Chapter 13

1 Introduction

(a) Newton's law of gravitation
The attractive force between two point masses and its application to extended objects
(b) The acceleration of gravity on the surface of the earth, above it, as well as below it
(c) Gravitational potential energy outside and inside the Earth
(d) Satellites (orbits, energy , escape velocity)
(e) Kepler's three laws on planetary motion
(f) Bohr model for the electron in the hydrogen atom
(g) Black-hole

2 Newton's law of universal gravitation

2.1 Inverse-square law

Every particle in the Universe attracts every other particle with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them

$$F = G\frac{m_1 m_2}{r^2}$$

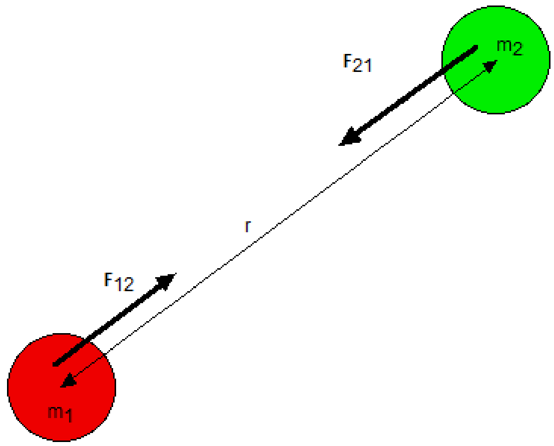
where G is the universal gravitational constant.

$G = 6.6742867 \times 10^{-11}\ \mathrm{Nm^2kg^{-2}}$

This is an example of an inverse square law; the magnitude of the force varies as the inverse square of the separation of the particles. The law can also be expressed in vector form

$$\boldsymbol{F}_{12} = -G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}_{12}, \qquad \boldsymbol{F}_{21} = G\frac{m_1 m_2}{r^2}\hat{\boldsymbol{r}}_{12}$$

The forces form a Newton's Third Law action-reaction pair. Gravitation is a field force that always exists between two particles, regardless of the medium between them. The force decreases rapidly as distance increases.

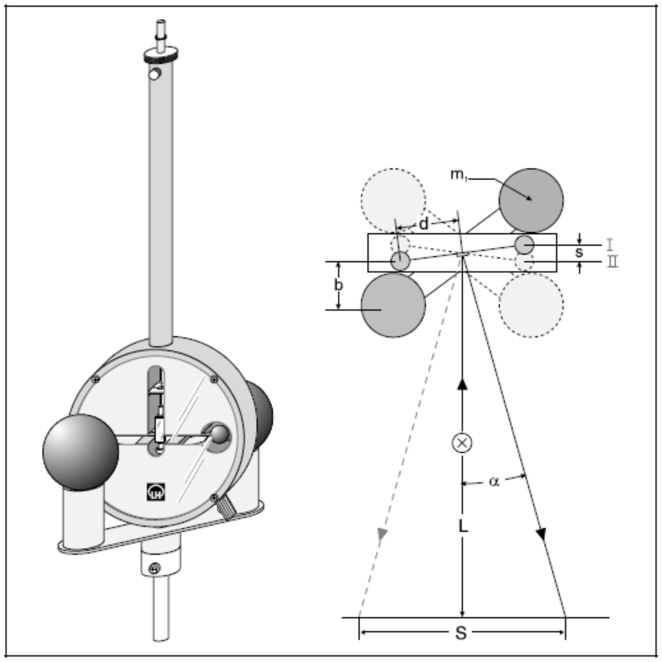


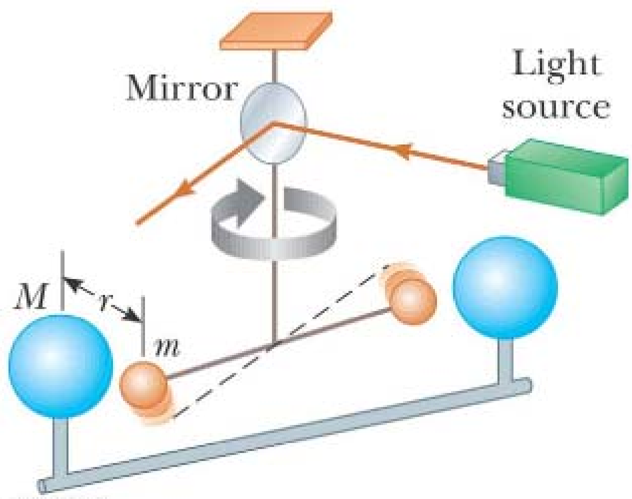
$\boldsymbol{F}_{12}$ is the force exerted by particle 1 on particle 2. The negative sign in the vector form of the equation indicates that particle 2 is attracted toward particle 1. $\boldsymbol{F}_{21}$ is the force exerted by particle 2 on particle 1

2.2 Cavendish balance

Phys.427/527 Senior Lab and Graduate Lab of Physcs

Henry Cavendish (1731 – 1810) measured the universal gravitational constant in an important *1798 experiment*. Cavendish apparatus consists of two small spheres, each of mass m, fixed to the ends of a light, horizontal rod suspended by a fine fiber or thin metal wire. When two large spheres, each mass *M*, are placed near the smaller ones, the attractive force between smaller and larger spheres causes the rod to rotate and twist the wire suspension to a new equilibrium orientation. The angle of rotation is measured by the deflection of a light beam reflected from a mirror attached to the vertical suspension.





The strength of the gravitational force depends on the value of *G*. The value of the gravitational constant can be determined using the Cavendish apparatus. Two small lead spheres of mass *m* are connected to the end of a rod of length *L* which is suspended from it midpoint by a fine fiber, forming a torsion balance. Two large lead spheres, each of mass *M*, are placed in the location indicated in Figure. The lead spheres will attract each other, exerting a torque on the rod. In the equilibrium position the gravitational torque is just balanced by the torque exerted by the twisted fiber. The torque exerted by the twisted wire is given by

$$\tau = \kappa\theta$$

The torque exerted by the gravitational force is given by

$$\tau = \frac{L}{2}F_g + \frac{L}{2}F_g = LF_g = L\frac{GmM}{R^2}$$

where R is the equilibrium distance between the center of the large and the small spheres. If the system is in equilibrium, the net torque acting on the rod is zero. Thus

$$L\frac{GmM}{R^2} = \kappa\theta$$

All of a sudden the large spheres are rotated to a new position (position B in Figure). The net torque acting on the twisted fiber is now not equal to zero, and the system will start to oscillate. The period of oscillation is related to the rotational inertia I and the torsion constant κ

$$T = 2\pi\sqrt{\frac{I}{\kappa}}$$

The angle between the two equilibrium positions is measured to be 2θ. This, combined with the measured torsion constant, is sufficient to determine the torque τ acting on the torsion balance due to the gravitational force. Measurements show that $G = 6.67 \times 10^{-11}$ Nm^2/kg^2.

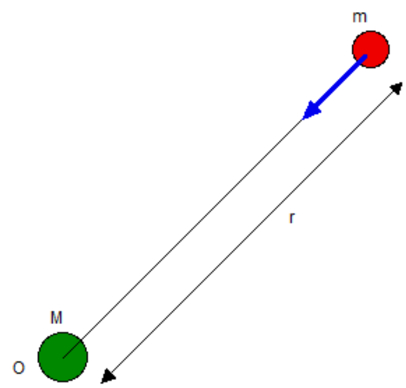
Link: see the article at the URL
http://www.leydenscience.org/physics/gravitation/cavend.htm

3 The potential energy

The attractive force (conservative) is given by

$$\boldsymbol{F}(r) = -\frac{GMm}{r^2}\hat{r}$$

This force is called a *central force*, since the direction of the force is radial.



We now consider the potential energy U defined by

$$\boldsymbol{F} = -\nabla U = -\frac{\partial U}{\partial \boldsymbol{r}} = -\frac{\partial U(r)}{\partial r}\hat{r}$$

or

$$\frac{dU(r)}{dr} = \frac{GMm}{r^2}$$

Then we have

$$U = -\int^{r} \boldsymbol{F}\cdot d\boldsymbol{r} = -\int^{r} \boldsymbol{F}\cdot \hat{r}dr = -\int^{r} (-\frac{GMm}{r^2}\hat{r})\cdot \hat{r}dr = \int^{r} \frac{GMm}{r^2}dr = -\frac{GMm}{r} + C$$

Here we choose $U = 0$ at $r = \infty$. Then we have final form of U as

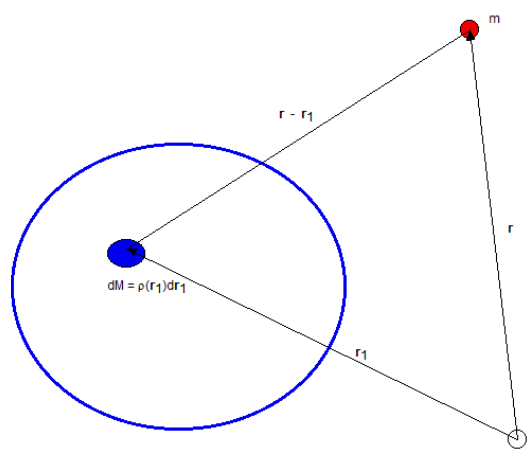
$$U = -\frac{GMm}{r}$$

Note that the sign of the *attractive potential is negative*.

In general, the potential energy of a localized mass distribution is given by

$$U(\boldsymbol{r}) = -Gm\int \frac{\rho(\boldsymbol{r}_1)}{|\boldsymbol{r} - \boldsymbol{r}_1|}d\boldsymbol{r}_1$$

where $\rho(\mathbf{r}_1)$ is the mass density at $\boldsymbol{r}_1$ and $d\mathbf{r}_1$ is the volume element.

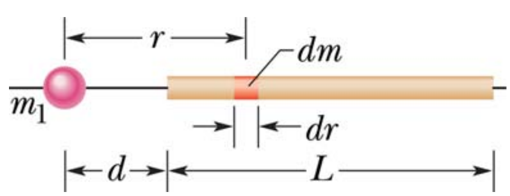


4. Typical calculations of gravitational forces and potentials

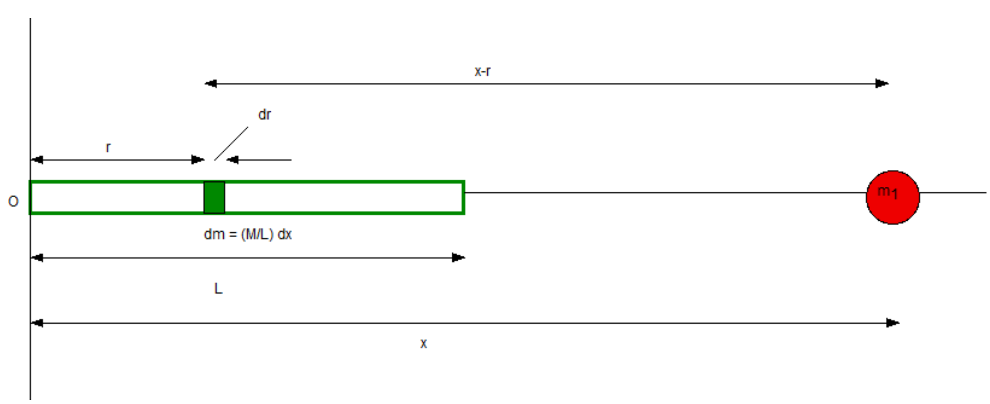
4.1 Example

Problem 13-16* (SP-13) (10-th edition)**

In Fig., a particle of mass is a distance cm from one end of a uniform rod with length and mass. What is the magnitude of the gravitational force on the particle from the rod?



((**Solution**))
For simplicity, we change this figure into the following figure.



