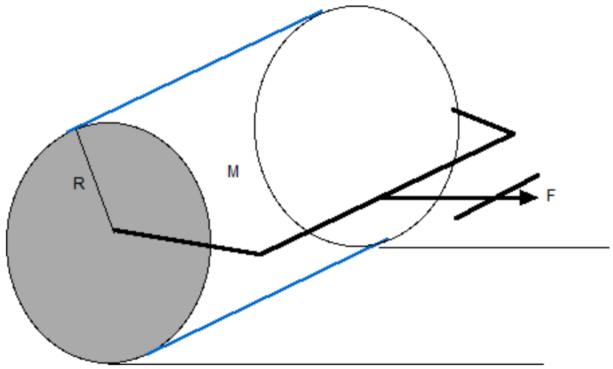
Fig. $\frac{I_{cm}}{MR^2} = \frac{1}{2}(1+\frac{r^2}{R^2})$ vs ratio r/R. The ratio $\frac{I_{cm}}{MR^2}$ is 1 for thin cylindrical shell (x = 1, v_{cm} is low) and is 0 for the solid cylinder (x = 0, v_{cm} is high)

Fig. WileyPlus. Physics demonstration (Chapter 11). Rolling bodies race down an incline

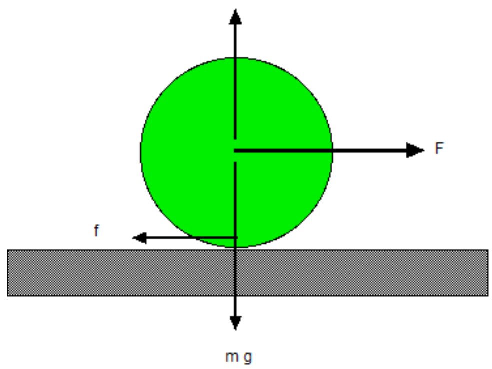
11.3
Serway 10-78

A constant horizontal force F is applied to a lawn roller in the form of a uniform solid cylinder of radius R and mass M. If the roller rolls without slipping on the horizontal surface, show that

(a) the acceleration of the center of mass is $2F/3M$, and
(b) the minimum coefficient of friction necessary to prevent slipping is $F/(3Mg)$. (Hint: take the torque with respect to the center of mass).



((**Solution**))



$$F - f = Ma_{cm}$$

$$Rf = I_{cm}\alpha = \frac{1}{2}MR^2\alpha$$

$$a_{cm} = R\alpha$$

(a)

$$f = \frac{1}{2}Ma_{cm}$$

$$F = \frac{1}{2}Ma_{cm} + Ma_{cm} = \frac{3}{2}Ma_{cm}$$

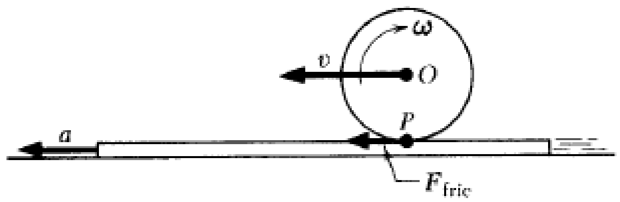
or $a_{cm} = \dfrac{2F}{3M}$

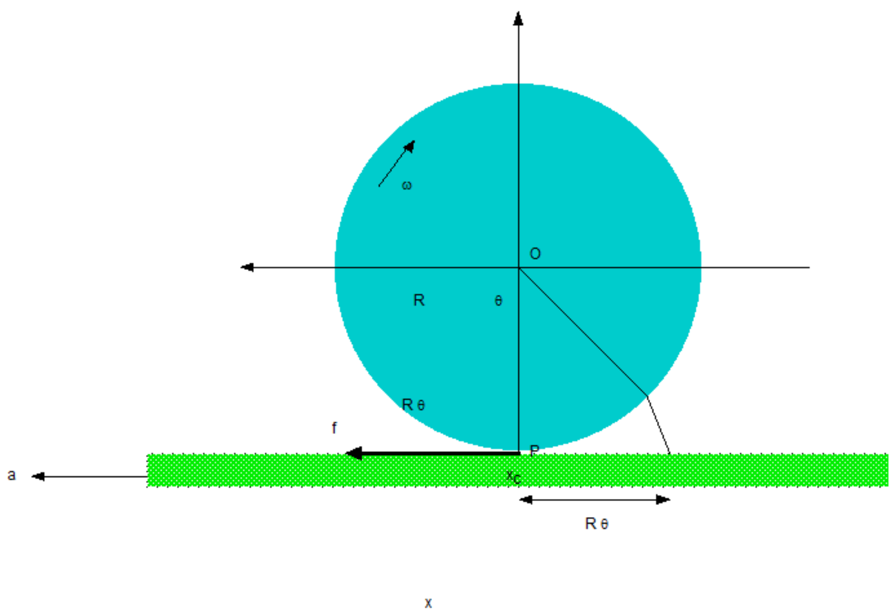
(b)

$$f = \frac{M}{2}\frac{2F}{3M} = \frac{F}{3}$$

11.4 Cylinder on an accelerated rough surface

We consider a cylinder (mass *M*) resting on a rough horizontal rug that is pulled out from under it with acceleration a perpendicular to the axis of the cylinder. What is the motion of the cylinder, assuming it does not slip? The only horizontal force on the cylinder is that of friction P.





The point O is the center of mass.

$$I\alpha = fR$$
$$Ma_{cm} = -f$$

or

$$a_{cm} = -\frac{R\alpha}{2} \qquad (1)$$

where a_{cm} is the acceleration of the center of mass and α is the angular acceleration. *I* is the moment of inertia,

$$I = \frac{1}{2}MR^2$$

We observe the motion of the cylinder from the ground (G).

$$x_{cylinder-G} = x_{cylindr-carpet} + x_{carpet-G}$$

We note that

$$x_{cylinder-G} = x_{cm}, \qquad x_{cylindr-carpet} = R\theta, \qquad x_{carpet-G} = x_p$$

So the position of the center of mass is expressed by

$$x_{cm} = x_P + R\theta$$

or

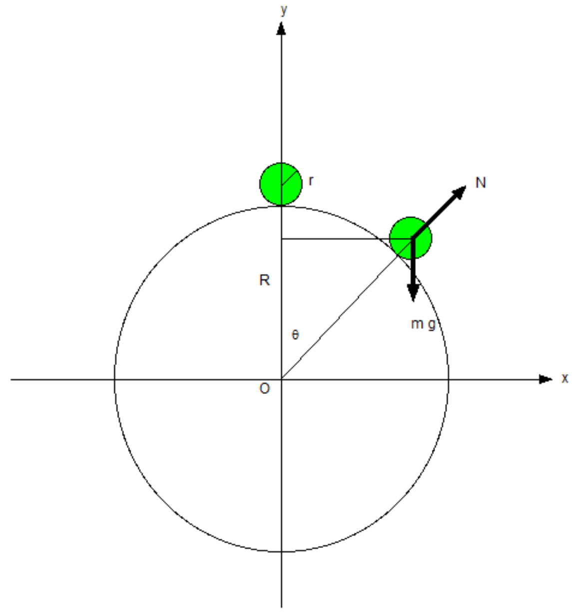
$$a_{cm} = -a + R\alpha \qquad (2)$$

where a>0. From Eqs.(1) and (2), we have

$$a = \frac{3}{2}R\alpha$$
$$a_{cm} = -\frac{1}{2}R\alpha = -\frac{1}{3}a$$
$$f = M\frac{a}{3}$$

11.5 Sphere rolling on a fixed sphere

A sphere of radius r and mass m on top of a fixed sphere of radius R. The first sphere is slightly displaced so that it rolls (without slipping) down the second sphere. What is the angle q at which the first sphere loses a contact with the second sphere.



