Chapter 12
Equilibrium and Elasticity

Fig. Balanced Rock in Arches National Park, Utah. This picture was taken by Dr. Itsuko Suzuki (July, 2008).

1 Rigid object in equilibrium

If a body is in **translational equilibrium** then d***P***/d*t* = 0, or

$$\frac{d\boldsymbol{P}}{dt} = \sum \boldsymbol{F}_{ext} = 0$$

If a body is in **rotational equilibrium** then d***L***/d*t* = 0, or

$$\frac{d\boldsymbol{L}}{dt} = \sum \boldsymbol{\tau}_{ext} = 0$$

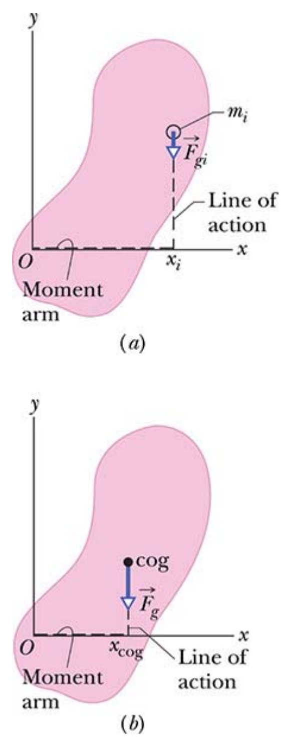
In summary

A rigid object is in equilibrium if and only if the resultant external forces acting on it is zero and the resultant external torque on it is zero about any axis.

$$\sum \boldsymbol{F} = 0$$
$$\sum \boldsymbol{\tau} = 0$$

The first condition is the condition for translational equilibrium.
The second condition is the condition for rotational equilibrium.

2 Center of gravity



The gravitational force $\boldsymbol{F}_g$ on a body effectively acts at a single point, called the **center of gravity** (cog) of the body.

If $\boldsymbol{g}$ is the same for all elements of a body, then the body's center of gravity (cog) is coincident with the body's center of mass (com).

((Proof))

$$(\sum_i m_i g_i) x_{cog} = \sum_i (m_i g_i x_i)$$

$$x_{cog} = \frac{\sum_i m_i g_i x_i}{\sum_i m_i g_i}$$

When $g_i = g$,

$$x_{cog} = \frac{\sum_i m_i x_i}{\sum_i m_i} = x_{com}$$

3 Problem solving strategy

Conceptualize

Identify the forces acting on the object
Think about the effect of each force on the rotation of the object, if the force was acting by itself

Categorize

Confirm the object is a rigid object in equilibrium

Analyze

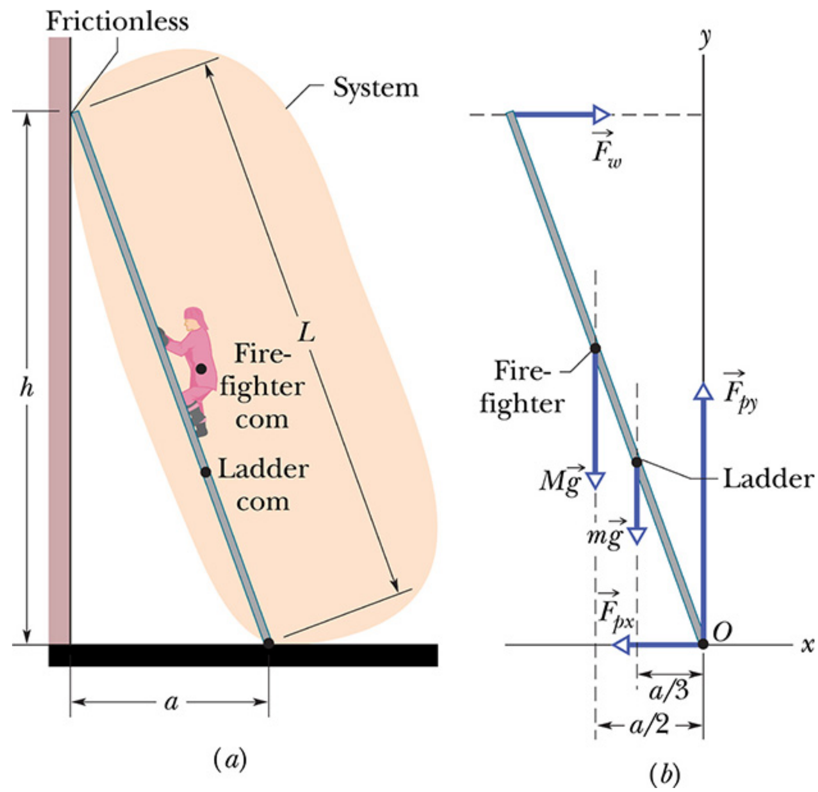
Draw a free body diagram
Label all external forces acting on the object
Resolve all the forces into rectangular components
Apply the first condition of equilibrium
Choose a convenient axis for calculating torques
Choose an axis that simplifies your calculations
Apply the second condition of equilibrium
Solve the simultaneous equations

Finalize

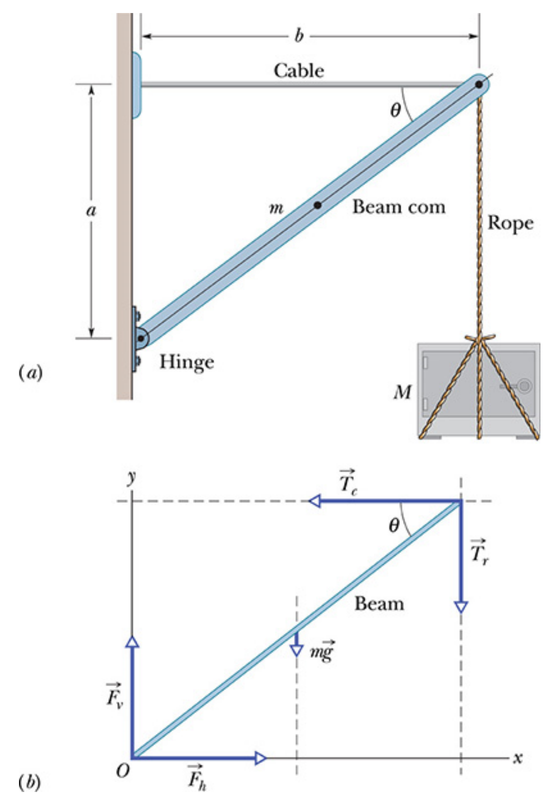
Be sure your results are consistent with the free body diagram

4 Free-body diagram

4.1 Example-1

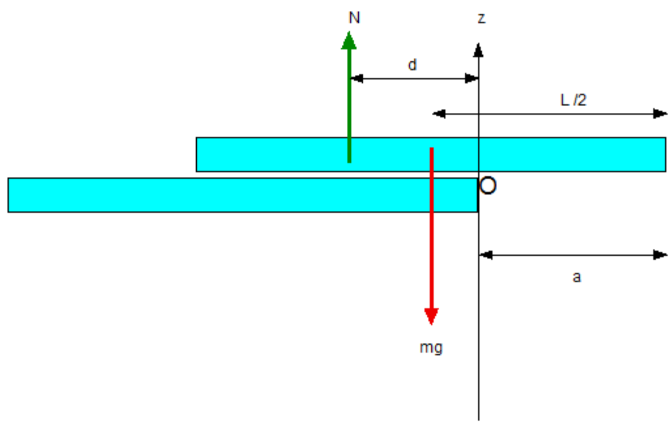

Frictionless
System
L
h
Fire-
fighter
com
Ladder
com
a
(a)
y
$\vec{F}_w$
Fire-
fighter
$\vec{F}_{py}$
$M\vec{g}$
Ladder
$m\vec{g}$
$\vec{F}_{px}$
O
x
a/3
a/2
(b)