<u>Calculation of the force</u>

$$dm = \frac{M}{L} dr \, .$$

The direction of the resultant force is along the positive x axis.

$$F_x = \int dF = -\int \frac{Gm_1 dm}{(x-r)^2} = -\int_0^L \frac{Gm_1}{(x-r)^2} \frac{M}{L} dr = -\frac{Gm_1 M}{L} \int_0^L \frac{1}{(x-r)^2} dr$$
$$= -\frac{Gm_1 M}{L} (\frac{1}{x-L} - \frac{1}{x})$$

$$F_x \mid_{x=d+L} = -\frac{GMm_1}{d(d+L)}$$

Calculation of the potential energy

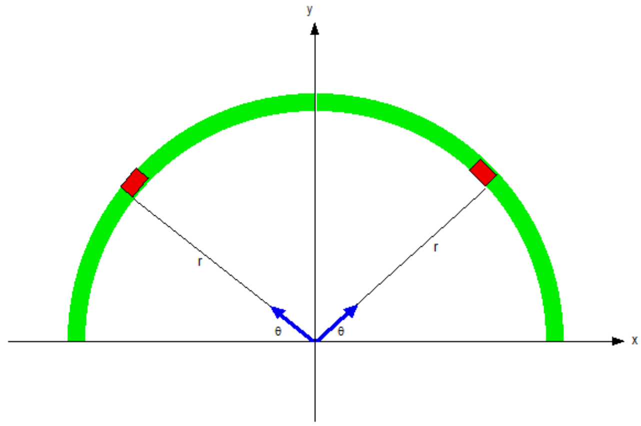
The potential energy U is given by

$$U = \int dU = -\int \frac{Gm_1 dm}{x-r} = -\int_0^L \frac{Gm_1}{x-r} \frac{M}{L} dr = \frac{Gm_1 M}{L} \ln(\frac{x-L}{x})$$

$$F_x = -\frac{dU}{dx} = -\frac{Gm_1 M}{L} (\frac{1}{x-L} - \frac{1}{x}) = -\frac{Gm_1 M}{x(x-L)}$$
$$F_x \mid_{x=L+d} = -\frac{Gm_1 M}{d(d+L)}$$

3.2 Gravitational force from a semicircle-shaped mass

Mass M is distributed uniformly over a semicircle of radius r. Find the gravitational force (magnitude and direction) between this semicircle mass and a particle of mass m located at the center of the semicircle.



The line density λ is

$$\lambda = \frac{M}{\pi r}$$

$$dm = \lambda r d\theta = \frac{M}{\pi r} r d\theta = \frac{M}{\pi} d\theta$$

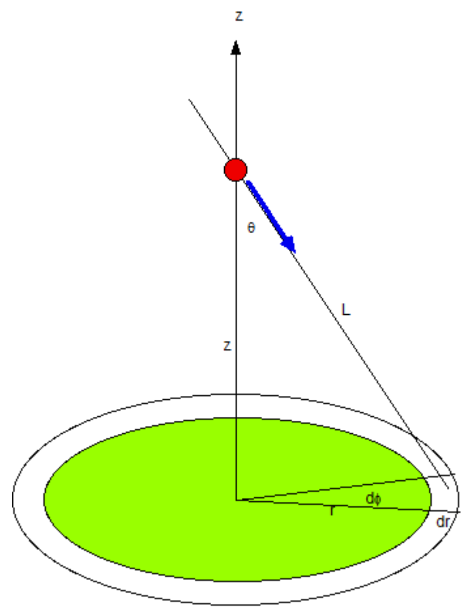
Calculation of the force for the particle with mass m_0 at the origin.

$$\begin{aligned}
F_y &= \int dF_y \\
&= \int 2\frac{Gm_0 dm}{r^2}\sin\theta \\
&= \frac{2GMm_0}{\pi r^2}\int_0^{\pi/2}\sin\theta d\theta \\
&= \frac{2GMm_0}{\pi r^2}[-\cos\theta]_0^{\pi/2} \\
&= \frac{2GMm_0}{\pi r^2}
\end{aligned}$$

$F_x = 0$ from the symmetry.

3.3 Gravitational force from a disk-shaped mass

Mass M is distributed uniformly over a disk of radius a. Find the gravitational force (magnitude and direction) between this disk-shaped mass and a particle of mass m located a distance z above the center of the disk.



Calculation of the force

In this figure

$$\cos\theta = \frac{z}{L}$$
$$M = \sigma\pi R^2$$
$$L = \sqrt{z^2 + r^2}$$

$$F_z = \int dF_z = -\iint \cos\theta \frac{Gm}{L^2}\sigma r d\phi dr = -\iint \frac{Gmz}{L^3}\frac{M}{\pi R^2} r d\phi dr = -\frac{2GmMz}{R^2}\int_0^R \frac{r}{(z^2 + r^2)^{3/2}} dr$$

or

$$F_z = -\frac{2GmMz}{R^2}\left(\frac{1}{z} - \frac{1}{\sqrt{z^2 + R^2}}\right) = -\frac{2GmM}{R^2}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right)$$

In the limit of $z\rightarrow\infty$,

$$F_z = -\frac{2GmMz}{R^2}\left(\frac{R^2}{2z^3}\right) = -\frac{GmM}{z^2}$$

We make a plot of

$$F_z = -\frac{2GmM}{R^2}\left(1 - \frac{z}{\sqrt{z^2 + R^2}}\right) = -\frac{2GmM}{R^2}\left(1 - \frac{\frac{z}{R}}{\sqrt{\frac{z^2}{R^2} + 1}}\right)$$

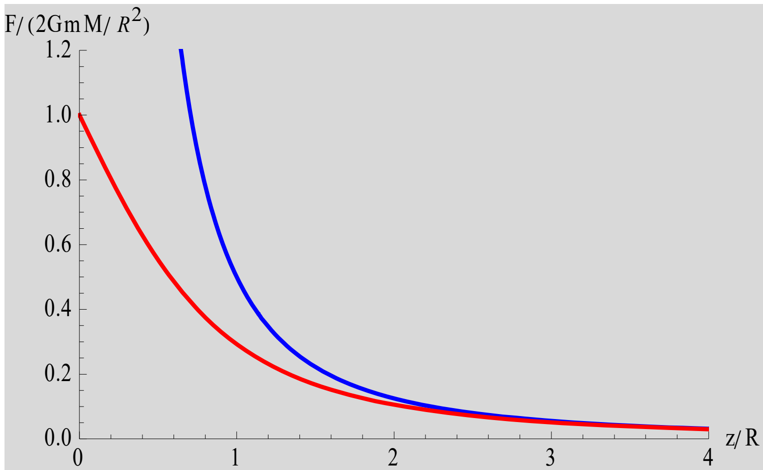
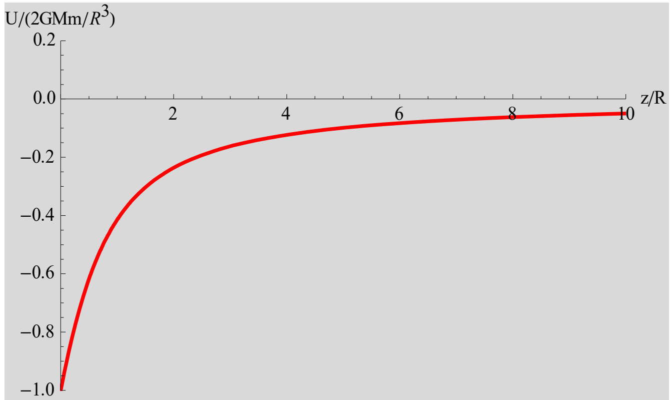


Fig. Red line for the force from the disk. Blue line for the force from a particle with mass *m* at *z* = 0.

Calculation of the potential energy

$$\begin{aligned}
U(z) &= -\iint \frac{Gm}{L}\sigma r d\phi dr \\
&= -\iint \frac{Gm}{L}\frac{M}{\pi R^2} r d\phi dr \\
&= -\frac{2GmM}{R^2}\int_0^R \frac{r}{(z^2 + r^2)^{1/2}} dr \\
&= -\frac{2GmM}{R^2}[\sqrt{R^2 + z^2} - z] \\
&= -\frac{2GmM}{R^3}\left(\sqrt{1 + \frac{z^2}{R^2}} - \frac{z}{R}\right)
\end{aligned}$$



The force F_z is obtained as

$$F_z = -\frac{dU(z)}{dz} = -\frac{2GmM}{R^2}[1 - \frac{z}{\sqrt{R^2+z^2}}]$$

((**Mathematica**))

```
∫_0^R r/(z^2 + r^2)^(3/2) ⅆr //
 Simplify[#, {R > 0, z > 0}] &
```

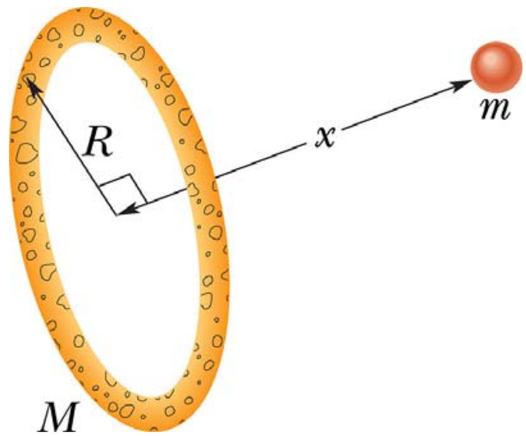
$$\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}$$

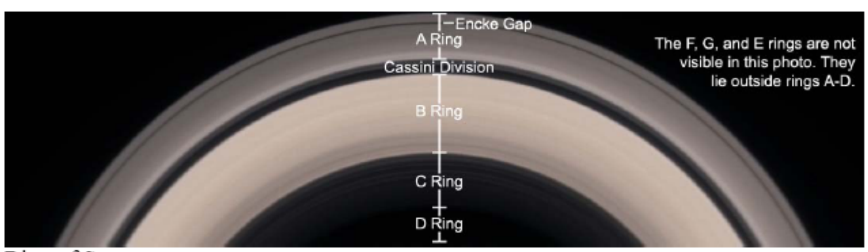
```
∫_0^R r/(z^2 + r^2)^(1/2) ⅆr //
 Simplify[#, {R > 0, z > 0}] &
```

$$-z + \sqrt{R^2+z^2}$$

3.4 Gravitational force from the planet

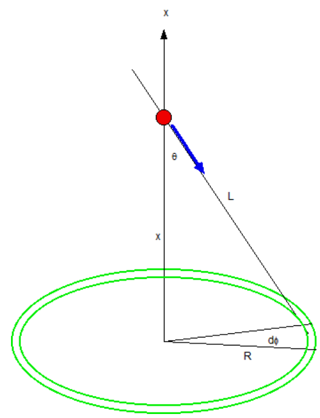
Several planets (Jupiter, Saturn, Uranus) are encircled by rings, perhaps composed of material that failed to form a satellite. In addition, many galaxies contain ring-like structures. Consider a homogeneous thin ring of mass M and outer radius R (Fig.). (a) What gravitational attraction does it exert on a particle of mass m located on the ring's central axis a distance x from the ring center? (b) Suppose the particle falls from rest as a result of the attraction of the ring of matter. What is the speed with which it passes through the center of the ring?