Energy-conservation law

$$mg(R+r) = mg(R+r)\cos\theta + \frac{1}{2}mv_{cm}{}^2 + \frac{1}{2}I_{cm}\omega^2$$

with

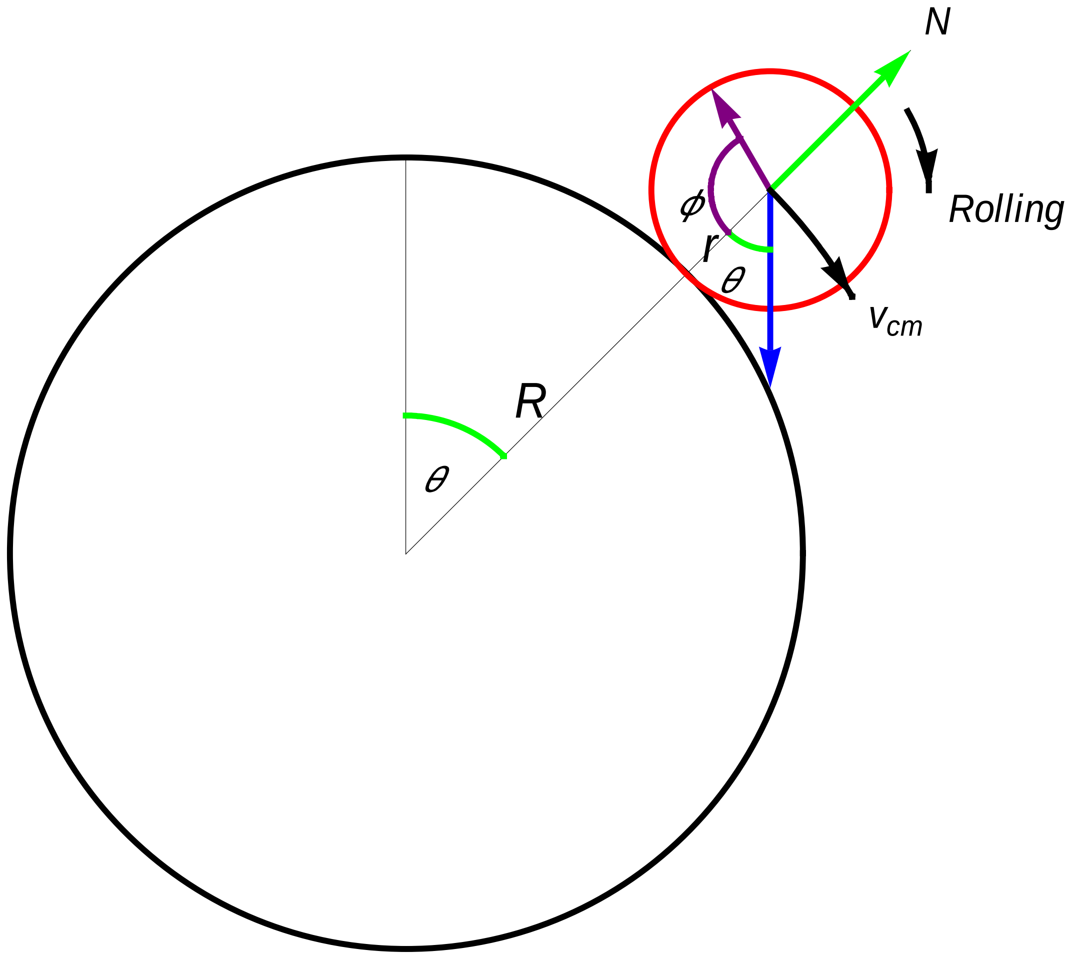
$$I_{cm} = Cmr^2$$

The condition for no slipping:

$$v_{cm} = \frac{ds}{dt} = (R+r)\dot{\theta} = (R+r)\frac{r}{R}\dot{\phi} = \omega r$$
$$R\dot{\theta} = r\dot{\phi}$$

where

$$\omega = r(\dot{\theta}+\dot{\phi}) = r(\frac{R+r}{R})\dot{\phi} \qquad \text{or} \qquad \dot{\phi} = \frac{R}{r(R+r)}\omega$$



where ϕ is the rotation angle of the ball.

Then we get

$$mg(R+r) = mg(R+r)\cos\theta + \frac{1}{2}mr^2\omega^2 + \frac{1}{2}Cmr^2\omega^2$$

$$g(R+r)(1-\cos\theta) = \frac{1}{2}\omega^2 r^2(1+C) \qquad (1)$$

Newton's second law (centripetal acceleration)

$$mg\cos\theta - N = m\frac{{v_{cm}}^2}{R+r} = m\frac{1}{R+r}r^2\omega^2$$

When $N = 0$,

$$g\cos\theta = \frac{r^2\omega^2}{R+r} \qquad (2)$$

From Eqs.(1) and (2), we have

$$g(R+r)(1-\cos\theta) = \frac{1}{2}\omega^2 r^2(1+C)$$

$$r^2\omega^2 = g\cos\theta(R+r)$$

leading to

$$\cos\theta = \frac{2}{3+C} = \frac{2}{3+\frac{I_{cm}}{mr^2}}$$

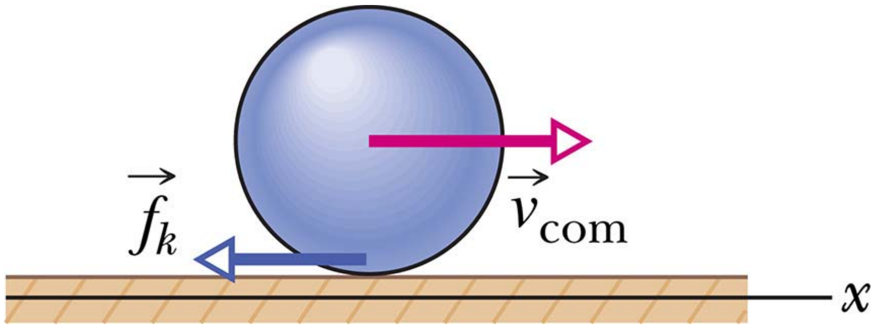
Sphere	C = 2/5.	$\theta = 53.97°$
Cylinder	$C = 1/2$	$\theta = 55.15°$
Hoop	C = 1	$\theta = 60.0°$

In other words, the take-off angle depends only on the value of *C* for the moment of inertia.

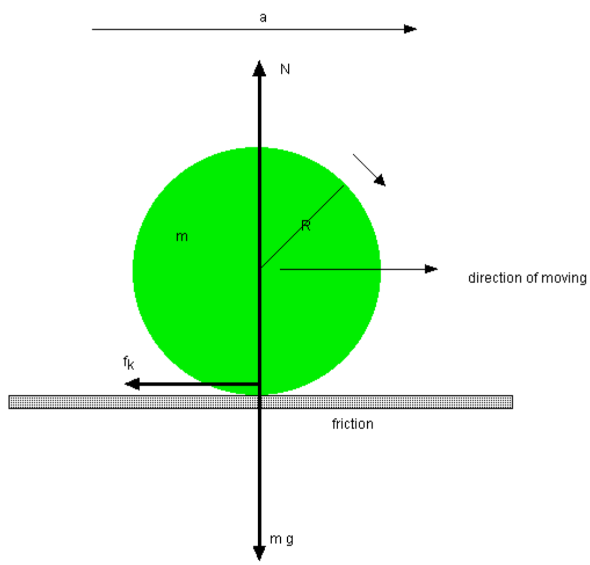
11.6 Bowling: Slipping at large velocity and rotating at small velocity

Problem 11-15* (10-th edition)**

A bowler throws a bowling ball of radius R = 11 cm along a lane. The ball (Fig) slides on the lane with initial speed v_{com}^0 = 8.5 m/s and initial angular speed ω_0 = 0. The coefficient of kinetic fiction between the ball and the lane is 0.21. The kinetic friction force f_k along the ball causes a linear acceleration of the ball while producing a torque that causes an angular acceleration of the ball. When speed v_{com} has decreased enough and angular speed ω has increased enough, the ball stops sliding and then rolls smoothly. (a) What then is v_{com} in terms of ω? During the sliding, what are the ball's (b) linear acceleration and (c) angular acceleration? (d) How long does the ball slide? (e) How far does the ball slide? (f) What is the linear speed of the ball when smooth rolling begins?



((Solution))



R = 11 cm, v_0 = 8.5 m/s, ω_0 = 0, μ_k = 0.21.