4.2 Example-2


b
Cable
θ
a
m
Beam com
Rope
Hinge
(a)
M
y
$\vec{T}_c$
θ
$\vec{T}_r$
Beam
$m\vec{g}$
$\vec{F}_v$
O
$\vec{F}_h$
x
(b)

4.3 Stacking blocks

Two bricks of length l and mass m are stacked. Using conditions of static equilibrium we can determine the maximum overhang of the top brick. The two forces acting on the top brick are the gravitational force mg and the normal force N, exerted by the bottom brick on the top brick. Both forces are directed along the z-axis. Since the system is in equilibrium, the net force acting along the z-axis must be zero.



If the top block is on the verge of falling down, it will rotate around O. The gravitational force mg acting on the whole block is replaced by a single force with magnitude mg acting on the center of mass of the top block. The normal force N acting on the whole contact area between the top and the bottom block is replaced by a single force N acting on a point a distance d away from the rotation axis O.

From the condition that $\sum F_z = 0$, we have

$$N = mg$$

From the condition that the total torque around the point O is zero, we have

$$mg\left(\frac{L}{2} - a\right) - Nd = 0$$

From these two equations, we get

$$d = \frac{L}{2} - a > 0$$

Since d>0,

$$a < \frac{L}{2}$$

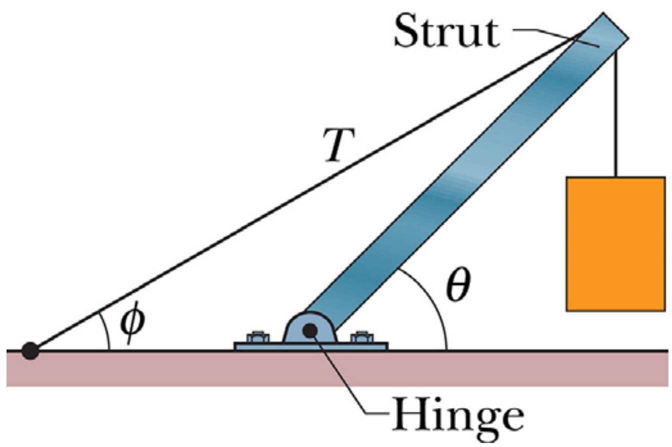
In other words, if the center of mass of the top brick is located to the left of the edge of the bottom brick, the system will be in equilibrium.

5 Selected Problems

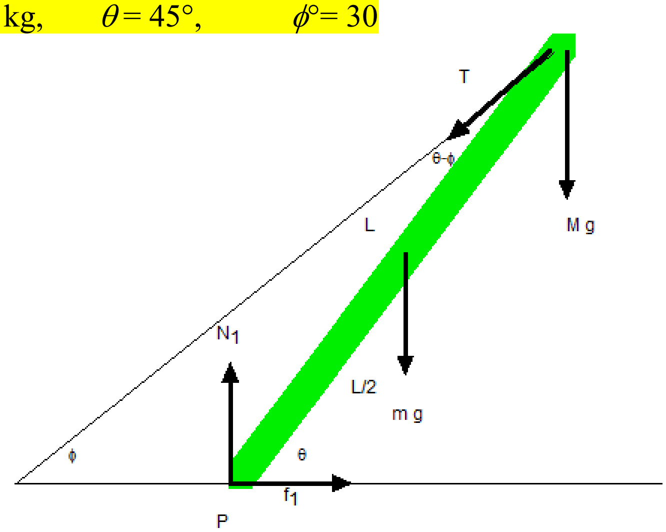
5.1

Problem 12-21 (SP-12) (10-th edition)**

The system in Fig. is in equilibrium. A concrete block of mass 2.25 kg hangs from the end of the uniform strut of mass 45.0 kg. For angles $\phi = 30.0^{\circ}$ and $\theta = 45.0^{\circ}$, find (a) the tension T in the cable and the (b) horizontal and (c) vertical components of the force on the strut from the hinge.



$M = 225$ kg, $m = 45$ kg, $\theta = 45°$, $\phi° = 30$

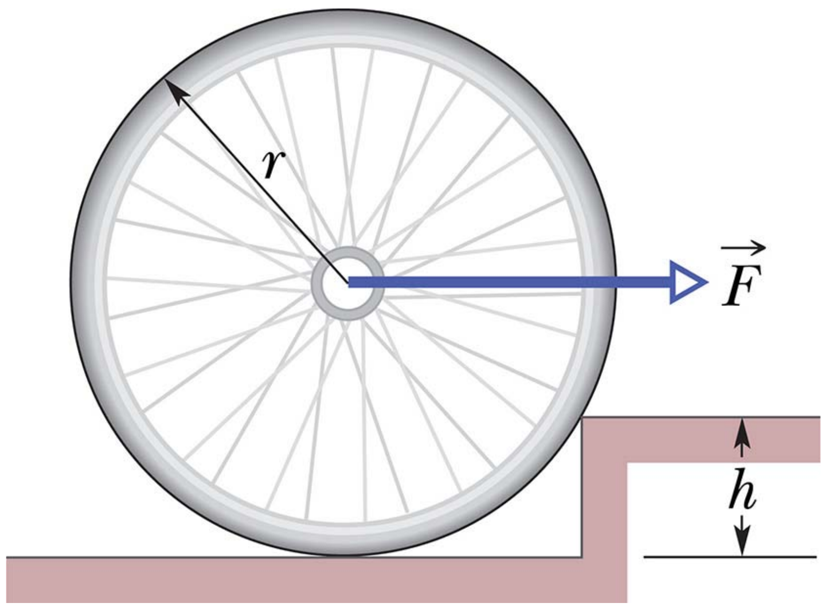


$$\sum F_x = f_1 - T\cos\phi = 0$$
$$\sum F_y = N_1 - mg - Mg - T\sin\phi = 0$$
$$\sum \tau = mg\frac{L}{2}\cos\theta + MgL\cos\theta - TL\sin(\theta - \phi) = 0$$

5.2

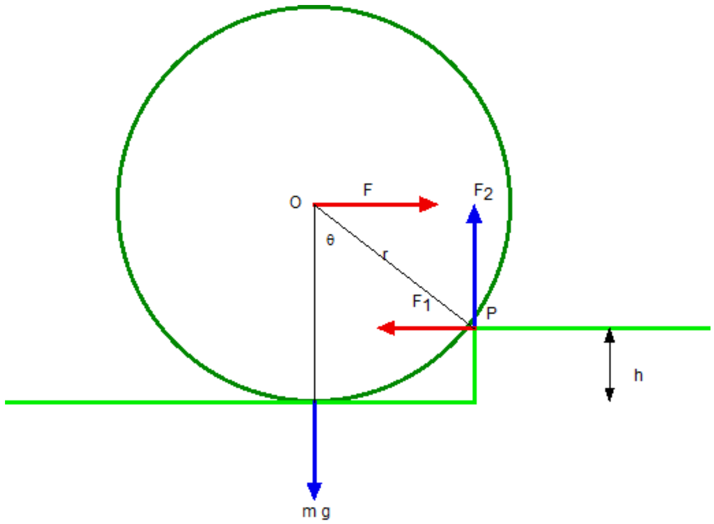
Problem 12-25 (SP-12) (10-th edition)**

In Fig., what magnitude of (constant) force F applied horizontally at the axle of the wheel is necessary to raise the wheel over an obstacle of height h = 3.00 cm? The wheel's radius is r = 6.00 cm, and its mass is m = 0.800 kg.