Ring of Saturn

Ring of Saturn
Link: see the article at the URL
http://en.wikipedia.org/wiki/Rings_of_Saturn



In this figure

$$\cos\theta = \frac{x}{L}$$
$$M = \lambda(2\pi R)$$
$$L = \sqrt{x^2 + R^2}$$

First we calculation the potential energy

$$U = \int dU = -\int \frac{Gm}{L}\lambda R d\phi = -\frac{Gm}{L}\frac{M}{2\pi R}2\pi R = -\frac{GMm}{L} = -\frac{GMm}{(x^2 + R^2)^{1/2}}$$

The total energy E is given by

$$E = \frac{1}{2}mv(x)^2 + U(x)$$

The energy conservation law:

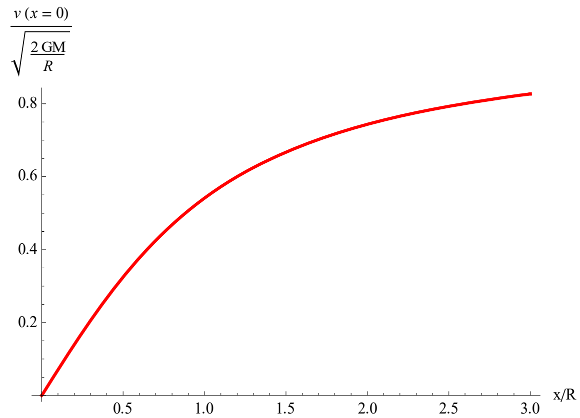
$$E = \frac{1}{2}mv(x=0)^2 + U(x=0) = \frac{1}{2}mv(x)^2 + U(x)$$

When $v(x) = 0$, then we have

$$\frac{1}{2}mv(x)^2 = U(x) - U(x=0) = -\frac{GMm}{(x^2 + R^2)^{1/2}} + \frac{GMm}{R}$$

or

$$v(x) = \sqrt{\frac{2GM}{R}}[1 - \frac{R}{(x^2 + R^2)^{1/2}}]^{1/2}$$



((Note)) Calculation of the force

$$F_x = -\frac{dU}{dx} = \frac{d}{dx}[\frac{GMm}{(x^2 + R^2)^{1/2}}] = -\frac{GMmx}{(x^2 + R^2)^{3/2}}$$

((Note)) Direct calculation

$$F_x = \int dF_x = -\int \cos\theta \frac{Gm}{L^2} \lambda R d\phi = -\frac{Gmx}{L^3}\frac{M}{2\pi R} 2\pi R = -\frac{GMmx}{L^3} = -\frac{GMmx}{(x^2 + R^2)^{3/2}}$$

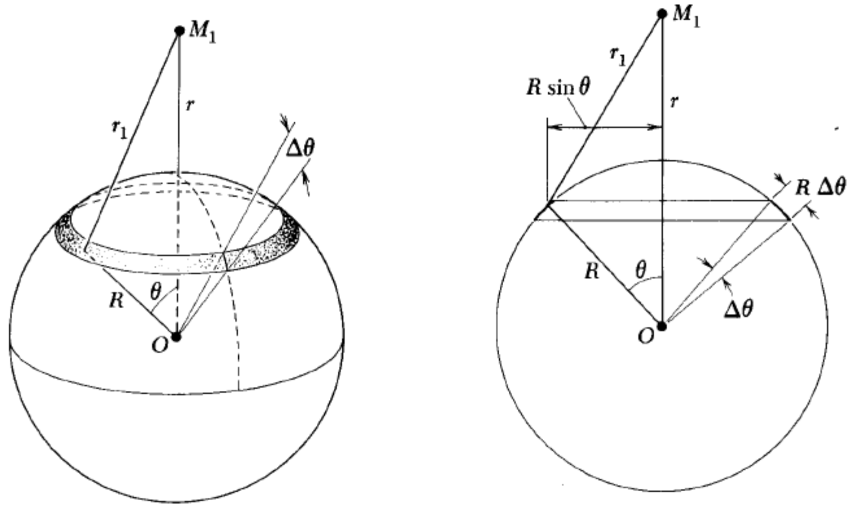
4. Potential energy and force between a point mass and a solid shell

4.1 The potential energy outside a shell

The force on a point test mass m (= M_1 in the Fig.) distant from the center of a uniform thin spherical shell of radius R is exactly the same at points $r>R$ out side the shell as if the entire mass of the shell were concentrated its center. For points $r<R$ inside the shell the force on the point mass is zero.

Let s be the mass per unit area of the shell. The total mass of the ring is

$$\Delta M = 2\pi R \sin\theta (R\Delta\theta)\sigma = 2\pi R^2 \sigma \sin\theta \Delta\theta$$



The potential energy ΔU of the test mass ($M_1 = m$) is obtained as

$$\Delta U = -\frac{Gm\Delta M}{r_1} = -\frac{Gm(2\pi R^2\sigma \sin\theta\Delta\theta)}{r_1}$$

where r_1 is the distance between the test mass and the ring,

$$r_1^2 = r^2 + R^2 - 2rR\cos\theta$$

Since

$$2r_1\Delta r_1 = 2rR\sin\theta\Delta\theta$$

$$\Delta U = -\frac{Gm(2\pi R^2\sigma)}{r_1}\frac{r_1\Delta r_1}{rR} = -\frac{Gm(2\pi R\Delta r_1\sigma)}{r}$$

The total potential energy U is

$$U = -\int_{r-R}^{r+R}\frac{Gm(2\pi R\sigma)}{r}dr_1 = -\frac{Gm(2\pi R\sigma)}{r}2R = -\frac{Gm(4\pi R^2\sigma)}{r} = -\frac{GmM}{r}$$

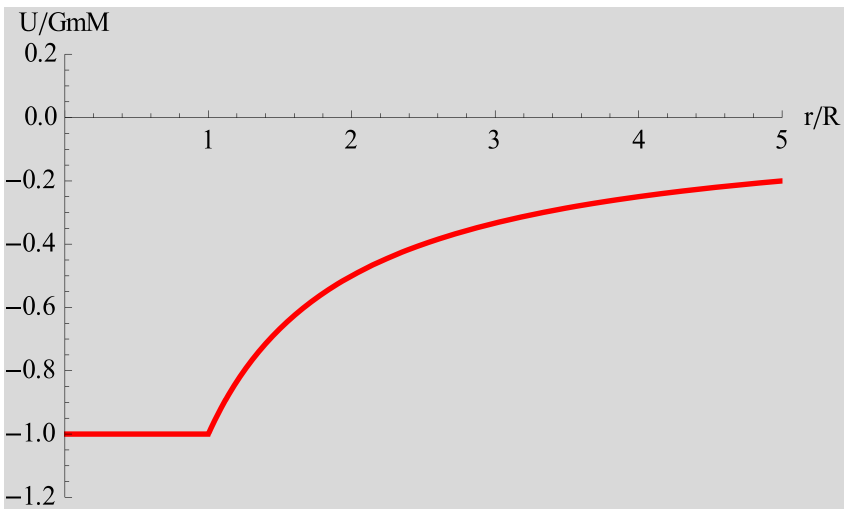
where $M = 4\pi R^2\sigma$.

4.2 The potential energy and the force inside a shell

If the test charge lies anywhere within the shell, the derivation is identical except that the range of summation of Δr_1 in U is from R - r to $R + r$.

$$U = -\int_{R-r}^{R+r} \frac{Gm(2\pi R\sigma)}{r} dr_1 = -\frac{Gm(2\pi R\sigma)}{r} 2r = -Gm(4\pi R\sigma)$$
$$= -Gm(4\pi R)\frac{M}{4\pi R^2}$$
$$= -\frac{GMm}{R}$$

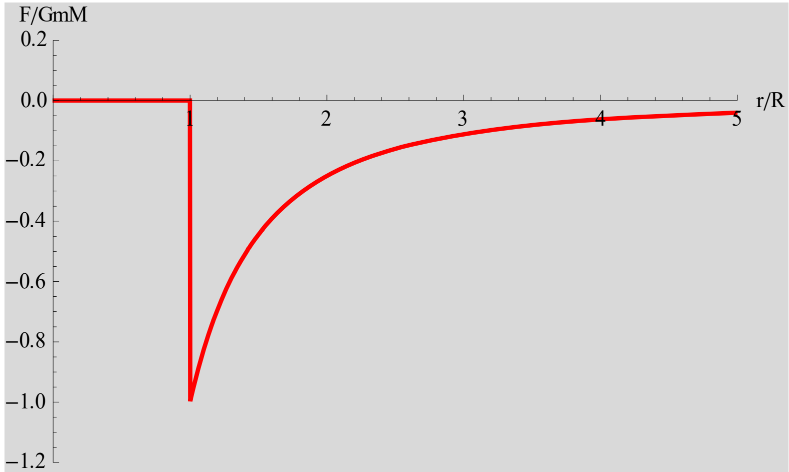
U is independent of r.



From the definition, the force F is obtained as

$$F = -\frac{\partial U}{\partial r} = -\frac{GMm}{r^2} \quad (r > R)$$
$$F = -\frac{\partial U}{\partial r} = 0 \quad (r < R)$$

for the spherical shell with radius R.



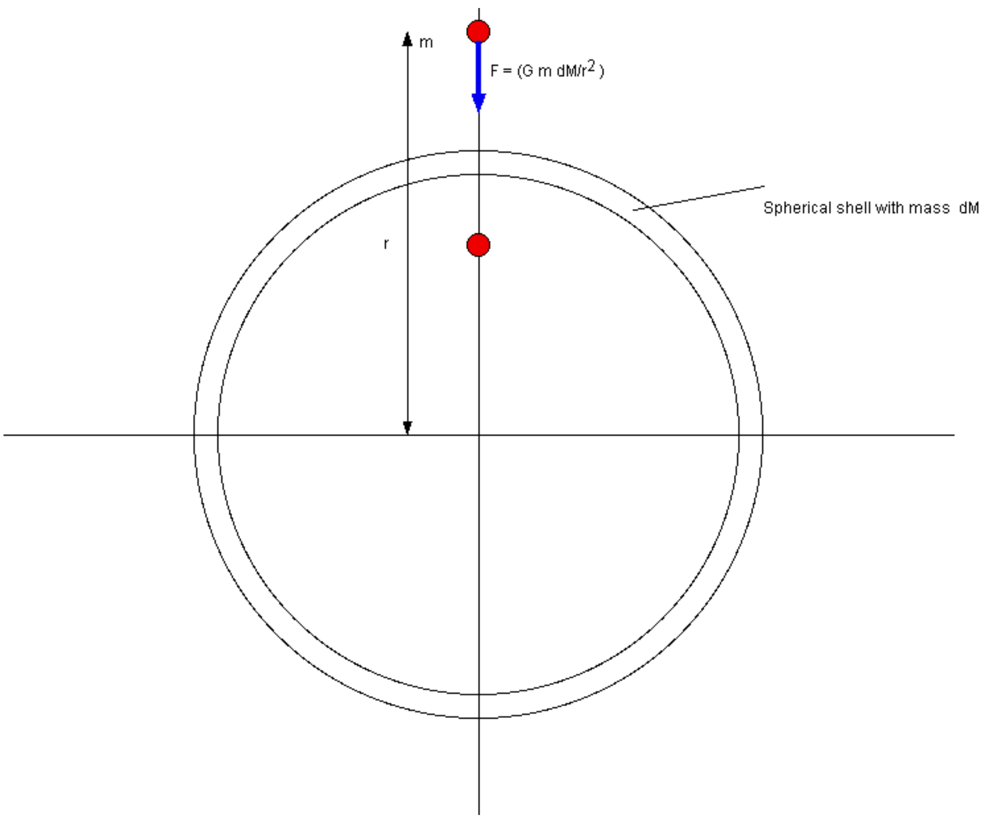


Fig. Force is equal to zero everywhere inside the spherical shell.

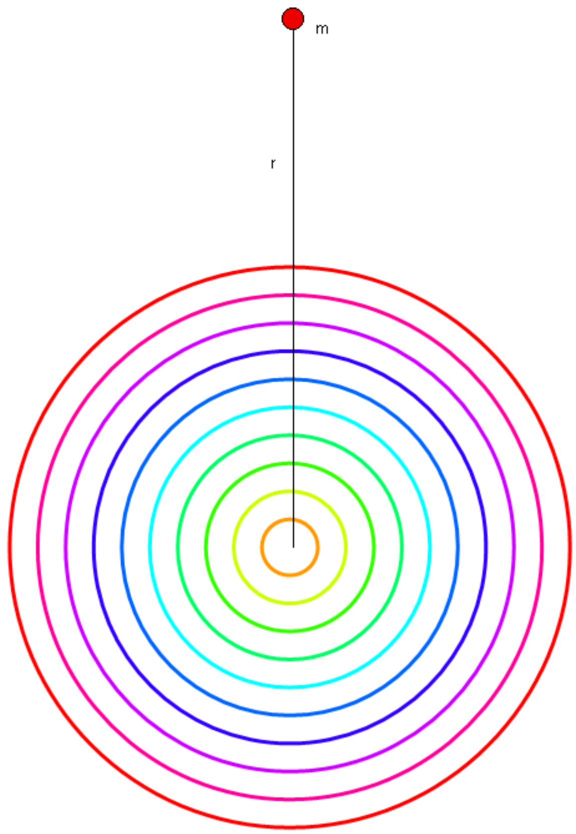
5. Potential energy and force between a point mass and a solid sphere

5.1 Gauss's theorem for gravitational force

The case of $r<R$.