The ball undergoes sliding as well as rotating, just after the ball is thrown.

$$I\alpha = f_k R$$

$$I = \frac{2}{5} mR^2$$

$$ma = -f_k$$

$$f_k = \mu_k N$$

$$N = mg$$

or

$$\alpha = \frac{5\mu_k g}{2R}$$

$$a = -\mu_k g$$

$$v_{CM} = at + v_0 \qquad \omega = \alpha t$$

$$x_{CM} = v_0 t + \frac{1}{2} a t^2 \qquad v_\omega = \omega r$$

Consider the condition of $v_{CM} = v_\omega$.

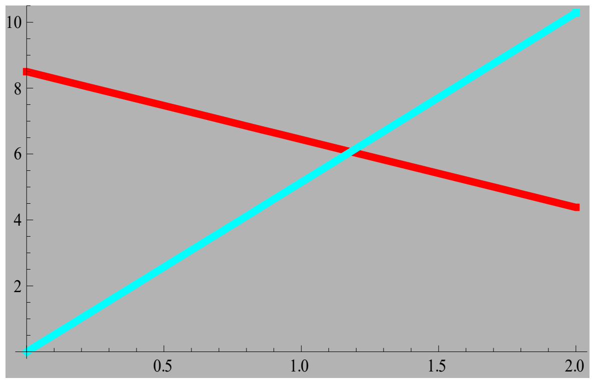


Fig. Plot of v_{cm} (red) and v_ω (blue) as a function of t.

((**Note**))
After the condition $v_{cm} = v_\omega = R\omega$, the friction becomes zero since the ball stops sliding. In other word, the acceleration for the center of mass becomes zero. The angular acceleration also becomes zero. Then the velocity for the center of mass is constant. The angular velocity also becomes constant. The reason is as follows. We assume that

$$a = R\alpha$$

Then we have

$$f_k = \frac{I\alpha}{R} = \frac{2}{5} mR\alpha = \frac{2}{5} ma$$

Since $ma = -f_k$, we have

$$ma = 0$$

In other words, we get

$$v_{cm} = v_\omega = R\omega$$

12. Gyroscope: angular momentum conservation
Topics from Feynman Lectures on Physics

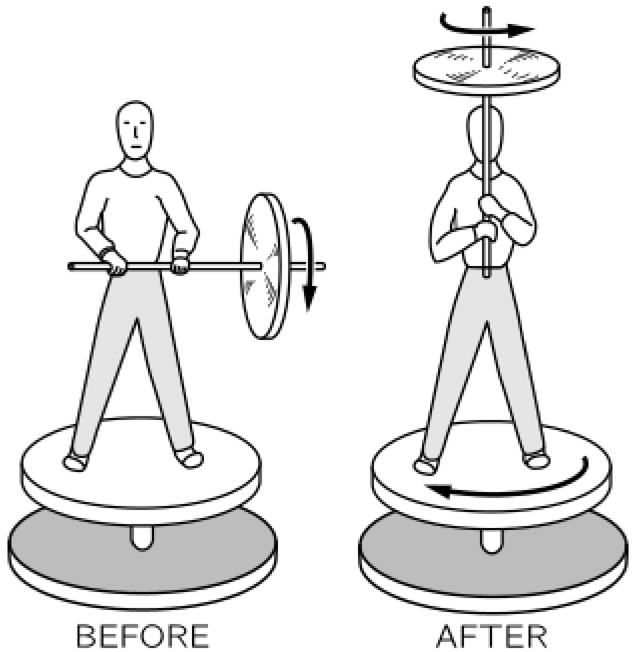


Fig. Before: axis is horizontal (the *y* axis); angular momentum about vertical axis (the *z* axis) = 0. After: the *z* axis is vertical; the angular momentum about the vertical axis is still zero; man and chair spin in direction opposite to spin of the wheel.

If we sit on a swivel chair and hold the spinning wheel with its axis horizontal (the *y* axis), the wheel has an angular momentum about the horizontal axis. Angular momentum around a vertical axis cannot change because of the frictionless pivot of the chair, so if we turn the axis of the wheel into the vertical (the *z* axis), then the wheel would have angular momentum about the vertical, because it is now spinning about this axis. But the system (wheel, ourselves, and chair) cannot have a vertical component, so we and chair have to turn in the direction opposite to the spin of the wheel, to balance it.

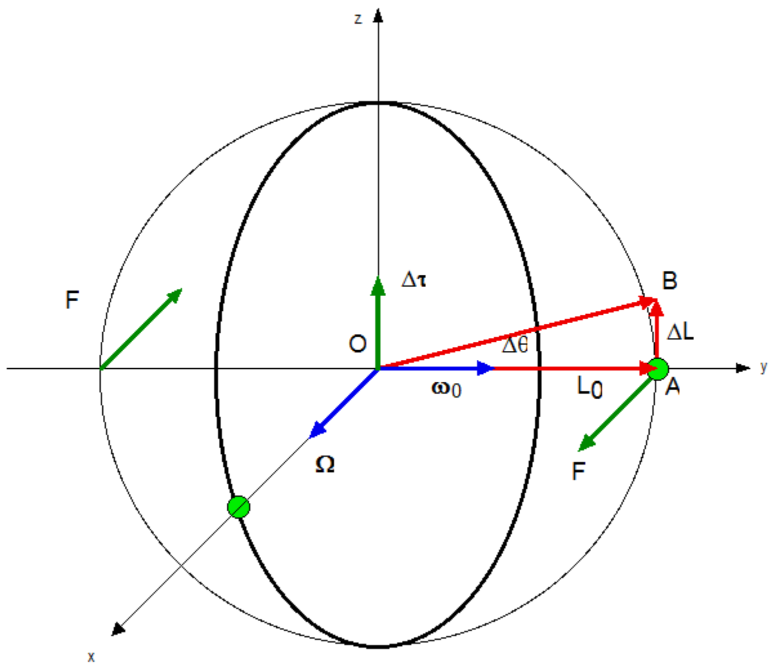


Figure shows the wheel spinning rapidly about the *y*-axis. Therefore its angular momentum is like-wise in that direction. Now suppose that we wish to rotate the axis of the wheel about the *x*-axis through the angle $\Delta\theta$. The rate of the rotation of L_0 is an angular velocity in the *x* direction, and is equal to $\Omega = \mathrm{d}\theta/\mathrm{d}t$. After a short time Δt, the axis has turned to a new position, tilted at an angle $\Delta\theta$ with the horizontal (the *y* axis). The change $\Delta \boldsymbol{L}$ is a vector pointing in the *z* direction.

$$\Delta L = L_0 \Delta\theta$$

$$\tau = \frac{\Delta L}{\Delta t} = L_o \frac{\Delta\theta}{\Delta t} = L_0 \Omega$$

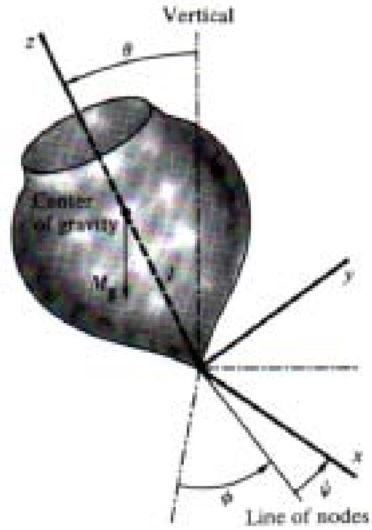
Taking the directions of the various quantities into account, we see that