((**Note**))
We consider the wheel as it leaves the lower floor. The floor no longer exerts a force on the wheel. So there is no friction force on the floor.

h = 3 cm, r = 6 cm, m = 0.8 kg



$$h = r(1-\cos\theta)$$

$$\sum F_x = F - F_1 = 0$$
$$\sum F_y = F_2 - mg = 0$$
$$\sum \tau = Fr\cos\theta - mgr\sin\theta = 0$$

$$F_2 = F\cot\theta = \sqrt{3}F$$
$$F_1 = F$$
$$mg = F\cot\theta$$

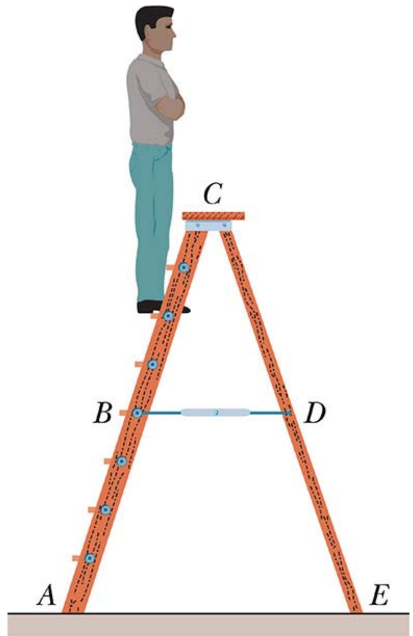
Then we have

$$\boxed{F = \frac{mg}{\cot\theta}}$$

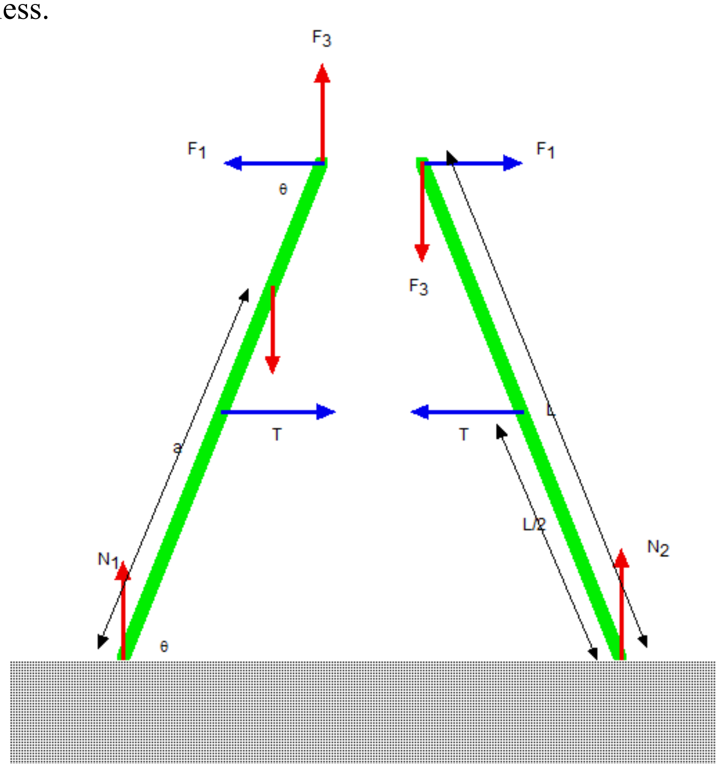
5.3

Problem 12-39* (SP-12) (10-th edition)**

For the stepladder shown in Fig., sides AC and CE are each 2.44 m long and hinged at C. bar BD is a tie-rod 0.762 m long, halfway up. A man weighing 854 N climbs 1.80 m along the ladder. Assuming that the floor is frictionless and neglecting the mass of the ladder, find (a) the tension in the tie-rod and the magnitudes of the forces on the ladder from the floor at (b) A and (c) E. (Hint: isolate parts of the ladder in applying the equilibrium conditions.)



The floor is frictionless.



The left ladder:

$$F_1 - T = 0$$
$$F_3 - mg + N_1 = 0 \qquad (1)$$
$$T\frac{L}{2}\sin\theta - N_1 L\cos\theta + (L-a)mg\cos\theta = 0$$

The right ladder

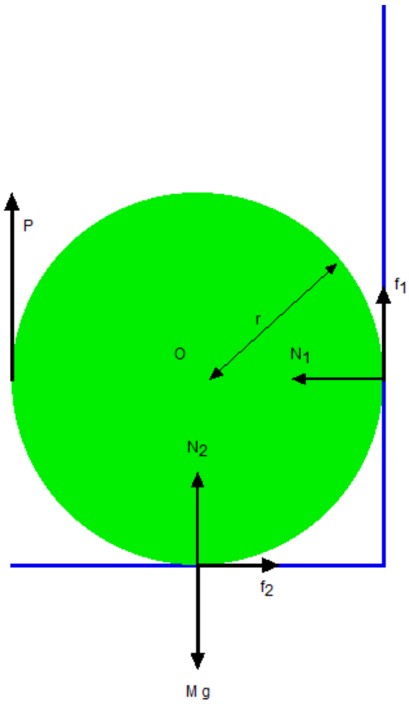
$$N_2 = F_3$$
$$-\frac{L}{2}T\sin\theta + N_2 L\cos\theta = 0, \qquad (2)$$

$$F_1 = \frac{amg}{L}\cot\theta$$
$$F3 = \frac{amg}{2L}$$
$$N_1 = g(m - \frac{am}{2L})$$
$$T = \frac{amg}{L}\cot\theta$$
$$N_2 = \frac{amg}{2L}$$

7. Advanced problems

7.1 Serway 12-60

Figure shows a vertical force applied tangentially to a uniform cylinder of weight F_g. The coefficient of static friction between the cylinder and all surfaces is 0.500. In terms of F_g, find the maximum force P that can be applied that does not cause the cylinder to rotate.