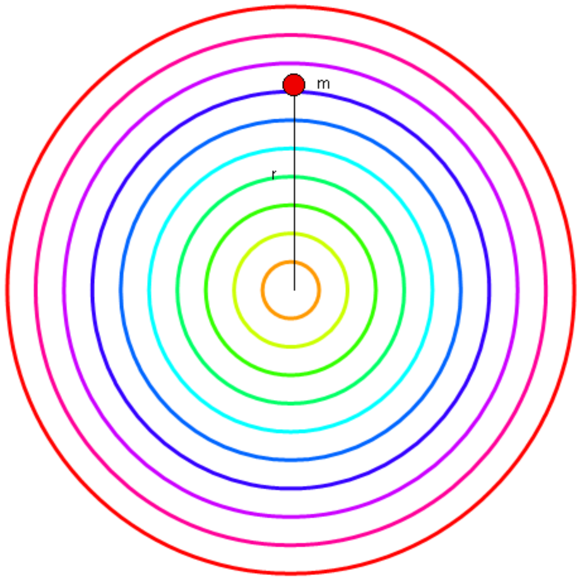
We may build up a solid sphere of mass *M* and radius *R* by adding up a series of concentric shells. For points outside the sphere, the force on the test mass *m* for *r*>*R* is given by

$$F_r = -\frac{Gm}{r^2}\sum M_{shell} = -\frac{GmM}{r^2}$$

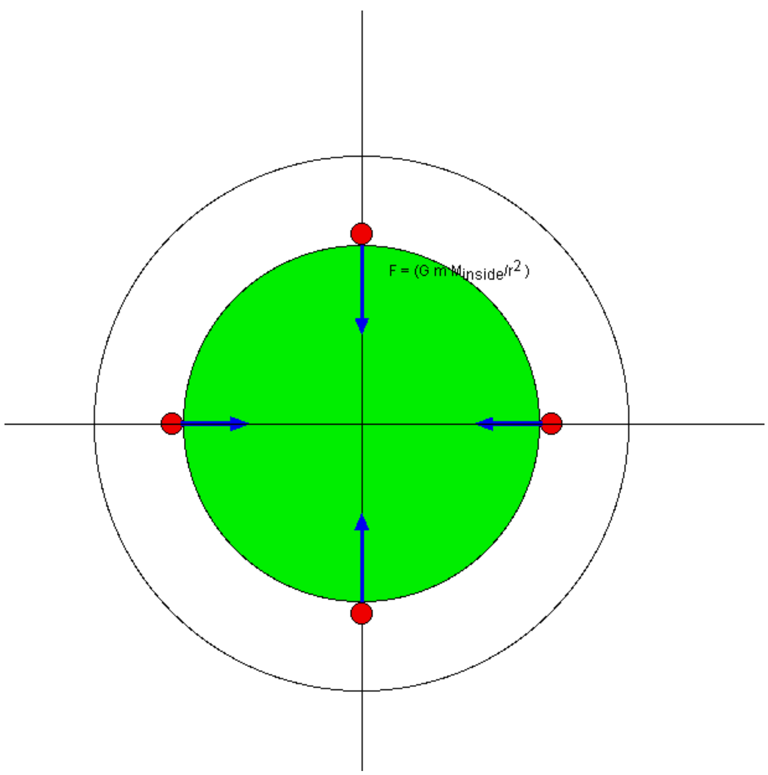


The case of *r*<*R*.

We now consider the case when a point mass is inside a solid sphere. We know that the mass in any spherical shell outside the test mass has no contribution to the force on the test mass. *Only the mass in all spherical shell inside the test mass contributes*. Then the force will be directed toward the center of the sphere and will be



$$F = -\frac{GmM_{inside}}{r^2}.$$



In conclusion, the force ***F*** between the test mass *m* and the center of sphere is given by

$$\boldsymbol{F} = -\frac{GmM_{enclosed}}{r^2}\hat{r}$$

where M_{enclosed} is the total mass inside the spherical surface area (radius r).

<u>Gauss's theorem</u>

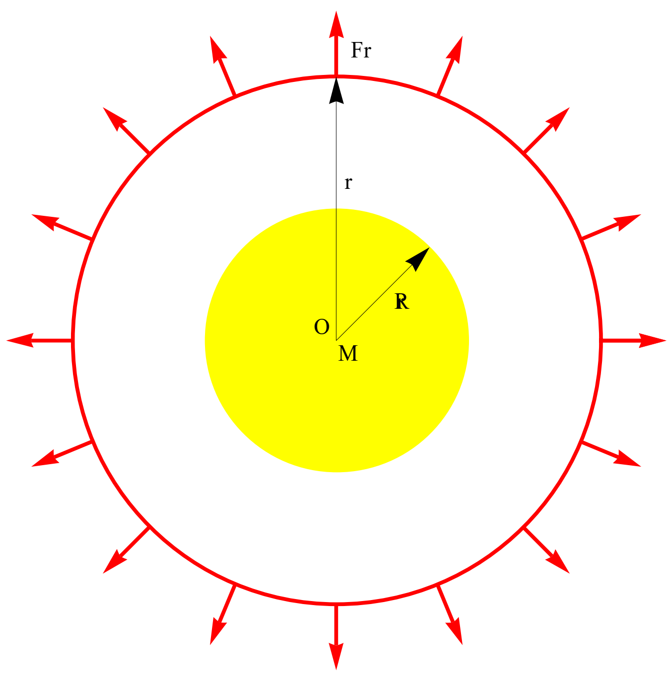
$$\oint \boldsymbol{F} \cdot d\boldsymbol{a} = F_r \cdot 4\pi r^2 = -4\pi GmM_{enlosed}$$

where d$\boldsymbol{a}$ is the surface element (with radius r) and is normal to the surface.

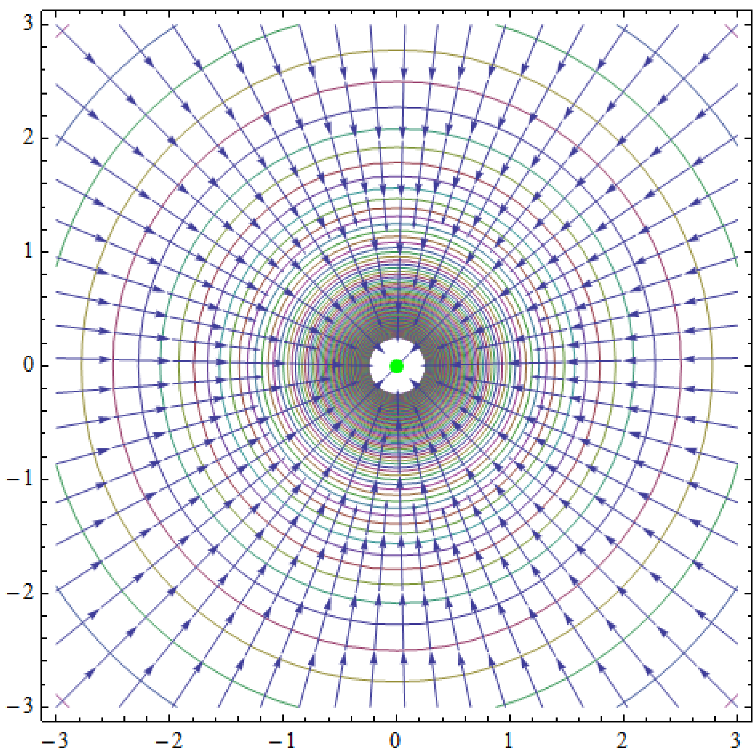
5.2 Application of Gauss's law to the gravitational force around sphere

We now calculate the gravitational force on a mass m outside and inside the sphere (Mass M, and radius R) such as Earth, using the above theorem. This theorem is applicable to the system such as sphere which is highly symmetric.

(a) Outside the sphere



$$\oint \boldsymbol{F} \cdot d\boldsymbol{a} = F_r \cdot 4\pi r^2 = -4\pi GmM_{enlosed} = -4\pi Gm(M)$$



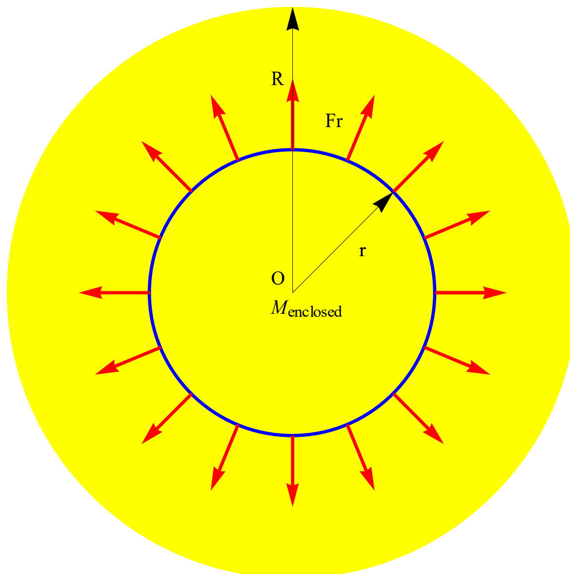
or

$$F_r = -\frac{GmM}{r^2} = -\frac{dU}{dr}$$

The potential energy $U(r)$ is given by

$$U(r) = -\int_{\infty}^{r} F(r)dr = \int_{\infty}^{r} \frac{GmM}{r^2} dr = -\frac{GmM}{r}$$

(b) Inside the sphere



$$\oint \boldsymbol{F} \cdot d\boldsymbol{a} = F_r \cdot 4\pi r^2 = -4\pi GmM_{inside}$$

If the sphere is of uniform density ρ then

$$M_{inside} = \frac{4\pi}{3}\rho r^3 = M\frac{r^3}{R^3}$$

where

$$M = \frac{4\pi}{3}\rho R^3$$

Then we have

$$F_r = -\frac{GmM}{r^2}\frac{r^3}{R^3} = -\frac{GmMr}{R^3} = -\frac{dU}{dr}$$

The potential energy is then given by

$$
\begin{aligned}
U(r) &= U(R) - \int_R^r \left(-\frac{GmM}{R^3} r\right) dr \\
&= -\frac{GmM}{R} + \int_R^r \frac{GmM_E}{R^3} r dr \\
&= -\frac{GmM}{R} + \frac{GmM_E}{2R^3}(r^2 - R^2) \\
&= -\frac{3GmM}{2R} + \frac{GmM}{2R^3} r^2 \\
&= -\frac{GmM}{R}\left(\frac{3}{2} - \frac{r^2}{2R^2}\right)
\end{aligned}
$$

5.2 Advanced Problem: gravity train

First consider a body of mass m outside the Earth. (a) What is the magnitude and direction of the gravitational force for the mass outside the Earth? Here r is the distance between the center of Earth and the body, R_E is the radius of the Earth, M_E is the mass of the Earth, and G is the gravitational constant. (b) What is the potential energy U for the mass outside the Earth? Note that $U = 0$ at $r = \infty$.

Next, see Fig.1, we consider the body of mass m inside the Earth. The density ρ of the Earth is homogeneous and is given by $\rho = M_E/(4\pi R_E^3/3)$. (c) What is the magnitude and direction of the gravitational force for the mass inside the Earth? Here r is the distance between the center of Earth and the body. (d) What is the potential energy U for the mass inside the Earth? Note for $r = R_E$, U inside the Earth equals to U outside the Earth.

Imagine that a hole is drilled through the center of the Earth to the other side along the x axis in Fig.1. An object of mass m at a distance r from the center of the Earth is pulled toward the center of the Earth only by the mass within the sphere of radius r. (e) Write Newton's second law of gravitation for an object at the distance r from the center of the Earth, and show that the force on it is of Hooke's law form $F_x = -kx$, where the effective force constant is $k = (4/3)\pi\rho Gm$.