$$\boldsymbol{\tau} = \boldsymbol{\Omega} \times \boldsymbol{L}_0$$

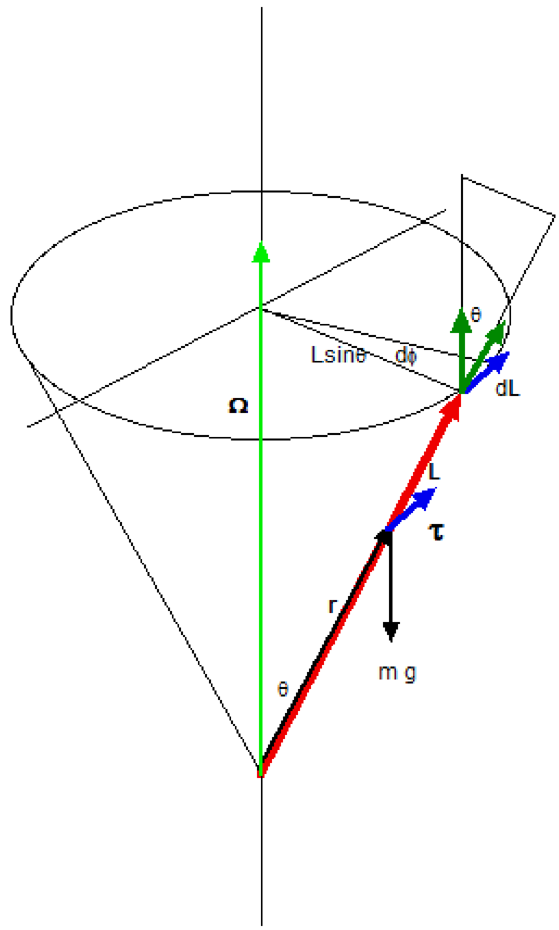
since the magnitude of $\boldsymbol{L}_0$ does not change. Thus, if Ω is along the *x* axis and $\boldsymbol{L}_0$ is along the *y* axis, the torque $\boldsymbol{\tau}$ is along the *z* axis. To produce such a torque, the man must exert equal and opposite forces (the horizontal forces $\boldsymbol{F}$ and $-\boldsymbol{F}$) in the *x* direction. We try to rotate the axis of the wheel into the *z* direction. But Newton's third law demands that equal and opposite forces (and equal and opposite torques) act on man. This causes man to rotate in the opposite sense about the vertical axis *z*.

13. Precession of gyroscope

Angular momentum is the basis of the operation of a gyroscope. A gyroscope is a spinning object used to control or maintain the orientation in space of the object or a system containing the object. Gyroscopes undergo precessional motion.



$L = I\omega$



$dL = L\sin\theta d\phi$

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = L\sin\theta\frac{d\phi}{dt}\hat{\phi} = L\sin\theta\Omega\hat{\phi}$$

or

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F} = -mgr(\hat{r} \times \hat{k}) = mgr\sin\hat{\phi}$$

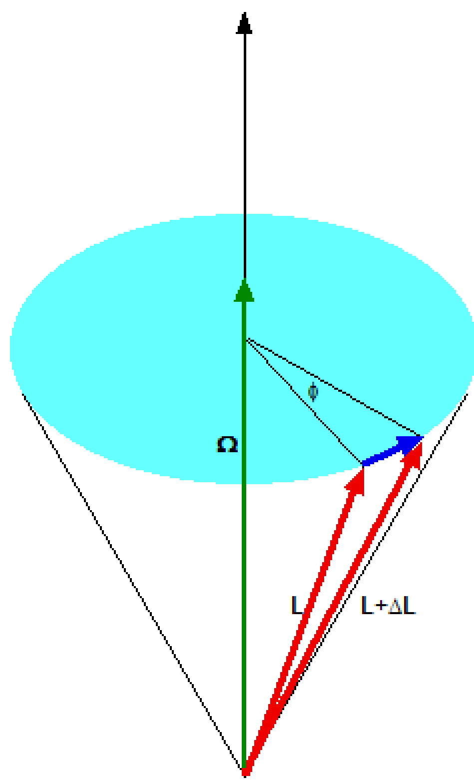
From these two equations, we have

$$mgr\sin\theta = L\sin\theta\Omega$$
$$mgr = L\Omega$$

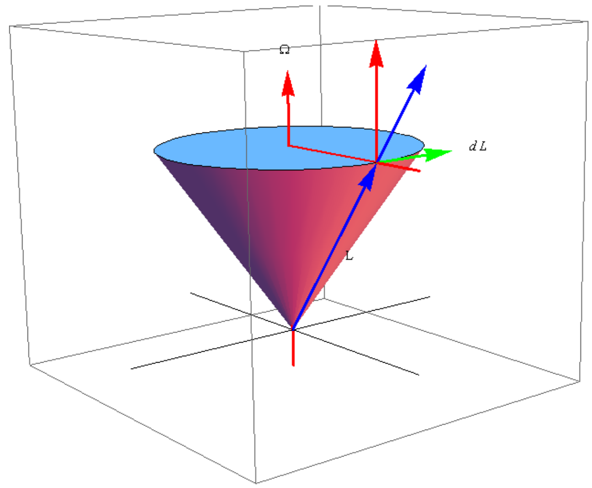
or

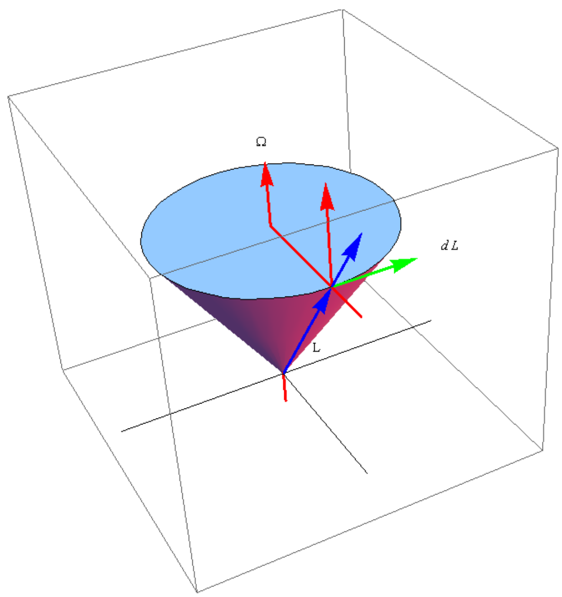
$$\frac{d\phi}{dt} = \Omega = \frac{mgr}{L} = \frac{mgr}{I\omega}$$

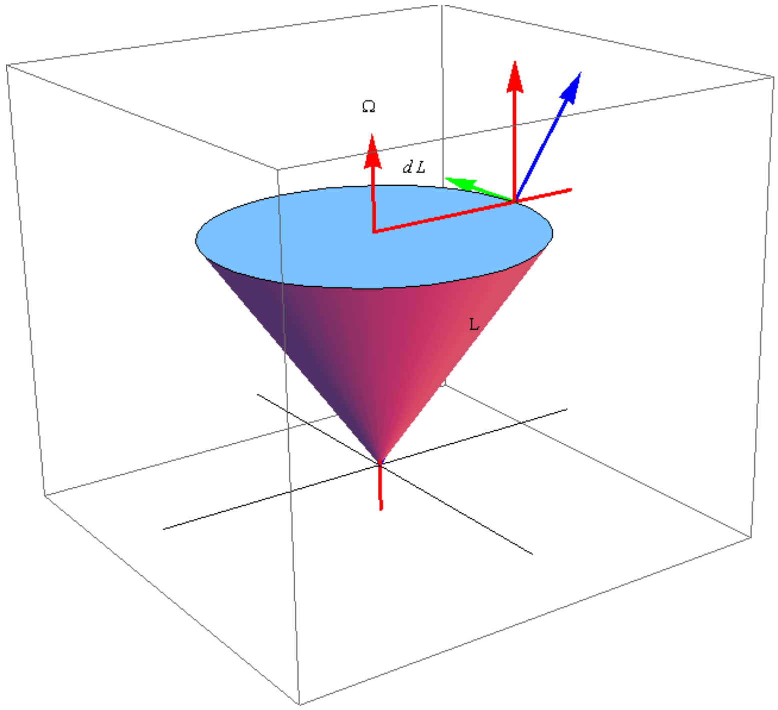
Ω is the angular frequency of precession.



((**Mathematica**)) Graphics3D

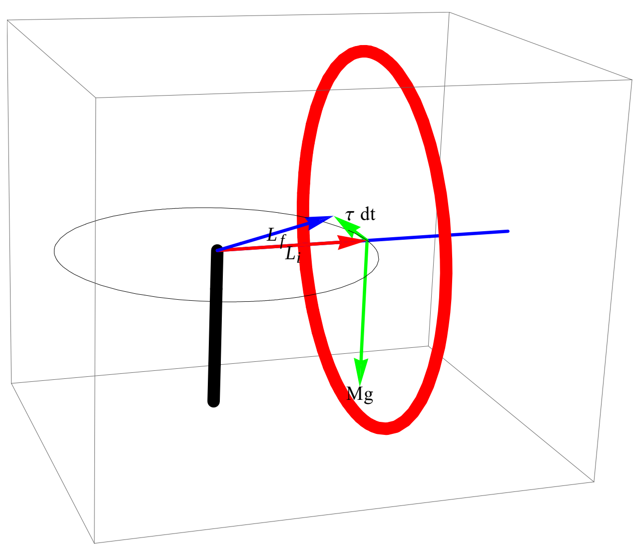

Ω
dL
L


Ω
dL
L



14. Gyroscope and precession

A gyroscope is supported at one end. When the flywheel spins, it and its axis float in the air while moving in a circle about the pivot. The horizontal circular motion of the flywheel is called precession. The angular speed of the precession is Ω.