$\mu = 0.5.$

$$\sum F_y = f_1 + P + N_2 - Mg = 0$$
$$\sum F_x = f_2 - N_1 = 0$$
$$\sum \tau = -rP + f_2 r + f_1 r = 0$$

$$f_1 \le \mu N_1$$
$$f_2 \le \mu N_2$$

From the above equations,

$$f_2 = N_1$$
$$f_1 = P - N_1$$
$$-N_1 + N_2 = Mg - 2P$$
$$f_1 \le \mu N_1 = 0.5 N_1$$
$$f_2 \le \mu N_2 = 0.5 N_2$$

or

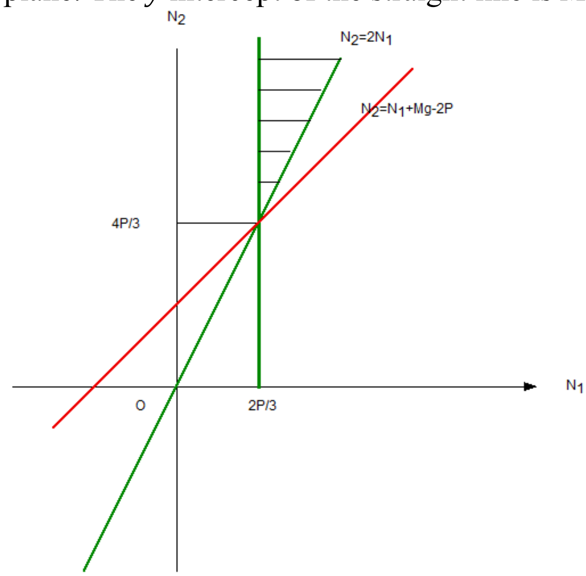
$$f_1 = P - N_1 \le \frac{1}{2} N_1$$

$$f_2 = N_1 \le \frac{1}{2} N_2$$

Then we have

$$\boxed{\begin{gathered} N_2 = N_1 + Mg - 2P \\ \frac{2}{3} P \le N_1 \le \frac{1}{2} N_2 \end{gathered}}$$

We consider the N_2 vs N_1 plane. The y intercept of the straight line is $Mg - 2P$.



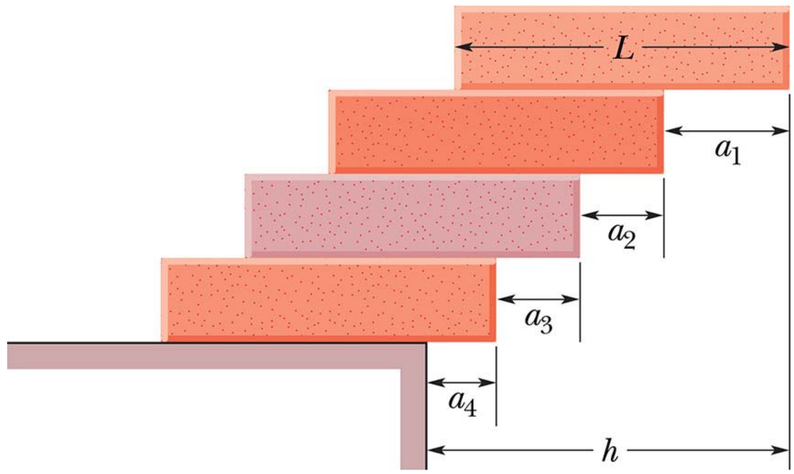
The value of P becomes maximum when $N_1 = 2P/3$ and $N_2 = 4P/3$.

$$P \le \frac{3Mg}{8}$$

7.2

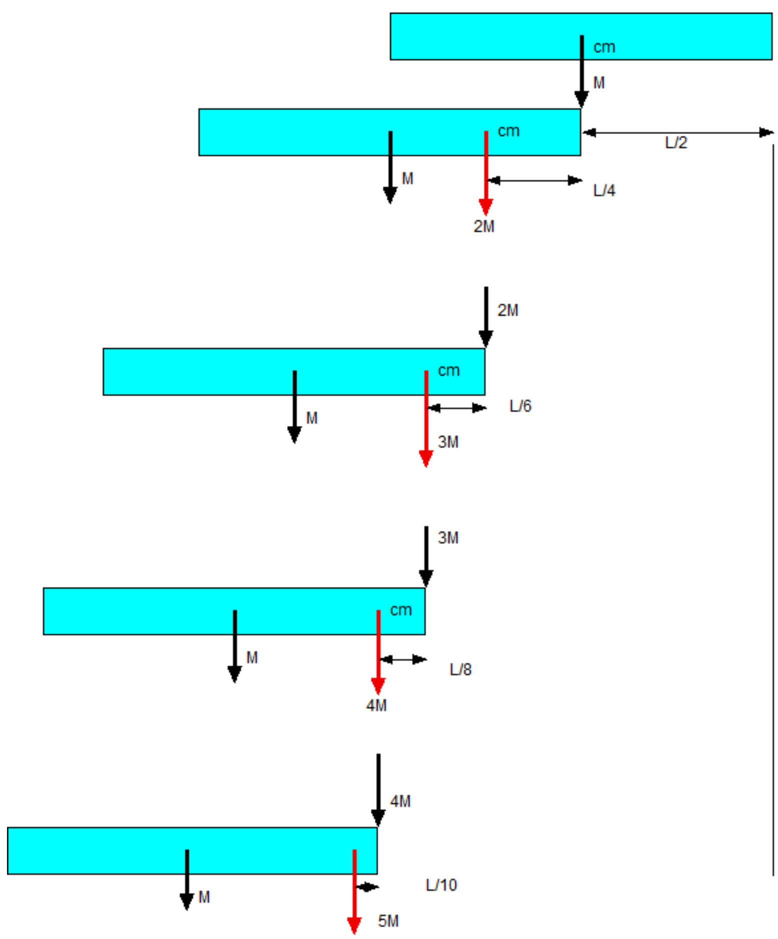
Problem 12-63 (10-th edition)

Four bricks of the length L, identical and uniform, are stacked on top of one another (Fig.) in such a way that part of each extends beyond the one beneath. Find, in terms of L, the maximum values of (a) a_1, (b) a_2, (c) a_3, (d) a_4, and (e) h, such that the stack is in equilibrium.



((**Solution**))

If the center of mass of the top brick is located to the left of the edge of the bottom brick, the system will be in equilibrium.



$$h = \frac{L}{2} + \frac{L}{4} + \frac{L}{6} + \frac{L}{8} = \frac{L}{2}\sum_{k=1}^{4}\frac{1}{k} = \frac{25L}{24}$$

What happens to *h* for many bricks?

((**Mathematica**))

```
In[1]:= h[n_] := L/2 Sum[1/k, {k, 1, n}]
```

$$h[n\_] := \frac{L}{2}\sum_{k=1}^{n}\frac{1}{k}$$

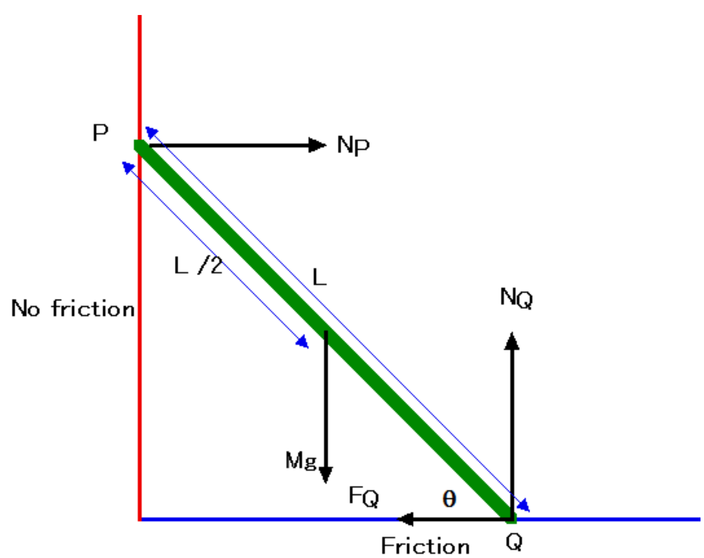
```
In[2]:= Table[{n, h[n]}, {n, 1, 10}] // TableForm
```