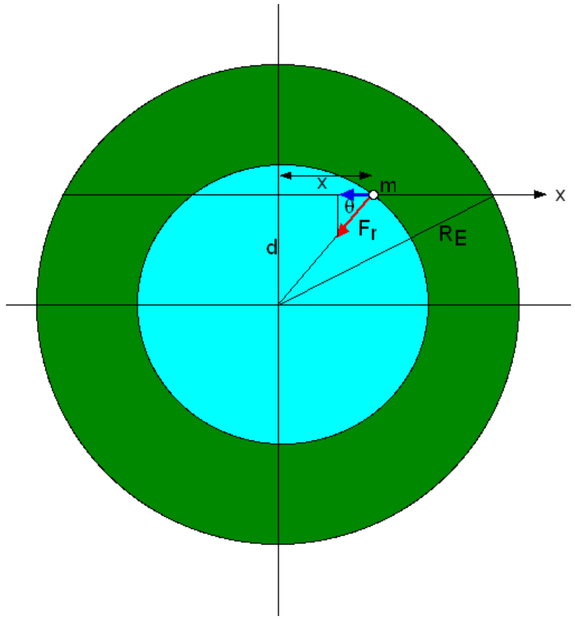


Fig. 1

((**Solution**))

We consider the gravitational force and the potential energy inside the Earth

$$M_E = \frac{4\pi}{3}\rho R_E^{\ 3}$$
$$M_r = \frac{4\pi}{3}\rho r^3$$
$$\frac{M_r}{M_E} = \frac{r^3}{R_E^{\ 3}}$$

(a) For $r>R_E$, the force is directed toward the center.

$$\boldsymbol{F} = -\frac{GmM_E}{r^2}\hat{r}.$$

(b) For $r>R_E$

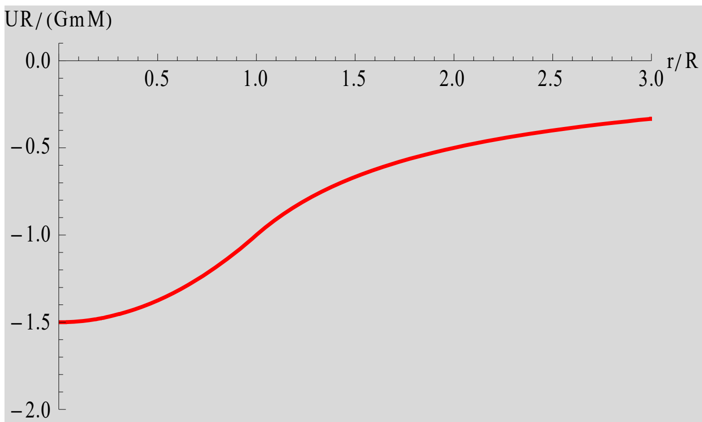
$$U(r) = -\frac{GmM_E}{r}.$$

(c) For $r<R_E$, the force is directed toward the center.

$$\boldsymbol{F} = -\frac{GmM_E}{R_E{}^3} r\hat{r},$$

(d) For $r<R_E$

$$U(r) = -\frac{GmM_E}{R_E}\left(\frac{3}{2} - \frac{r^2}{2R_E{}^2}\right)$$



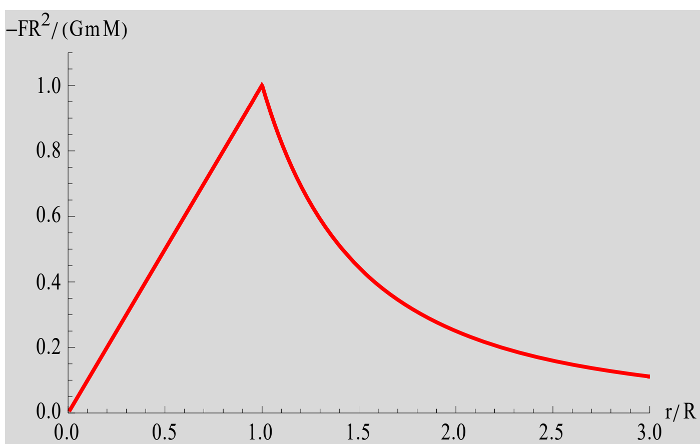


Fig. Plot of the potential energy and the gravitational force as a function of r/R

(e) The force is directed toward the center.

$$\boldsymbol{F} = -\frac{GmM_r}{r^2}\hat{r} = -\frac{GmM_E}{r^2}\frac{r^3}{R_E{}^3}\hat{r} = -\frac{GmM_E}{R_E{}^3} r\hat{r} = F_r\hat{r}$$

The equation of motion for the particle on the tunnel along the *x*-axis.

$$m\ddot{x} = F_r \cos\theta = -\frac{GmM_E}{R_E{}^3} r\cos\theta = -\frac{GmM_E}{R_E{}^3} x = -kx \qquad (7)$$

where

$$k = \frac{GmM_E}{R_E{}^3} = \frac{4\pi}{3} R_E{}^3 \rho \frac{Gm}{R_E{}^3} = \frac{4\pi}{3} Gm\rho$$

or

$$\ddot{x} = -\omega^2 x \qquad \text{(Simple harmonics)}$$

where

$$\omega = \sqrt{\frac{GM_E}{R_E{}^3}} = \sqrt{\frac{g}{R_E}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E{}^3}{GM_E}} = 2\pi\sqrt{\frac{R_E}{g}} = 5061.43s$$

or

$$\frac{T}{2} = 2530.7s = 42.2\,\text{min}$$

((**Note**)) **Use of Mathematica**

Suppose that the expression for the force is given as a function of *r* for each region ($r<R$ and $r>R$). We need to get the expression for the potential energy U, such that

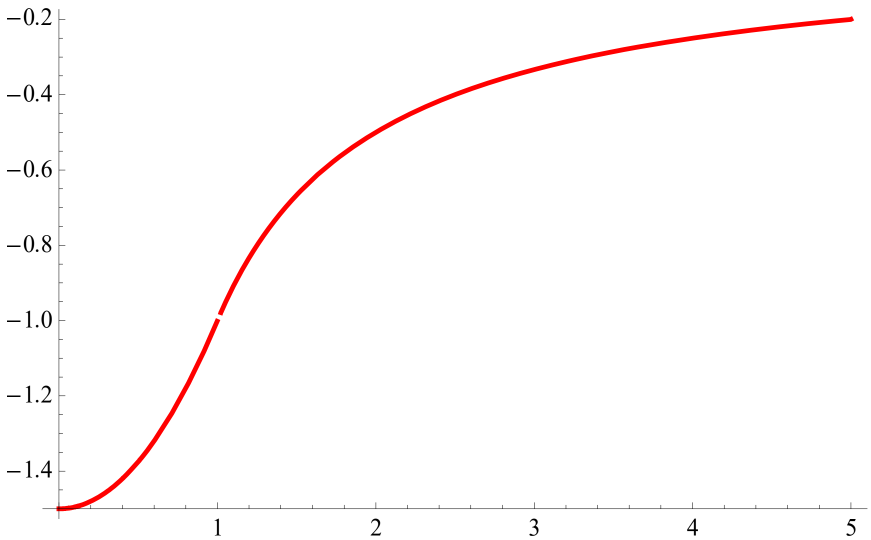
$$F_r = -\frac{dU}{dr}.$$

The use of the Mathematica makes it easier to calculate the form of *U* and to make a plot of *U* as a function of *r*. We add constant such that the potential energy becomes zero at the infinity.

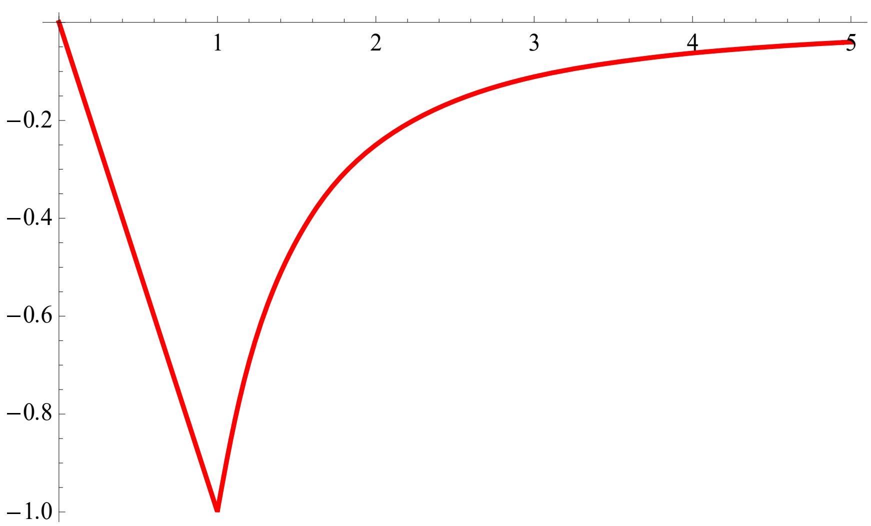
```
Clear["Global`*"];
f1 = Which[0 < x ≤ 1, -x, 1 ≤ x, -1/x^2];
f2 = Integrate[-f1, x]
```

$$\begin{cases} 0 & x \leq 0 \\ \frac{x^2}{2} & 0 < x \leq 1 \\ \frac{3}{2} - \frac{1}{x} & \text{True} \end{cases}$$

```
Plot[f2 - 3 / 2, {x, 0, 5}, PlotStyle → {Red, Thick}]
```




```
Plot[f1, {x, 0, 5}, PlotStyle → {Red, Thick}]
```



5.3 Gravitational force on Earth

Consider an object of mass m near the earth's surface. The gravitational field at some point has the value of the free fall acceleration

$$mg = G\frac{M_E m}{R_E{}^2}$$

or

$$g = G\frac{M_E}{R_E{}^2} = 9.8 m/s^2$$

where

$M_E = 5.9736 \times 10^{24}$ kg and $R_E = 6.372 \times 10^6$ m

The average density ρ_E of the Earth can be estimated as follows.

$$M_E = \frac{gR_E{}^2}{G}$$

$$\rho_E = \frac{M_E}{V_E} = \frac{\frac{gR_E{}^2}{G}}{\frac{4\pi}{3}R_E{}^3} = \frac{3}{4\pi}\frac{g}{GR_E} = 5.51\times 10^3 kg/m^3 = 5.51 g/cm^3$$

5.4 Example

A hole is drilled from the surface of the earth to its center of the earth. Ignore the earth's rotation and air resistance. If the particle is dropped from rest at the surface of the earth, what is its speed when it reaches the center of the earth?

((**Solution**))

The energy conservation:

$$E_C = \frac{1}{2}mv_C{}^2 - \frac{3GmM_E}{2R_E}, \qquad \text{at the center of the earth}$$

$$E_S = -\frac{GmM_E}{R_E} \qquad \text{at the surface of the earth,}$$

where v_C is the velocity

Since $E_S = E_C$ (the energy conservation), we get

$$\frac{1}{2}m{v_C}^2 - \frac{3GmM_E}{2R_E} = -\frac{GmM_E}{R_E}$$

or

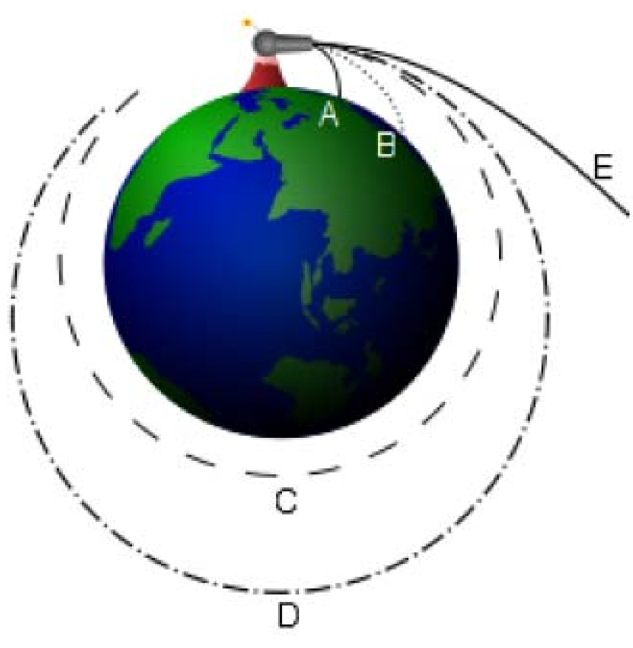
$$v_C = \sqrt{\frac{GM_E}{R_E}} = \sqrt{gR_E} = 7.90493 \text{ km/s.}$$

5.5 Escape velocity

The total energy of the system is given by

$$E = \frac{1}{2}mv^2 - \frac{GM_E m}{r}$$

where v is the velocity.