We see what happens if the flywheel is spinning initially, so the initial angular momentum $\boldsymbol{L}_i$ is is not zero. Since the flywheel rotates around its symmetry axis, $\boldsymbol{L}_i$ lies along the axis. But each change in angular momentum is perpendicular to the axis because the torque

$$\boldsymbol{\tau} = \boldsymbol{r} \times M\boldsymbol{g}$$

is perpendicular to the axis. This causes the direction of L to change, but not its magnitude.

$$\boldsymbol{L}_f = \boldsymbol{L}_i + \boldsymbol{\tau} dt$$

The changes $\boldsymbol{\tau} dt$ are always in the horizontal *x-y* plane, so the angular momentum vector and the flywheel axis with which it moves are always horizontal. In other words, the axis does not fall. It just presses. Note that the only external force on the gyroscope are the normal force N acting at the pivot (assumed to be frictionless) and the weight Mg of the flywheel that acts at its center of mass.

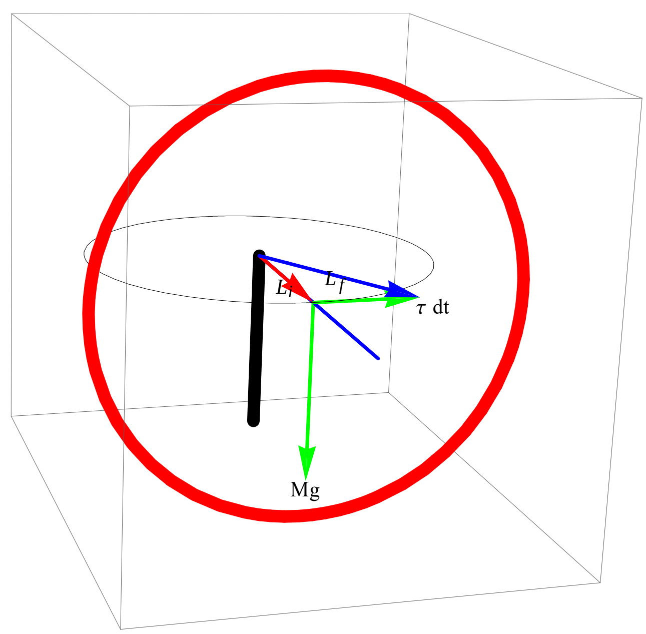
Suppose that the angle between $\boldsymbol{L}_i$ and $\boldsymbol{L}_f$ is $d\phi$.

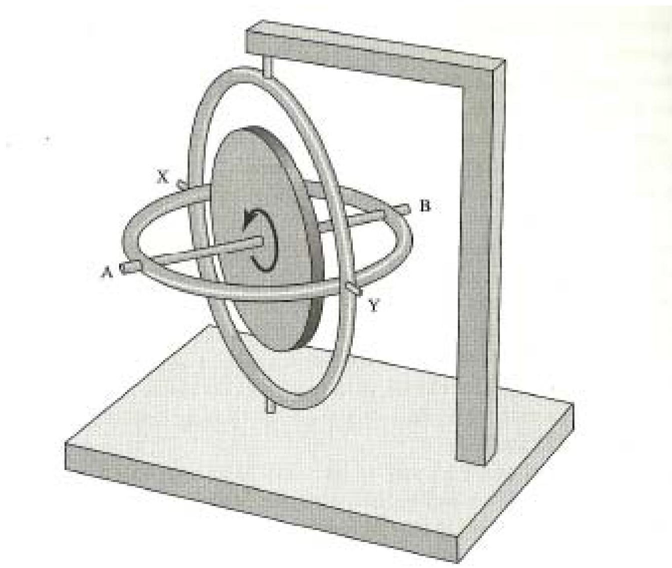
$$L_i d\phi = I\omega d\phi = \tau dt = Mgrdt$$

The rate at which the axis moves, Ω (= $d\phi/dt$) is called the precession angular speed.

$$\Omega = \frac{d\phi}{dt} = \frac{Mgr}{I\omega}$$

where ω is the angular velocity of the flywheel.





A demonstration gyroscope

14. Link

Cycloid
http://mathworld.wolfram.com/Cycloid.html

Angular momentum
http://en.wikipedia.org/wiki/Angular_momentum

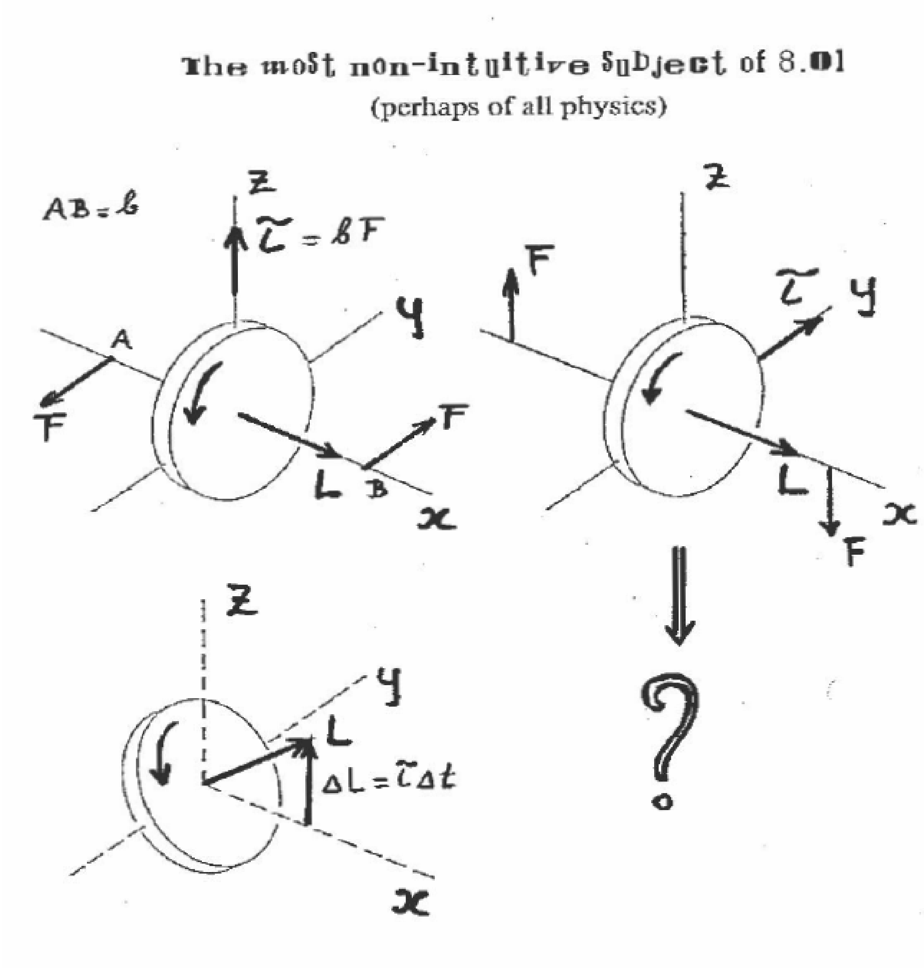
WileyPlus
http://edugen.wiley.com/edugen/secure/instructor/index.uni?&protocol=http

Appendix

Angular momentum and torque (from Walter Levine, MIT Lecture); it is none of intuitive. http://www.youtube.com/watch?v=NeXIV-wMVUk&feature=related

Rotation gyroscope
http://ocw.mit.edu/courses/physics/8-01-physics-i-classical-mechanics-fall-1999/video-lectures/lecture-24/

((Experiments by Walter Levin)) This figure is copied from the note written by Walter Levin.



A.1 Experiment I

This is the excellent demonstration of the Newton's second law for the angular momentum.