Out[2]//TableForm=

1	$\frac{L}{2}$
2	$\frac{3L}{4}$
3	$\frac{11L}{12}$
4	$\frac{25L}{24}$
5	$\frac{137L}{120}$
6	$\frac{49L}{40}$
7	$\frac{363L}{280}$
8	$\frac{761L}{560}$
9	$\frac{7129L}{5040}$
10	$\frac{7381L}{5040}$

7.4 Stability of ladder ((Example given in the lecture of Walter Levin, MIT Classical Mechanics)

http://ocw.mit.edu/courses/physics/8-01-physics-i-classical-mechanics-fall-1999/video-lectures/lecture-25/
9:36 min



What is the critical angle θ_{cr} at the ladder is stable, hanging on the horizontal frictional floor and the vertical frictionless wall?
For the condition of static equilibrium

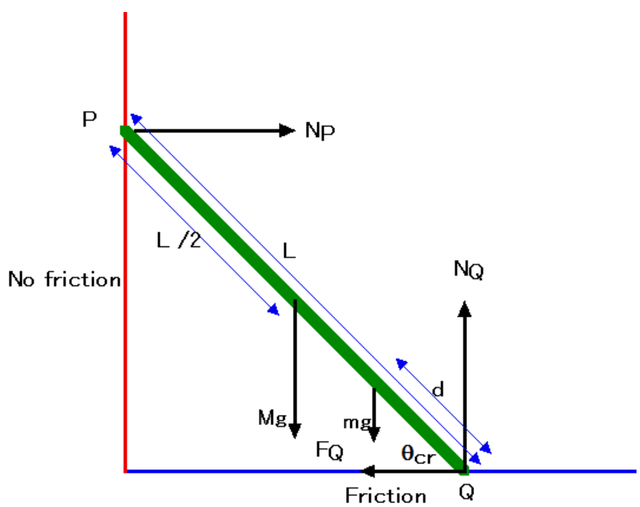
$$\sum F_x = N_P - F_Q = 0$$
$$\sum F_y = N_Q - Mg = 0$$
$$\sum \tau_Q = N_P(L\sin\theta) - Mg(\frac{L}{2}) = 0$$
$$F_Q \le \mu_s N_Q$$

From these equations we have

$$\tan\theta \ge \frac{1}{2\mu_s}$$

$$\tan\theta_{cr} = \frac{1}{2\mu_s} \qquad \text{or} \qquad \cot\theta_{cr} = 2\mu_s$$

Now we assume that $\theta = \theta_{cr}$. Newly we put a mass on the ladder (distance d from the point Q).



For the condition of static equilibrium,

$$\sum F_x = N_P - F_Q = 0$$
$$\sum F_y = N_Q - (M+m)g = 0$$
$$\sum \tau_Q = N_P(L\sin\theta_{cr}) - Mg(\frac{L}{2})\cos\theta_{cr} - mgd\cos\theta_{cr} = 0$$
$$F_Q \le \mu_s N_Q$$

or

$$g\cot\theta_{cr}(\frac{M}{2}+\frac{md}{L})\le\mu_s(M+m)g$$

Since $\cot\theta_{cr}=2\mu_s$, we get

$$2\mu_s(\frac{M}{2}+\frac{md}{L})\le\mu_s(M+m)$$

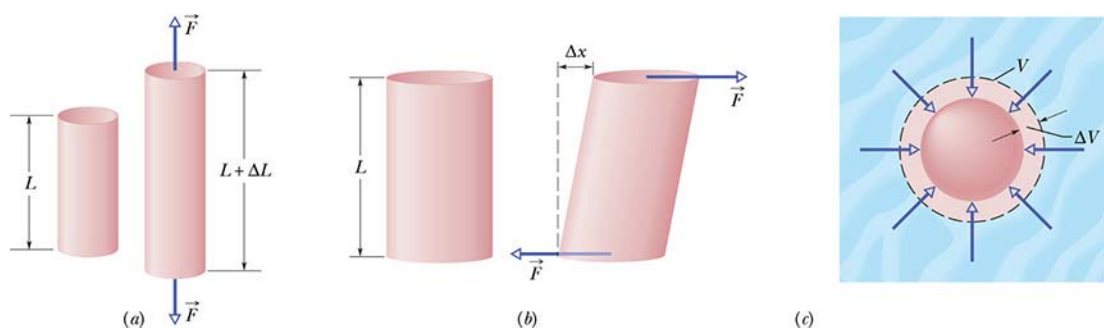
or

$$M+\frac{2md}{L}\le M+m$$

which leads to the condition

$$d\le\frac{L}{2}$$

8 Elasticity



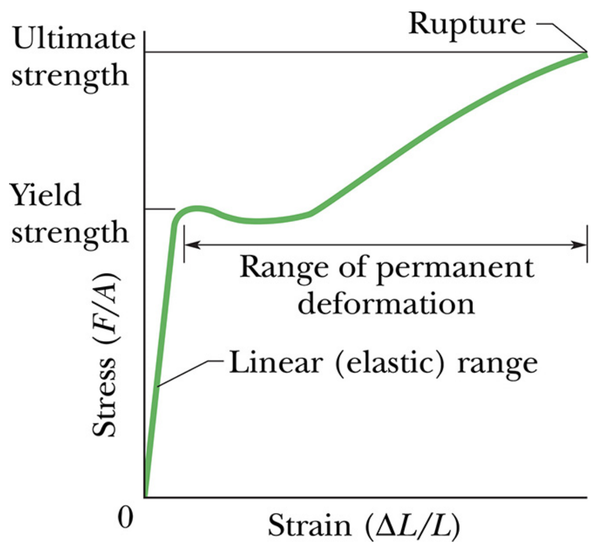
Elasticity describes the deformation of solids and liquids under stress.

8.1 Tensile stress and Young's modulus

F/A: tensile stress
$\Delta L/L$; strain

$$\frac{F}{A}=E\frac{\Delta L}{L}$$

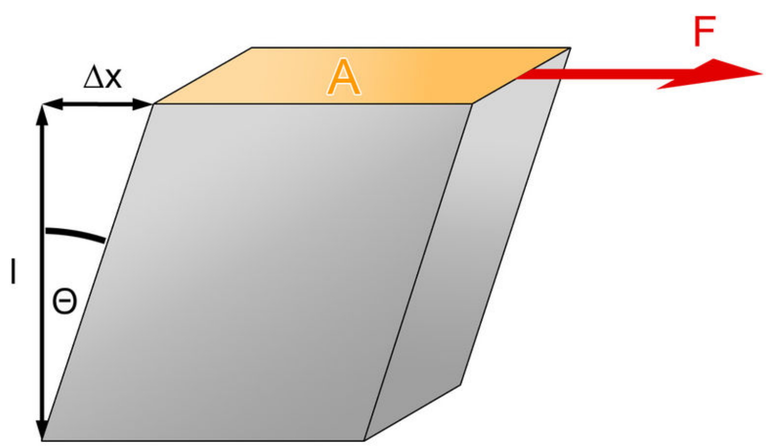
where A is the cross-section area. The proportionality constant is called Young's modulus. It is directly related to the spring constant of the molecular bonds, so it depends on the material from which object is made but not on the object's geometry.