Suppose that $v = 0$ in the limit of $r \to \infty$. Then we have $E = 0$. The escape velocity v_{esc} can be estimated as

$$E = 0 = \frac{1}{2} m v_{esc}{}^{2} - \frac{G M_E m}{R_E}$$

or

$$v_{esc} = \sqrt{\frac{2 G M_E}{R_E}} = 11.187 \text{ km/s}$$

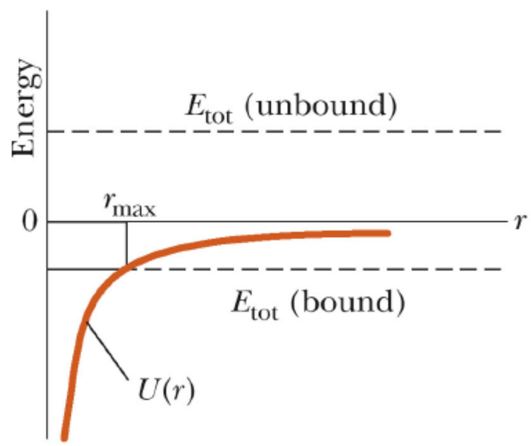
Similarly the escape velocity for the sun is given by

$$v_{esc} = \sqrt{\frac{2 G M_{Sun}}{R_{Sun}}} = 617.5 \text{ km/s}$$

5.6 Circular motion (satellite)

The mechanical energy of the satellite (E) is given by

$$E = \frac{1}{2} m v^2 - \frac{G M_E m}{r}$$



Newton's second law (condition of the circular orbit):

$$m\frac{v^2}{r} = \frac{G M_E m}{r^2}, \qquad \text{or} \qquad m v^2 = \frac{G M_E m}{r}$$

Then E is derived as

$$E = -\frac{G M_E m}{2r} < 0 \qquad \text{(circular orbit)}$$

The velocity is obtained as

$$v = \sqrt{\frac{GM_E}{r}} .$$

When $r = R_E$, we have

$$v = \sqrt{\frac{GM_E}{R_E}} = 7.910 \text{ km/s}$$

The period T is

$$T = \frac{2\pi R_E}{v} = 2\pi\sqrt{\frac{R_E{}^3}{GM_E}} = 5061.43 \text{ sec} = 1 \text{ hour } 24 \text{ min } 21 \text{ s}$$

((**Note**))
A.R.P. Rau, The Beauty of Physics: Patterns, Principles, and Perspective (Oxford 2014). p.12-13

Since

$$gR_E{}^2 = GM_E ,$$

the period T can be rewritten as

$$T = 2\pi\sqrt{\frac{R_E}{g}} .$$

This time period is that of a pendulum of length l equal to the radius of the Earth. This coincides with the time it takes a near-Earth satellite such as the International Space Station to go once around in a circular orbit.

5.7 Evaluation of the physical quantities by Mathematica

```
Physconst = {G → 6.6742867 10^-11,
  Mea → 5.9736 10^24, Rea → 6.372  10^6 ,
  Msun → 1.988435 10^30, Rsun → 6.9599 10^8}
```

$\{G \to 6.67429\times 10^{-11},$
$Mea \to 5.9736\times 10^{24},\ Rea \to 6.372\times 10^{6},$
$Msun \to 1.98844\times 10^{30},\ Rsun \to 6.9599\times 10^{8}\}$

$$g1 = G\,\frac{Mea}{Rea^2}\ /.\ Physconst$$

9.8195

$$\rho 1 = \frac{3\,g1}{4\,\pi\,G\,Rea}\ /.\ Physconst$$

5512.14

$$vcir = \sqrt{\frac{G\,Mea}{Rea}}\ /.\ Physconst$$

7910.11

$$vesc = \sqrt{\frac{2\,G\,Mea}{Rea}}\ /.\ Physconst$$

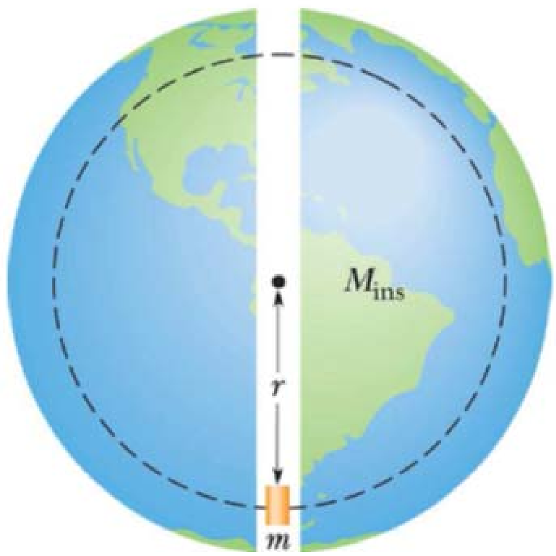
11186.6

$$T1 = 2\,\pi\,\sqrt{\frac{Rea^3}{G\,Mea}}\ /.\ Physconst$$

5061.43

5.8 Simple harmonic oscillation of the apple inside the earth

Suppose we make a tunnel inside the earth. This tunnel passes through the center of the earth. The apple is dropped from rest at the surface of the earth without any resistance including air. We assume that the density is uniform inside the earth. We find that the apple undergoes the motion of simple harmonics.



Inside the earth, we have a force directed toward the center,

$$F_x = -\frac{GmM_E}{x^2}\frac{x^3}{R^3} = -\frac{GmMx}{R^3} = -\frac{dU}{dx}$$

where we use *x* instead of *r*. We set up the equation of motion for the system inside the tunnel of the earth.

$$m\ddot{x} = F_x = -\frac{GmM_E}{{R_E}^3}x \qquad \text{or} \qquad \ddot{x} = -\frac{GM_E x}{{R_E}^3} = -\frac{g}{R_E}x = -\omega^2 x$$

where

$$g = \frac{GM_E}{{R_E}^2}, \qquad \omega = \sqrt{\frac{g}{R_E}}$$

So we find that the system undergoes the motion of simple harmonics with the period

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E}{g}} = 5061.43s$$

The total energy is a sum of the kinetic energy and the potential energy,

$$E = K + U = \frac{1}{2}m\dot{x}^2 + \frac{GmM_E}{2R_E{}^3}x^2 - \frac{3GmM_E}{2R_E},$$

The velocity of the apple at the center of the earth can be obtained as

$$v_c = \sqrt{\frac{GM_E}{R_E}} = \sqrt{gR_E} = 7.90493 \text{ km/s.}$$

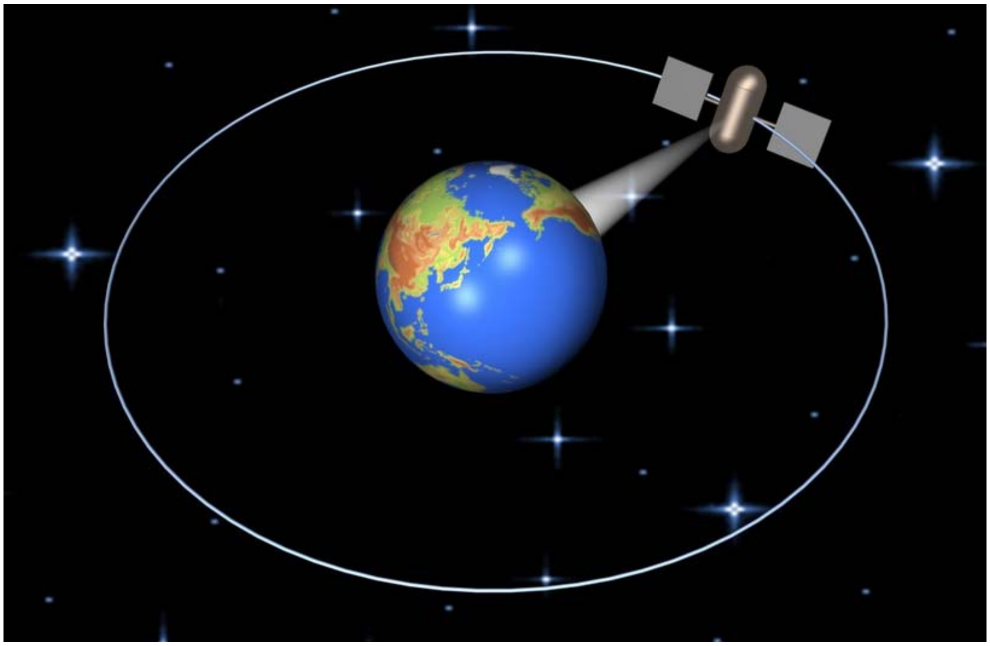
from the energy conservation law;

$$E = \frac{1}{2}mv_c{}^2 - \frac{3GmM_E}{2R_E} \quad \text{at the center}$$

$$E = -\frac{GmM_E}{R_E} \qquad \text{at the surface}$$

5.9 Geosynchronous orbit

A **geosynchronous orbit** (sometimes abbreviated GSO) is an orbit around the Earth with an orbital period of one sidereal day, intentionally matching the Earth's sidereal rotation period (approximately 23 hours 56 minutes and 4 seconds). The synchronization of rotation and orbital period means that, for an observer on the surface of the Earth, an object in geosynchronous orbit returns to exactly the same position in the sky after a period of one sidereal day.



https://i.ytimg.com/vi/sj7zsGkpZxg/maxresdefault.jpg

$$m\frac{v^2}{r} = G\frac{mM_E}{r^2}, \qquad \text{or} \qquad v^2 r = GM_E$$

We also note that

$$mg = G\frac{Mm}{R_E{}^2}, \qquad \text{or} \qquad gR_E{}^2 = GM$$

on the Earth surface. From these two equations, we get

$$v = R_E\sqrt{\frac{g}{r}} \qquad \text{(velocity)}$$

$$T = \frac{2\pi r}{v} = \frac{2\pi}{R_E}\sqrt{\frac{r^3}{g}} \qquad \text{(period)}$$

$$r = g^{1/3}\left(\frac{TR_E}{2\pi}\right)^{2/3} \qquad \text{(radius)}$$

((Example))

Period: T = 23 hours 56 min 4 sec = 86164 s

Radius: r = 42,149.1 km

Height: 35777.1 km =22,235.6 mile

Velocity 3.07356 km/s (1.90982 mile/s = 6875 miles/hour)

6 The Potential energy in many-body system

If the system contains more than two particles, the **principle of superposition** applies. In this case we consider each pair and the total potential energy is equal to the sum of the potential energies of each pair. In calculating the total potential energy of a system of particles **one should take great care not to double count the interactions**. The total potential energy of a system of particles is sometimes called **the binding energy of the system**. The total potential energy is the amount of work that needs to be done to separate the individual parts of the system and bring them to infinity.

6.1 The system of two particles

The potential energy associated with any pair of particles of mass m_1 and m_2 separated by a distance r_{12} is given by

$$U = -\frac{Gm_1m_2}{r_{12}}$$

6.2 The system of three particles

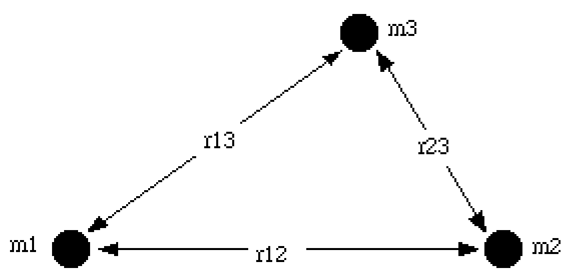


Figure: A system of three particles.