We consider the wheel rotating around the axis. The angular frequency of the spinning wheel is ω. The moment of inertia of the wheel around the center of mass is I. When such a wheel is hanged to the string as shown below. In this case the wheel undergoes a precession around the string.

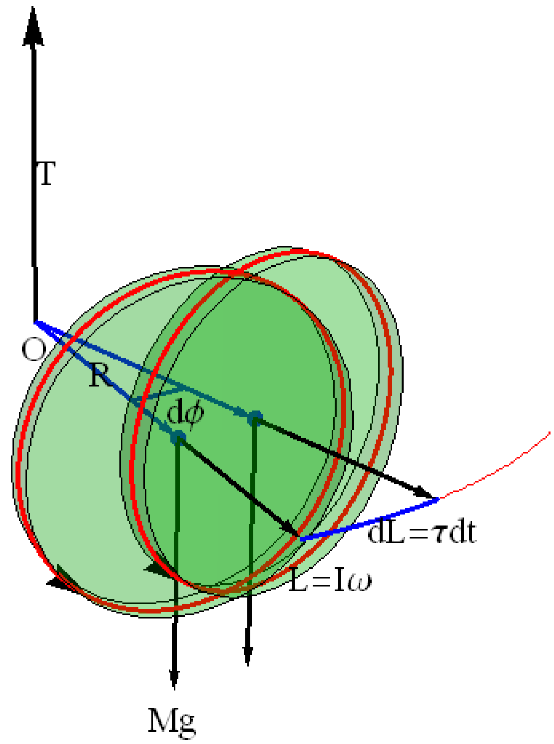
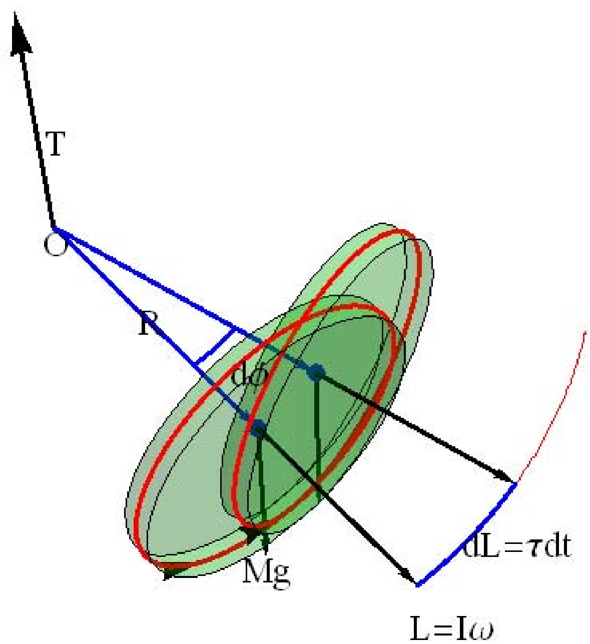


Fig. Precession of the spinning wheel around the string (the *z* axis) under the presence of the torque (*MgR*). Note that $\boldsymbol{F}(=m\boldsymbol{g})\perp\boldsymbol{\tau}$ and $\Delta\boldsymbol{L}//\boldsymbol{\tau}$.



The precession angular frequency is calculated as follows. The change of the angular momentum is given by

$$dL = \tau dt$$

where τ is the torque and is given by

$$\tau = MgR\ .$$

Note that the direction of the vector d***L*** is parallel to that of the torque (vector). The magnitude of dL is obtained as

$$dL = Ld\phi = MgRdt\ ,$$

where dϕ is the change of the angle as a result of the precession of the wheel around the string (the z axis). Then the precession angular frequency is evaluated as

$$\Omega = \frac{d\phi}{dt} = \frac{MgR}{L} = \frac{MgR}{I\omega}.$$

Note that the tension T is given by

$$T = Mg$$

A2. Experiment II

We assume that the rotation wheel is supported by both hands at the points A and B. The wheel rotates in counter-clock wise. Thus the angular momentum is directed along the OB direction. Suppose that the force F is applied at the point A and point B as shown in the figure during the short time Δt. The direction of the torque is denoted by the purple arrow. Then the angular momentum is changed from $\boldsymbol{L}$ to $\boldsymbol{L} + d\boldsymbol{L}$. The wheel rotates around the vector $\boldsymbol{L} + d\boldsymbol{L}$ in counter-clockwise in the y-z plane

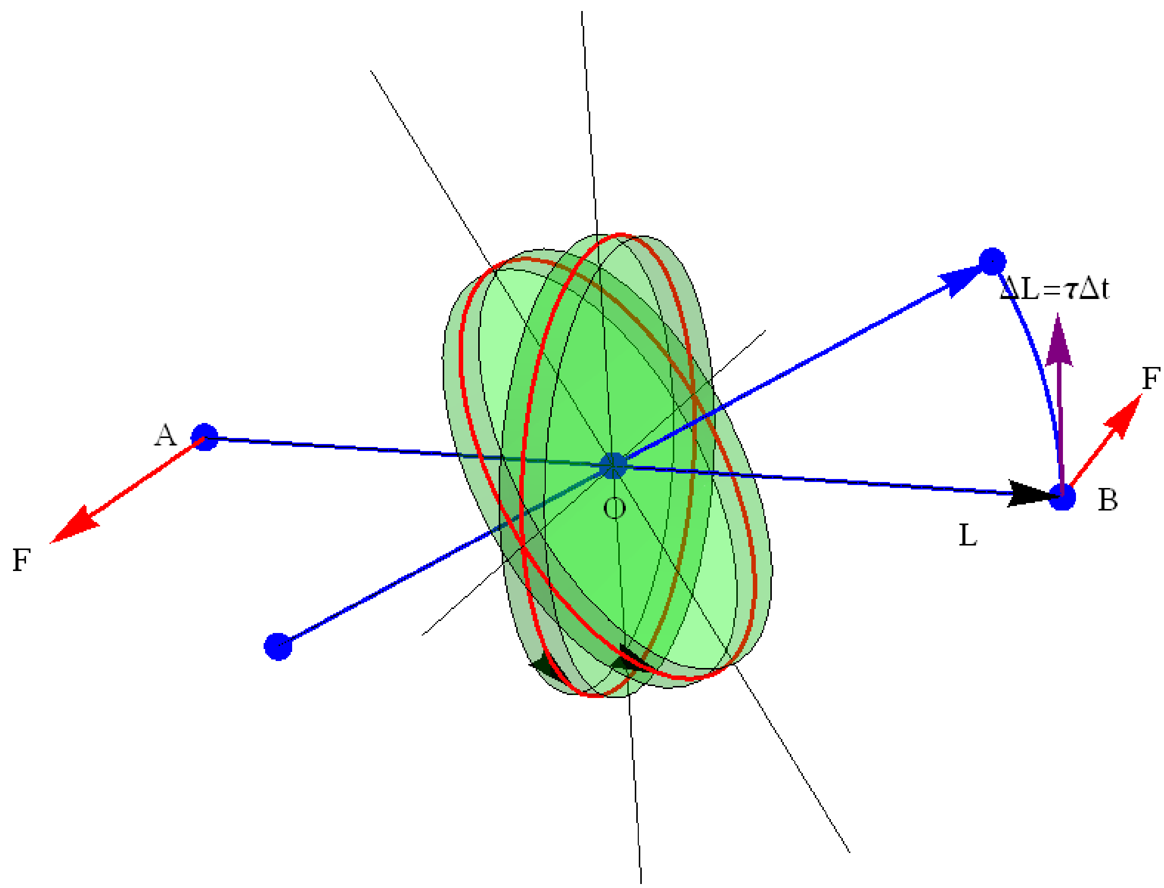


Fig. The effect of the force $\boldsymbol{F}$ at the point A and point B during the short time Δt, on the rotating wheel. The direction of $\boldsymbol{F}$ is horizontal (the negative x axis) at the point B. $\Delta \boldsymbol{L} = \boldsymbol{\tau}\Delta t$ with $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$. After the time *Δt*, the angular momentum changes from $\boldsymbol{L}$ to $\boldsymbol{L} + \Delta\boldsymbol{L}$. Note that $\boldsymbol{F} \perp \boldsymbol{\tau}$ and $\Delta \boldsymbol{L} // \boldsymbol{\tau}$.