((**Note**))

It is named after the 19th-century British scientist Thomas Young (the name is well known from his Young's double slits experiment). However, the concept was developed in 1727 by Leonhard Euler, and the first experiments that used the concept of Young's modulus in its current form were performed by the Italian scientist Giordano Riccati in 1782, pre-dating Young's work by 25 years. http://en.wikipedia.org/wiki/Young%27s_modulus

8.2 Shear modulus: elasticity of shape



Another type of deformation occurs when an object is subjected to a force parallel to one of its faces while the opposite face is held fixed by another force.

F/A: Shear stress

$\Delta x / L$: Shear strain
G: Shear modulus

$$\frac{F}{A} = G\frac{\Delta x}{L} = G(\Delta\theta)$$

Note:

$$\tan(\Delta\theta) = \sin(\Delta\theta) = \Delta\theta$$

8.3 Volume elasticity: volume stress and the bulk modulus

Young's modulus characterizes the response of an object to be pulling in one direction. We consider an object under a water, which is squeezed from all sides by the water pressure.

F/A: volume stress applied to all surfaces of a object (the volume stress is really the same as the pressure).
$\Delta V / V$ volume strain (it is a negative number because the volume stress decreases the volume)
B: bulk modulus

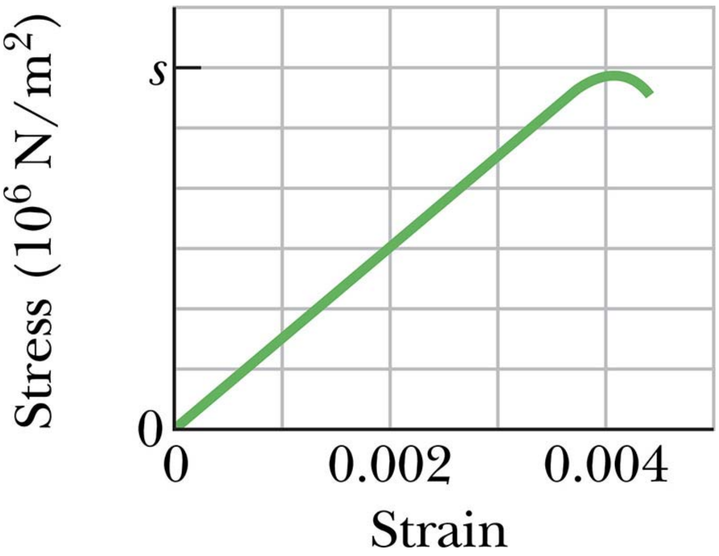
$$\frac{F}{A} = P = -B\frac{\Delta V}{V}$$

9. Example

9.1

Problem 12-44* **(10-th edition)**

Figure shows the stress-strain curve for a material. The scale of the stress axis is set by s = 300, in units of 10^6 N/m^2. What are (a) the Young's modulus and (b) the approximate yield strength for this material?



$$\frac{F}{A} = E\frac{\Delta L}{L}$$

Stress = F/A,   strain = $\Delta L/L$,

(a) The Young's modulus is given by

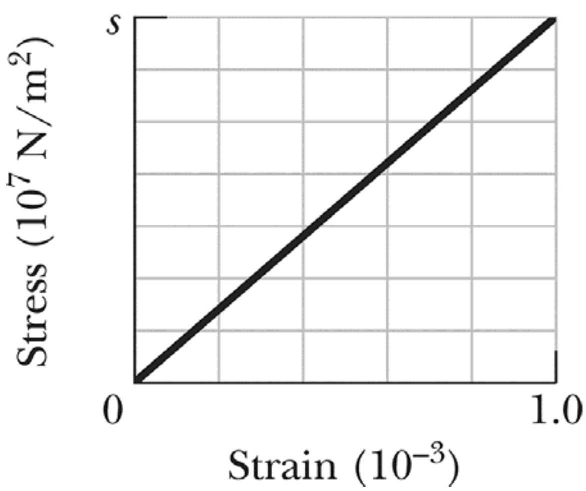
$$E = \frac{\text{stress}}{\text{strain}} = \text{slope of the stress-strain curve} = \frac{150\times10^6\,\text{N/m}^2}{0.002} = 7.5\times10^{10}\,\text{N/m}^2.$$

(b) Since the linear range of the curve extends to about 2.9×10^8 N/m^2, this is approximately the yield strength for the material.

9.2

Problem 12-48   (10-th edition)**

Figure shows the stress versus strain for an aluminum wire that is stretched by a machine pulling in opposite directions at the two ends of the wire. The scale of the stress axis is set by s = 7.0, in units of 10^7 N/m^2. The wire has an initial length of 0.800 m and an initial cross-sectional area of 2.00 x 10^{-6} m^2. How much work does the force from the machine do on the wire to produce a strain of 1.00 x 10^{-3}?



A = 2.00 x 10^{-6} m^2
L = 0.800 m.
maximum strain $s_{max} = x_{max}/L$ = 1.0 x 10^{-3}
maximum stress $S_{max} = F_{max}/A$ = 7.0 x 10^7 N/m^2

The Young modulus E is

$$E = \frac{S_{\max}}{s_{\max}} = \frac{7.0\times10^{7}}{1.0\times10^{-3}} = 7.0\times10^{10}\,Pa = 70GPa$$

Where Pa = N/m^2, GPa=10^9 N/m^2.

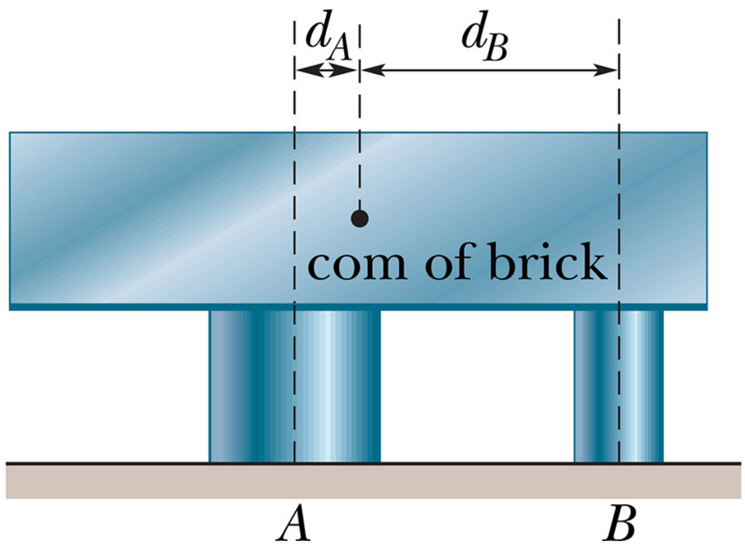
$$S = \frac{F}{A} = E\frac{\Delta L}{L}$$
$$x = \Delta L$$
$$F = \frac{AE}{L}x$$