The total potential energy of the three-particle system is given by

$$U = -\frac{1}{2}[(\frac{Gm_1m_2}{r_{12}} + \frac{Gm_3m_1}{r_{31}}) + (\frac{Gm_1m_2}{r_{12}} + \frac{Gm_2m_3}{r_{23}}) + (\frac{Gm_2m_3}{r_{23}} + \frac{Gm_3m_1}{r_{31}})]$$
$$= -\frac{Gm_1m_2}{r_{12}} - \frac{Gm_2m_3}{r_{23}} - \frac{Gm_3m_1}{r_{31}}$$

where the factor 2 is needed because of double counting.

6.3 General case (many body systems)

The potential energy of N discrete masses due to their mutual gravitational attraction is equal to the sum of the potential energy of all pairs of masses.

$$U = -\frac{1}{2}\sum_{i=1}^{n}\sum_{\substack{j=1 \\ i\neq j}}^{n}\frac{Gm_i m_j}{r_{ij}} = -\sum_{\substack{All \\ pairs \\ i\neq j}}^{n}\frac{Gm_i m_j}{r_{ij}}$$

((**Example**)) **Estimation of the gravitational energy of the galaxay**.

We approximate that the gross composition of the galaxy by N stars of mass M, and with each pair of stars at a mutual separation of the order of R. Then we have

$$U = -{}_nC_2\frac{GM^2}{R} = -\frac{N(N-1)}{2}\frac{GM^2}{R}$$

where ${}_nC_2 = \dfrac{N(N-1)}{2}$

Here we assume that

$N = 1.6 \text{ x } 10^{11}$, $R = 10^{21}$ m, and $M = 2 \text{ x } 10^{30}$ kg.

Then we have

$$U = -3.4\times 10^{51} J$$

((**Mathematica**))

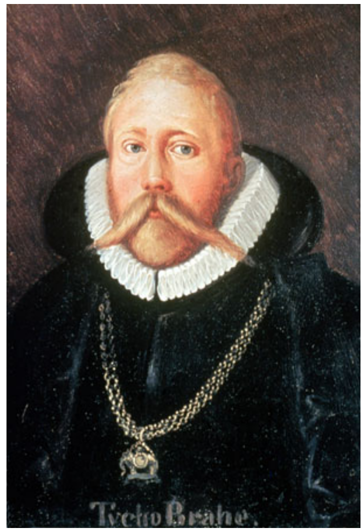
```
Physconst = {G → 6.6742867 10^-11, M → 2.0 10^30, N → 1.6 10^11, R → 1.0 10^21}
{G → 6.67429×10^-11, M → 2.×10^30, N → 1.6×10^11, R → 1.×10^21}

U1 = - (N (N - 1))/2 (G M^2)/R /. Physconst
-3.41723×10^51
```

7 Kepler's laws

Johannes Kepler (December 27, 1571 – November 15, 1630)

Johannes Kepler (December 27, 1571 – November 15, 1630) was a German mathematician, astronomer and astrologer, and key figure in the 17th century astronomical revolution. He is best known for his eponymous laws of planetary motion, codified by later astronomers based on his works Astronomia nova, Harmonices Mundi, and Epitome of Copernican Astronomy.
http://en.wikipedia.org/wiki/Johannes_Kepler

Tycho Brahe (14 December 1546– 24 October 1601)



Tycho Brahe (14 December 1546– 24 October 1601), born **Tyge Ottesen Brahe**, was a Danish nobleman known for his accurate and comprehensive astronomical and planetary observations. He was born in Scania, then part of Denmark, now part of modern-day Sweden. Tycho was well known in his lifetime as an astronomer and alchemist and has been described more recently as "the first competent mind in modern astronomy to feel ardently the passion for exact empirical facts."
http://en.wikipedia.org/wiki/Tycho_Brahe

((**Kepler's First Law**))

Each planet in the Solar System moves in an elliptical orbit with the Sun at one focus (F_1).

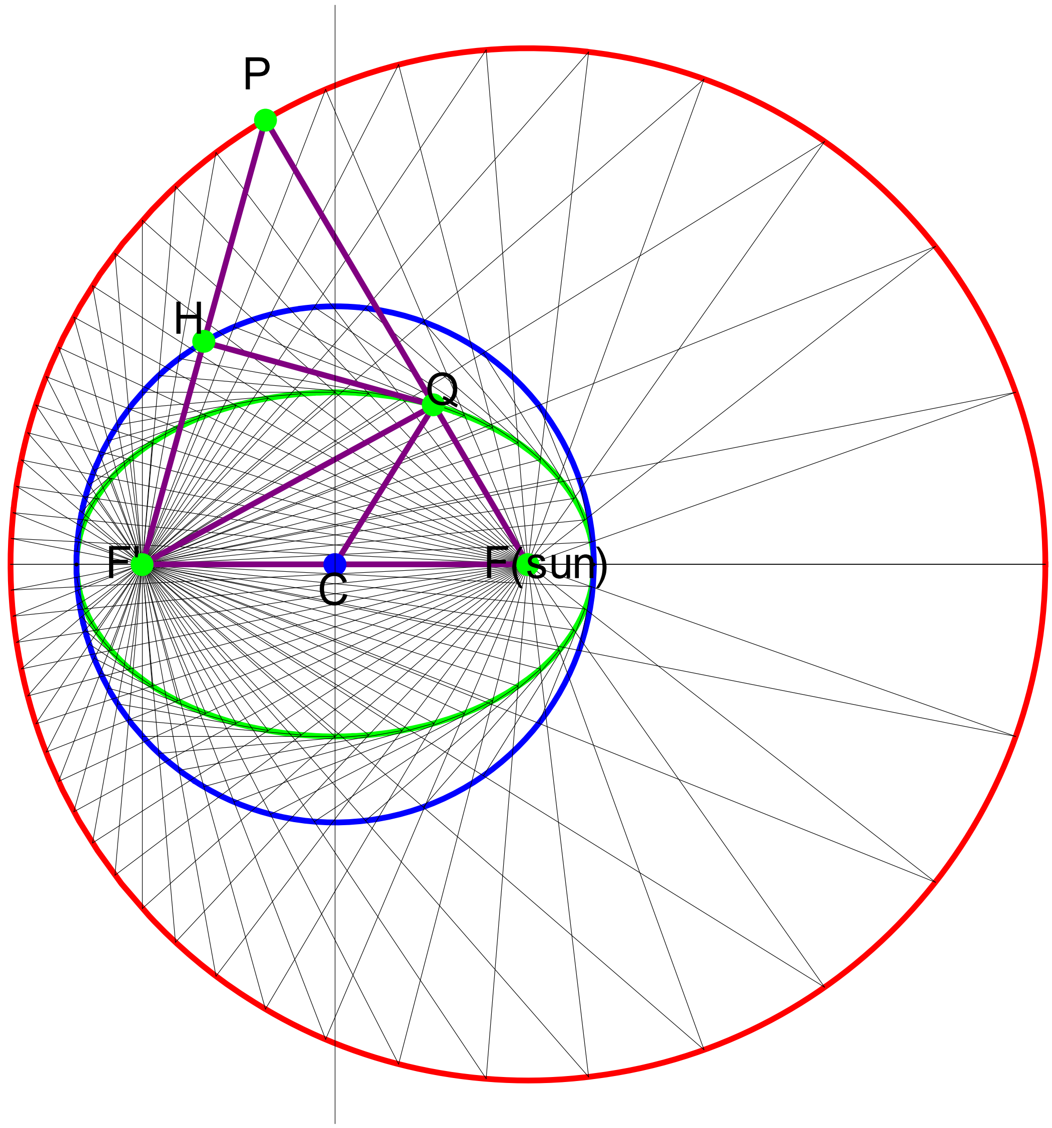
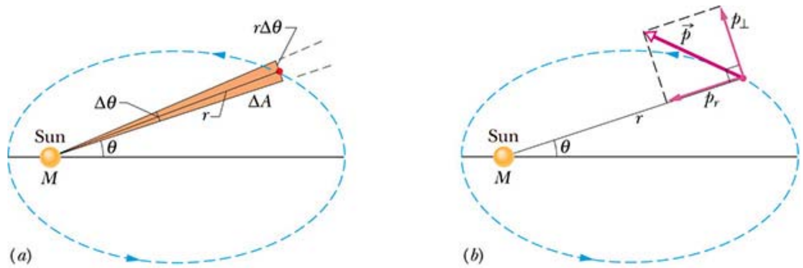
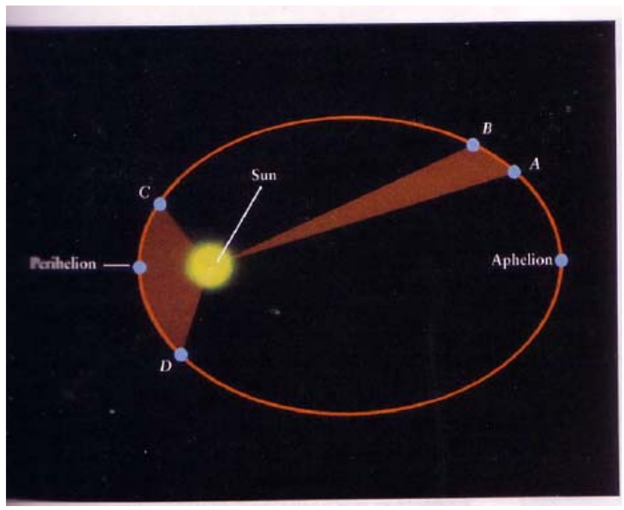


Fig. Two focal points F_1 (sun) and F_2. The planet (Q) on the ellipse orbit (green).

((**Kepler's Second Law**))

The radius vector drawn from the Sun to a planet sweeps out equal areas in equal time intervals. It is a direct consequence of the law of conservation of angular momentum.





((**Kepler's Third Law**))

The square of the orbital period of any planet is proportional to the cube of the semimajor axis of the elliptical orbit.

$$T^2 = \frac{4\pi^2}{GM_{sun}} a^3$$

For Earth, T = 1 year and a = 1 AU = 1.49597870 x 10^{11} m (astronomical units).

In other words,

$$(1 year)^2 = \frac{4\pi^2}{GM_{sun}} (1AU)^3$$

Then the Kepler's third law can be rewritten as

$$[T(year)]^2 = [a(AU)]^3$$

((**Mathematica**))

```
Physconst = {G → 6.6742867 10^-11, Msun → 1.988435 × 10^30,
    Rsun → 6.9599 × 10^8,   AU → 1.49597870 10^11,
    year → 365.25 × 3600 × 24};
```

$$\frac{4\,\pi^2}{G\,Msun}\,AU^3\ /.\ Physconst$$

9.95909×10^{14}

$$year^2\ /.\ Physconst$$

9.95882×10^{14}

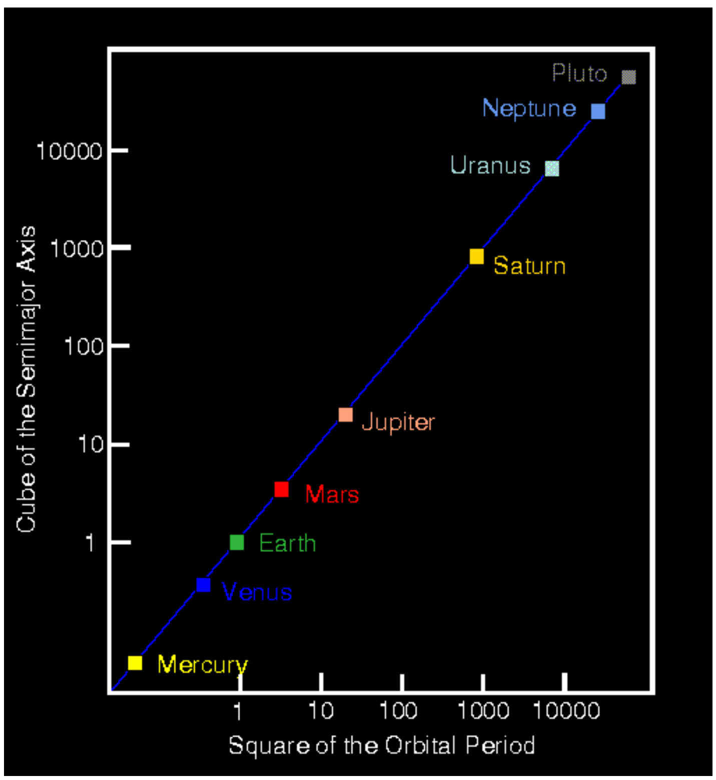


Figure: $[a(\text{AU})]^3$ vs $[T(\text{year})]^2$ for the solar system

Table 4-3 A Demonstration of Kepler's Third Law

Planet	Sidereal period P (years)	Semimajor axis a (AU)	P^2	a^3
Mercury	0.24	0.39	0.06	0.06
Venus	0.61	0.72	0.37	0.37
Earth	1.00	1.00	1.00	1.00
Mars	1.88	1.52	3.53	3.51
Jupiter	11.86	5.20	140.7	140.6
Saturn	29.46	9.54	867.9	868.3
Uranus	84.01	19.19	7,058	7,067
Neptune	64.79	30.06	27,160	27,160
Pluto	248.54	39.53	61,770	61,770