A3. Experiment III

We assume that the rotation wheel is supported by both hands at the points A and B. The wheel rotates in counter-clock wise. Thus the angular momentum is directed along the OB direction. Suppose that the force *F* is applied at the point A and point B as shown in the figure during the short time *Δt*. The direction of the torque is denoted by the purple arrow. Then the angular momentum is changed from $\boldsymbol{L}$ *to* $\boldsymbol{L} + d\boldsymbol{L}$. *The wheel rotates around the vector* $\boldsymbol{L} + d\boldsymbol{L}$ *in counter-clockwise in the x-y* plane.

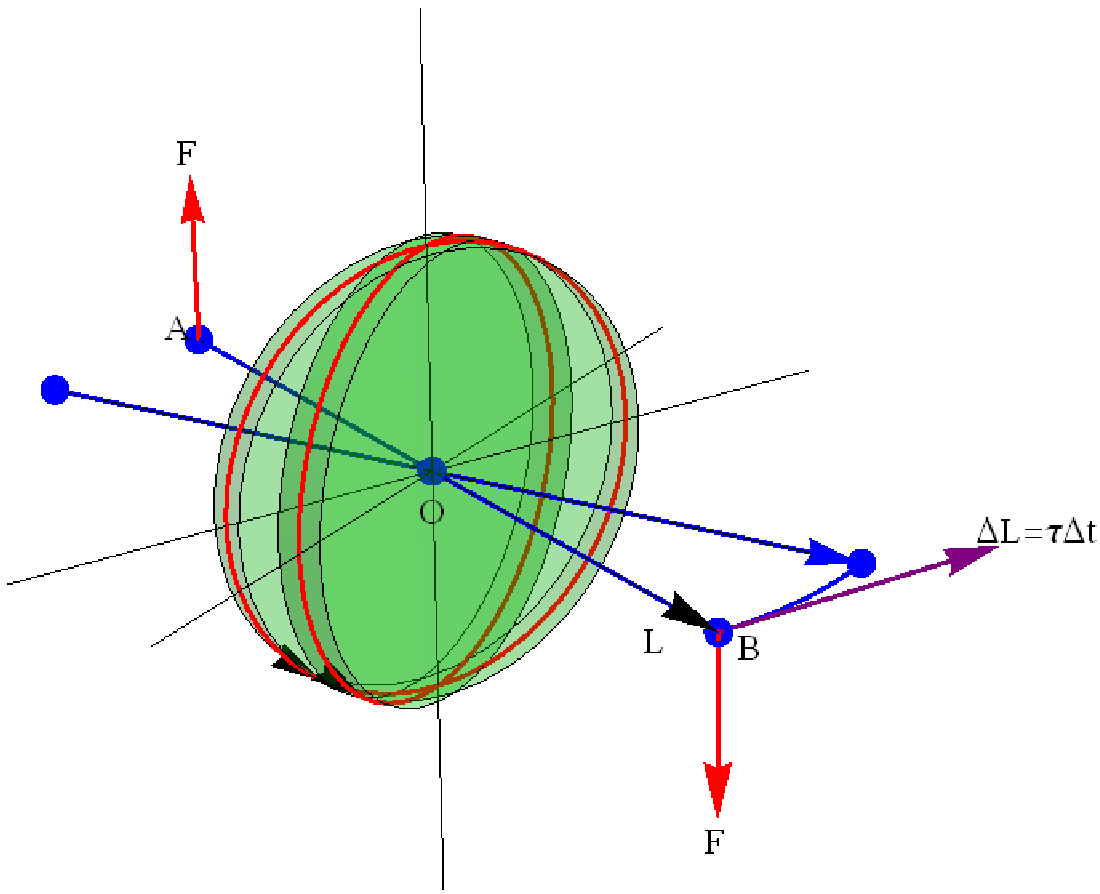


Fig. The effect of the force $\boldsymbol{F}$ at the point A and point B during the short time *Δt*, on the rotating wheel. The direction of $\boldsymbol{F}$ is downward at the point B. $\Delta \boldsymbol{L} = \boldsymbol{\tau}\Delta t$ with $\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F}$. After the time *Δt*, the angular momentum changes from $\boldsymbol{L}$ to $\boldsymbol{L} + \Delta\boldsymbol{L}$. Note that $\boldsymbol{F} \perp \boldsymbol{\tau}$ and $\Delta \boldsymbol{L} // \boldsymbol{\tau}$.

C. Physics on white dwarf, neutron star and black hole

C1. Degeneracy pressure

In quantum mechanics, the electrons have very large velocity in the limit of small space. When L is the length of space, the momentum p is described by

$$p = \frac{h}{L}$$

where h is the Planck's constant. The electrons are also fermions with spin 1/2. According to the Pauli's exclusion principle, each electron has a different state. No same state exists. As the density increases, the region (L) where each electron can move is decreased. Because of the increased momentum, the kinetic energy of electrons increases, leading to a pressure. This pressure is called a degeneracy pressure. The degeneracy pressure is expressed by

$$P_f = \frac{6}{5}(\frac{\pi^4}{3})^{1/3}\frac{\hbar^2}{2m_e}n_e^{5/3}$$

where n_e is the density of electrons and m_e is the mass of electron. Note that protons and neutrons are also fermions and are certainly degenerate. The mass of the fermion in question appears in the denominator of the degenerate pressure. Since nucleons are some 2000 times more massive than electrons, their degeneracy pressure is negligible.

C2. Gravitational Pressure

The mass of the star is almost totally accounted for with N nucleons – neutrons or protons.

$$P_g = \frac{1}{5}(\frac{4\pi}{3})^{1/3}G(Nm_p)^2V^{-4/3} \qquad (1)$$

C. Balancing the degeneracy pressure against the gravitational pressure

We assume that n_e is given by

$$n_e = \frac{N}{V}$$

Then the degeneracy pressure is rewritten as

$$P_f = \frac{6}{5}(\frac{\pi^4}{3})^{1/3}\frac{\hbar^2}{2m_e}N^{5/3}V^{-5/3} \qquad (2)$$

We consider a star with the mass of $k\ M_{sun}$, where $k = 1 – 50$. We have

$$N = \frac{kM_{sun}}{m_p}$$

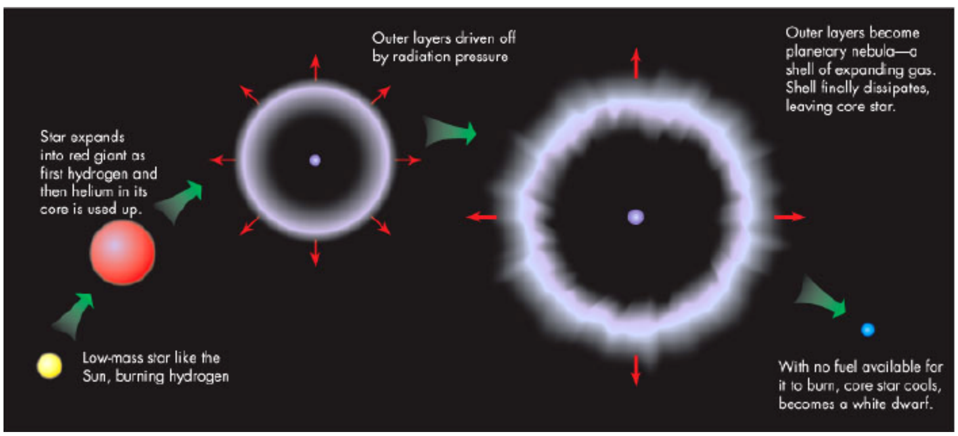
When $P_f = P_g$, the radius R is estimated as

$$R = \frac{3(\frac{3}{2})^{1/3}\pi^{2/3}}{2k^{1/3}} \frac{\hbar^2}{Gm_e m_p^{5/3} M_{sun}^{1/3}}$$

or

$$R = \frac{2.27325\times 10^4}{k^{1/3}} km$$

Stars that reach the degenerate state described here are familiar to astronomers as white dwarf. A white dwarf with the mass of the sun is on the same order as that of Earth.