Then the work is given by

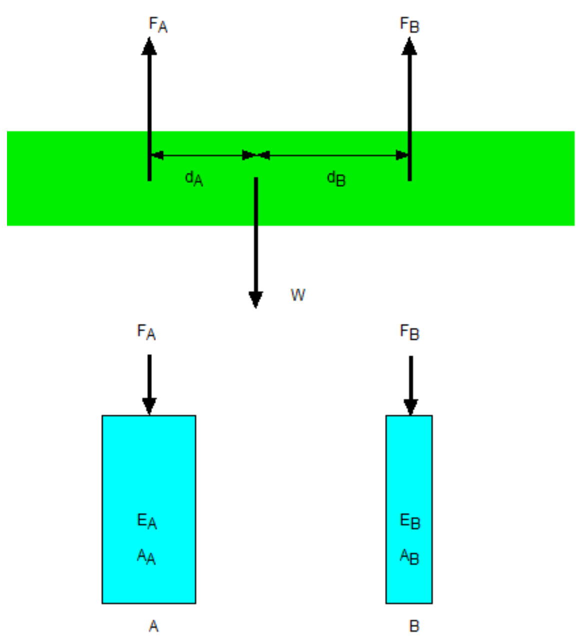
$$dW = Fdx = (\frac{AE}{L}x)dx = \frac{AE}{L}xdx$$
$$W = \int dW = \int_0^{x_{\max}} \frac{AE}{L}xdx = \frac{AE}{L}\frac{1}{2}x_{\max}{}^2 = \frac{AE}{L}\frac{1}{2}(Ls_{\max})^2 = \frac{1}{2}AELs_{\max}{}^2 = 0.056J$$

9.3
Problem 12-45**   **(10-th edition)**

In Fig, a lead brick rests horizontally on cylinders A and B. The areas of the top faces of the cylinders are related by $A_A = 2A_B$; the Young's moduli of the cylinders are related by $E_A = 2\ E_B$. The cylinders had identical lengths before the brick was placed on them. What fraction of the brick's mass is supported (a) by cylinder A and (b) by cylinder B? The horizontal distance between the center of mass of the brick and the centerlines of the cylinders are d_A for cylinder A and d_B for cylinder B. (c) What is the ratio d_A/d_B?



$E_A = 2\ E_B$, $A_A = 2\ A_B$,



$$F_A + F_B = W$$
$$d_A F_A = d_B F_B$$
$$\frac{F_A}{A_A} = E_A \frac{\Delta l}{l}$$
$$\frac{F_B}{A_B} = E_B \frac{\Delta l}{l}$$

From these equations, we have

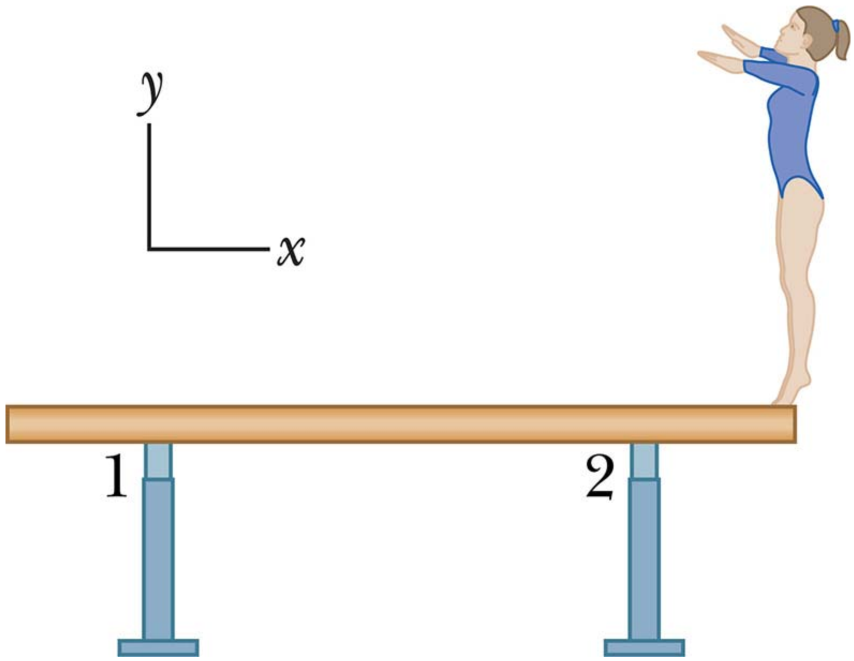
$$\frac{F_A}{F_B} = 4$$
$$F_A = 0.8W$$
$$F_B = 0.2W$$
$$\frac{d_A}{d_B} = \frac{1}{4}$$

10 HW-12 and SP-12

10.1.

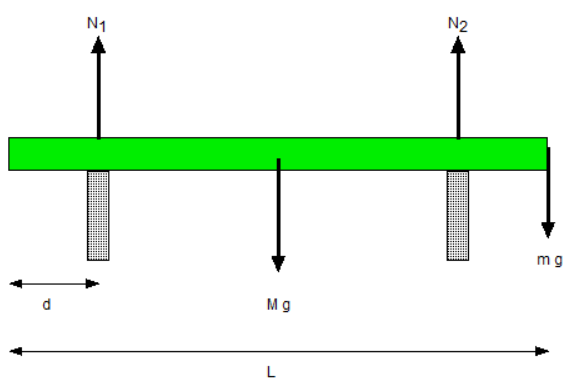
Problem 12-76 (HW-12) (10-th edition)

A gymnast with mass 46.0 kg stands on the end of a uniform balance beam as shown in Fig. The beam is 5.00 m and has a mass of 250 kg (excluding the mass of the two supports). Each support is 0.540 m from its end of the beam. In unit-vector notation, what are the forces on the beam due to (a) support 1 and (b) support s?



((**Solution**))

$L = 5.0$ m, $d = 0.54$ m, $M = 250$ kg, $m = 46.0$ kg,



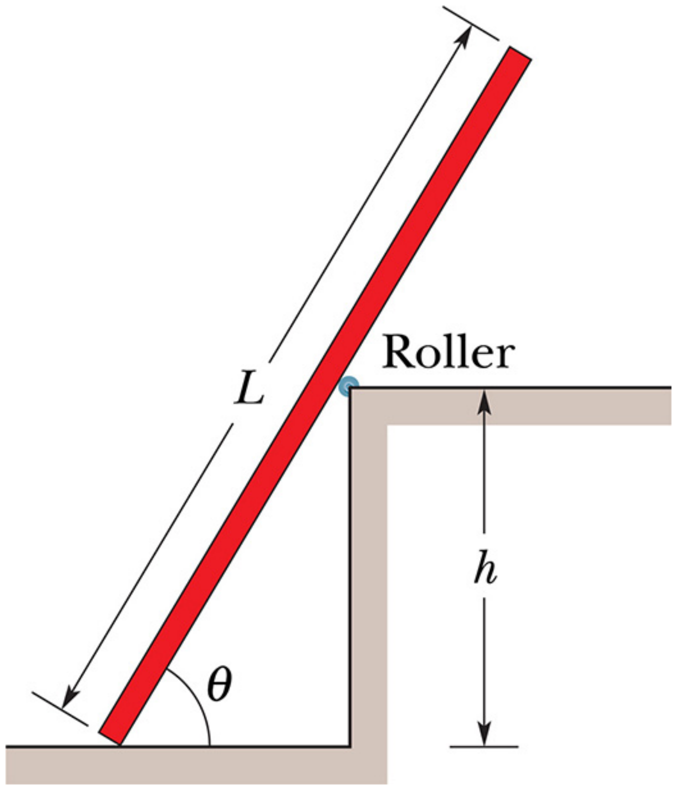
$$\sum F_y = N_1 + N_2 - Mg - mg = 0$$

$$\sum \tau = N_1 d - \frac{L}{2} Mg + N_2 (L - d) - mgL = 0$$

10.2

Problem 12-37 (HW-12) (10-th edition)

In Fig. a uniform plank, with a length L of 6.10 m and a weight of 445 N, rests on the ground and against a frictionless roller at the top of a wall of height h = 3.05 m. The plank remains in equilibrium for any value of $\theta \geq 70°$ but slips if $\theta < 70°$. Find the coefficient of static friction between the plank and the ground.