((**Mathematica**))

White dwarf, neutron star and black hole

$$\text{Physconst} = \{g \to 9.80665,\ G \to 6.6742867\ 10^{-11},\ me \to 9.1093821545\ 10^{-31},\ eV \to 1.602176487\ 10^{-19},\ MeV \to 1.602176487\ 10^{-13},\ qe \to 1.602176487\ 10^{-19},\ c \to 2.99792458\ 10^{8},\ mn \to 1.674927211\ 10^{-27},\ mp \to 1.672621637\ 10^{-27},\ h \to 6.62606896\ 10^{-34},\ \hbar \to 1.05457162853\ 10^{-34},\ Msun \to 1.988435\ 10^{30},\ Rsun \to 6.9599\ 10^{8}\}$$

$$\{g \to 9.80665,\ G \to 6.67429 \times 10^{-11},\ me \to 9.10938 \times 10^{-31},\ eV \to 1.60218 \times 10^{-19},\ MeV \to 1.60218 \times 10^{-13},\ qe \to 1.60218 \times 10^{-19},\ c \to 2.99792 \times 10^{8},\ mn \to 1.67493 \times 10^{-27},\ mp \to 1.67262 \times 10^{-27},\ h \to 6.62607 \times 10^{-34},\ \hbar \to 1.05457 \times 10^{-34},\ Msun \to 1.98844 \times 10^{30},\ Rsun \to 6.9599 \times 10^{8}\}$$

Density of neutron stars

$$\text{rule1} = \{T \to 33\ 10^{-3}\}$$

$$\left\{T \to \frac{33}{1000}\right\}$$

$$\rho = \frac{3\pi}{T^2 G}\ /.\ \text{rule1}\ /.\ \text{Physconst}$$

$$1.2967 \times 10^{14}$$

Condition for the Black - hole; the escape velocity is equal to the velocity of light

$$Rb = k\frac{2\,G\,Msun}{c^2}\ /.\ \text{Physconst}$$

$$2953.28\,k$$

Balancing between degenerate pressure and gravitational pressure. N is the number of protons for the system with mass having k time larger than Sun

$$\text{rule1} = \left\{N \to k\frac{Msun}{mp}\right\}$$

$$\left\{N \to \frac{k\,Msun}{mp}\right\}$$

Number of protons for Sun

$$\frac{Msun}{mp}\ /.\ \text{Physconst}$$

$$1.18881 \times 10^{57}$$

Degenerate pressure

$$Pf = \frac{6}{5}\left(\frac{\pi^4}{3}\right)^{1/3}\frac{\hbar^2}{2\,me}(N)^{5/3}\,V^{-5/3}$$

$$\frac{3^{2/3}\,N^{5/3}\,\pi^{4/3}\,\hbar^2}{5\,me\,V^{5/3}}$$

$$Pf1 = Pf\ /.\ \text{rule1}\ /.\ \text{Physconst}\ //\ \text{Simplify}$$

$$\frac{3.11767 \times 10^{57}\,k^{5/3}}{V^{5/3}}$$

Gravitational pressure

$$\text{Pg} = \frac{1}{5}\left(\frac{4\pi}{3}\right)^{1/3} G\,(N\,\text{mp})^2\,V^{-4/3}$$

$$\frac{2^{2/3}\,G\,\text{mp}^2\,N^2\left(\frac{\pi}{3}\right)^{1/3}}{5\,V^{4/3}}$$

```
Pg1 = Pg /. rule1 /. Physconst // Simplify
```

$$\frac{8.50786\times 10^{49}\,k^2}{V^{4/3}}$$

Radius when the degenerate pressure is equal to gravitational pressure

$$\text{s1} = \text{Solve}\left[\text{Pf} == \text{Pg} /. V \to \frac{4\pi}{3}R^3, R\right] // \text{Simplify}$$

$$\left\{\left\{R \to \frac{3\left(\frac{3}{2}\right)^{1/3}(-\pi)^{2/3}\hbar^2}{2\,G\,\text{me}\,\text{mp}^2\,N^{1/3}}\right\}, \left\{R \to -\frac{3\left(-\frac{3}{2}\right)^{1/3}\pi^{2/3}\hbar^2}{2\,G\,\text{me}\,\text{mp}^2\,N^{1/3}}\right\}, \left\{R \to \frac{3\left(\frac{3}{2}\right)^{1/3}\pi^{2/3}\hbar^2}{2\,G\,\text{me}\,\text{mp}^2\,N^{1/3}}\right\}\right\}$$

```
R1 = R /. s1[[3]] /. rule1 /. Physconst
```

$$\frac{2.27325\times 10^{7}}{k^{1/3}}$$

```
R12 = R /. s1[[3]] /. rule1 // PowerExpand
```

$$\frac{3\left(\frac{3}{2}\right)^{1/3}\pi^{2/3}\hbar^2}{2\,G\,k^{1/3}\,\text{me}\,\text{mp}^{5/3}\,\text{Msun}^{1/3}}$$

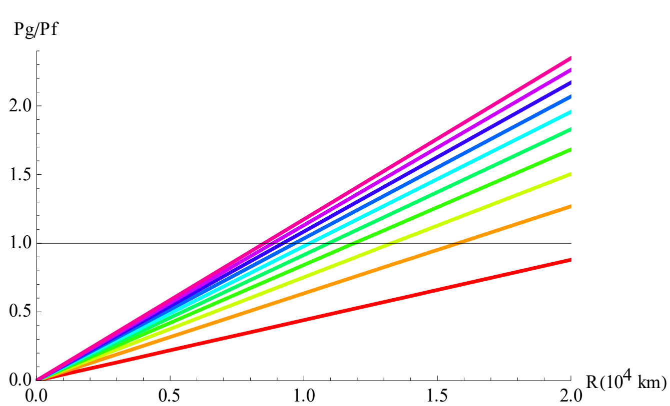
Ratio Pg/Pf

R2 is the radius in unts of 10^4 km

$$\text{A1} = \frac{\text{Pg1}}{\text{Pf1}} /. \left\{V \to \frac{4\pi}{3}R^3\right\} /. \{R \to 10^7\,\text{R2}\} // \text{PowerExpand}$$

$$0.439899\,k^{1/3}\,\text{R2}$$

```
p1 = Plot[Evaluate[Table[A1, {k, 1, 20, 2}]], {R2, 0, 3},
  PlotStyle → Table[{Thick, Hue[0.1 i]}, {i, 0, 10}],
  PlotRange → {{0, 2}, {0, 2.4}}, AxesLabel → {"R(10^4 km) ", "Pg/Pf"}];
p2 = Plot[1, {R2, 0, 3}, PlotStyle → {Thin, Black},
  PlotRange → {{0, 2}, {0, 2.4}}];
Show[p1, p2]
```



APPENDIX Scattering: Semiclassical argument for the angular momentum,

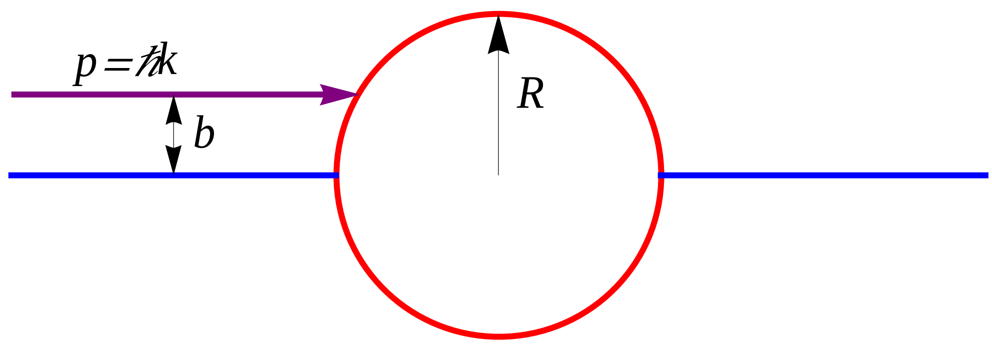
The particles with the impact parameter b possesses the angular momentum L given by

$$L = pb\,,$$

where p $(=\hbar k)$ is the linear momentum of the particles. Only particles with impact parameter b less than or equal to the range R of the potential energy would interact with the target;

$$L \le L_{\max} = \hbar kR$$

since $b<R$.



When energy is low, L_{max} is small. Partial waves for higher l are, in general, unimportant. That is why the partial wave expansion is useful in the case of low energy incident particle. The main contribution to the scattering is the S-wave ($l = 0$). The P-wave ($l = 1$) does not contribute in typical cases.