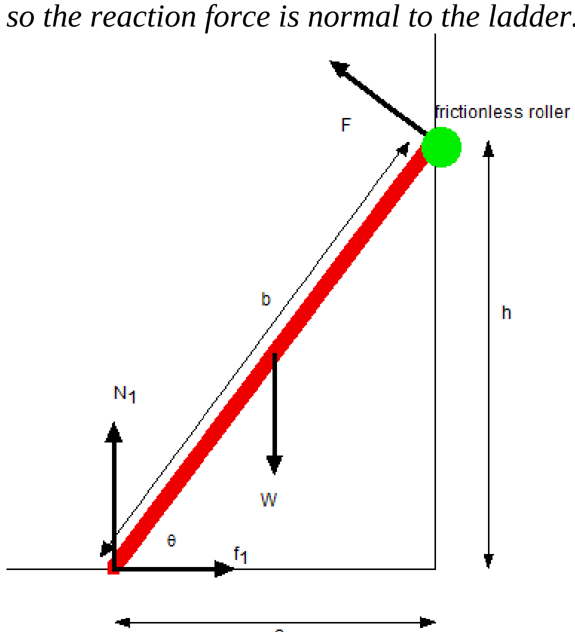


((**Solution**))

$L = 6.10$ m, $W = 445$ N, $h = 2.05$ m

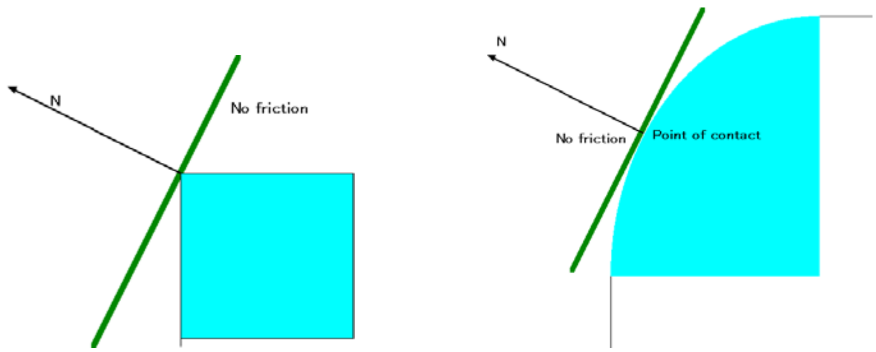
$$a = \frac{h}{\tan\theta}, \qquad b = \frac{h}{\sin\theta}$$

The roller is free to rotate so the reaction force is normal to the ladder.



((Note)) Direction of the normal force at the edge with no friction

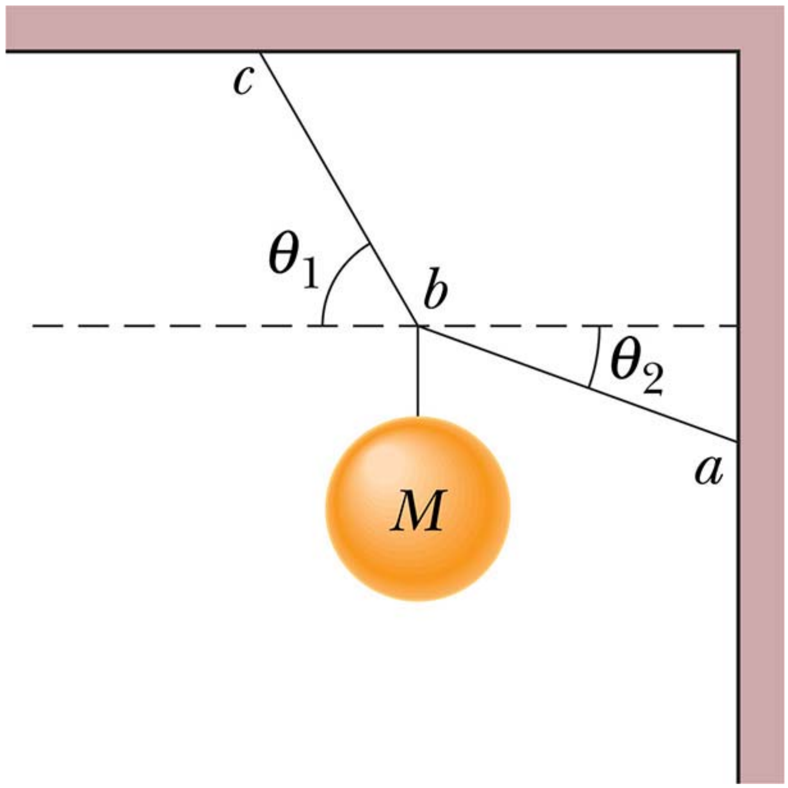
We assume that there is no friction between the edge of table and the rod. Even for the edge with right angle in the table, on a microscopic scale (see Fig.), the direction of the normal force is perpendicular to the tangential line at the point (the same as the axis of the rod) where the rod contacts with the table surface.



10.3

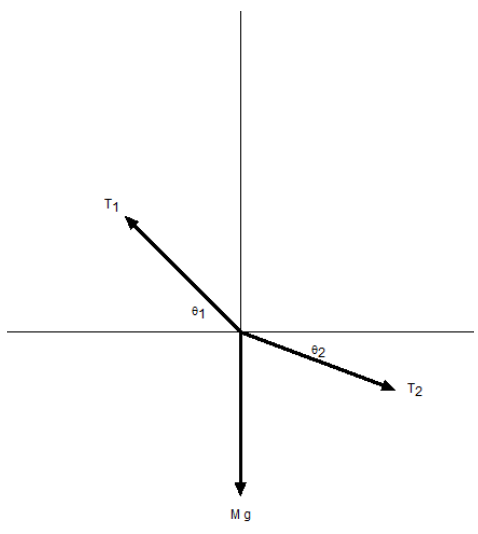
Problem 12-72 (HW-12) (10-th edition)

The system in Fig. is in equilibrium. The angles are θ_1 = 60º and θ_2 = 20º, and the ball has mass M = 2.0 kg. What is the tension in (a) string ab and (b) string bc?



((**My solution**))

$\theta_1 = 60°$, $\theta_2 = 20°$, $M = 2.0$ kg



11. Link

Young modulus and shear modulus
http://www.youtube.com/watch?v=jqFoHaGtkig&feature=related