8 Kepler problem

8.1 Definition of ellipsoid

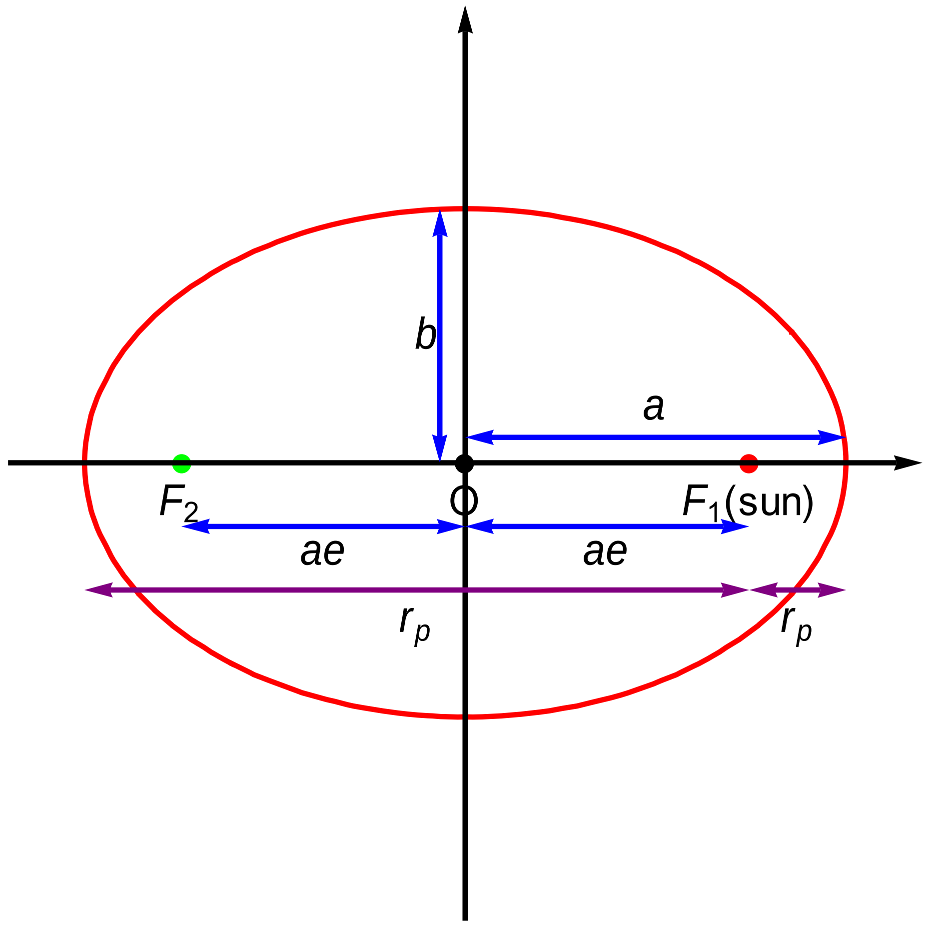
The Sun is at the one focus of the ellipse (the Earth orbit). The ellipse orbit is described by

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

a is the semimajor axis.
b is the semiminor axis
e is the eccentricity ($0<e<1$)

The essentricity e is defined by

$$e=\frac{\sqrt{a^2-b^2}}{a}$$



The focus is at (ae,0) and (-ae,0). For simplicity, we assume that Sun is located at focus (ae,0). p is the *semi latus rectum* and is defined by

$$p = a(1-e^2).$$

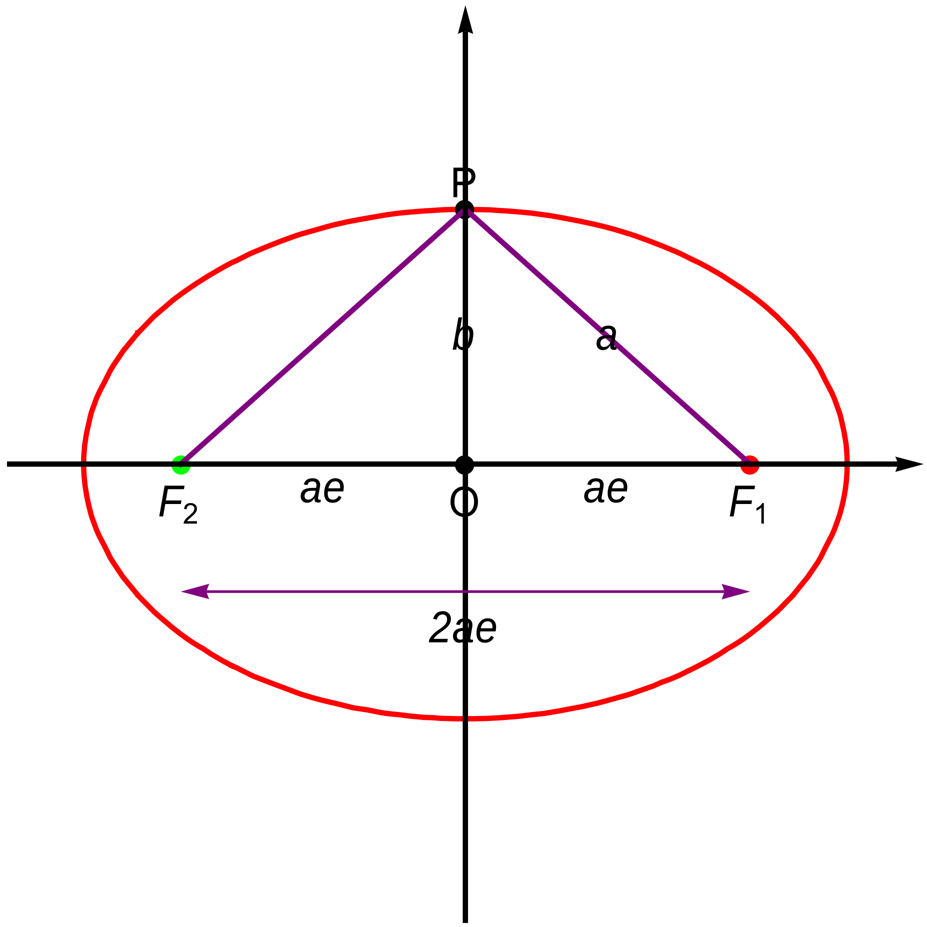
Sun is at the focal point (F_1).

$$r_p = a(1-e), \qquad r_a = a(1+e).$$

Perihelion (r_p) the point nearest the Sun
Aphelion (r_a) the point farthest the Sun

From the Pythagorean theorem for ΔOPF_1, we have

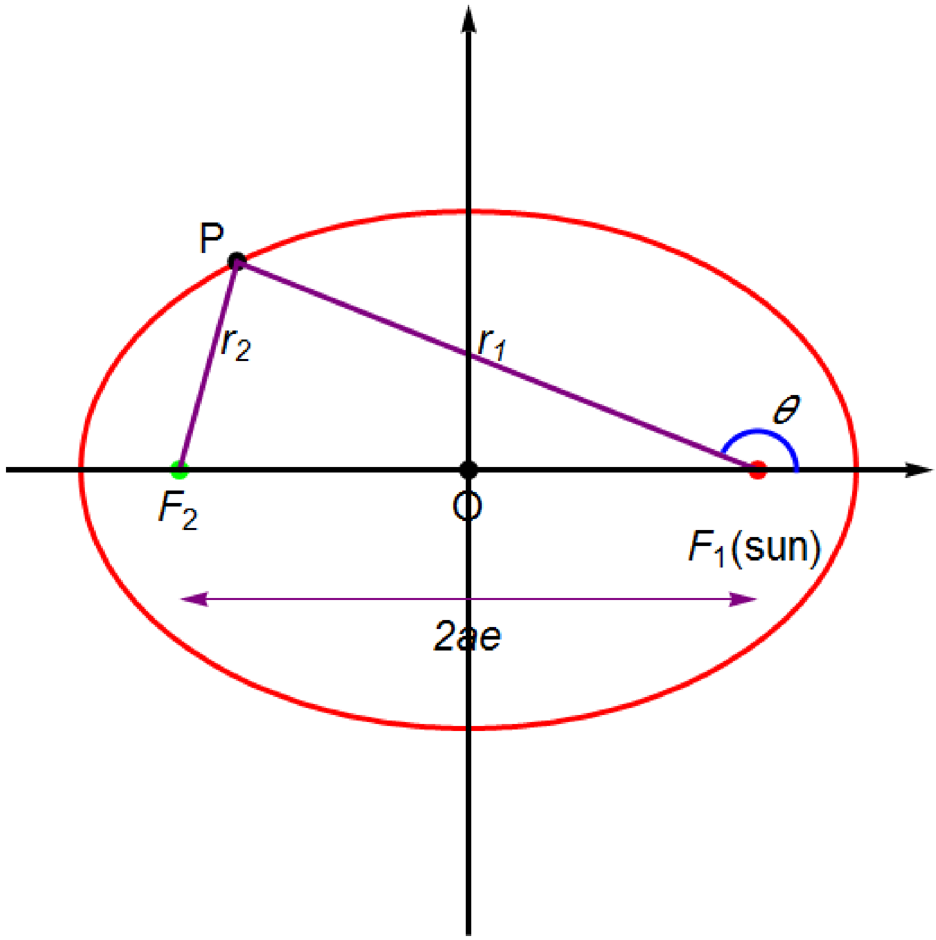
$$a^2 = b^2 + a^2e^2, \qquad b = a\sqrt{1-e^2}.$$



We note that the area of the ellipse orbit is given by

$$A = \pi ab = \pi a^2 \sqrt{1-e^2}$$

We now discuss the dependence of r_1 on the angle θ.



In the triangle ΔF_1F_2P, we have

$$r_1 + r_2 = 2a,$$

from the definition of the ellipse. Using the cosine law, we get

$$r_2^{\;2} = r_1^{\;2} + 4a^2e^2 - 4aer_1\cos(\pi - \theta) = r_1^{\;2} + 4a^2e^2 + 4aer_1\cos\theta .$$

From these two equations, we have

$$(2a - r_1)^2 = r_1^{\;2} + 4a^2e^2 + 4aer_1\cos\theta$$

or

$$4a^2 - 4ar_1 + r_1^{\;2} = r_1^{\;2} + 4a^2e^2 + 4aer_1\cos\theta$$

or

$$r_1 = \frac{p}{1 + e\cos\theta}$$

with

$$p = a(1 - e^2)$$

Note that r_1 is an even function of θ.

(i) For $\theta = 0$ the planet is at the perihelion at minimum distance

$$r_1 = r_p = \frac{p}{1+e} = \frac{a(1-e^2)}{1+e} = a(1-e)$$

(ii) For $\theta = 90°$: $r_1 = p$.

(iii) For $\theta=\pi$, the planet is at the aphelion at maximum distance,

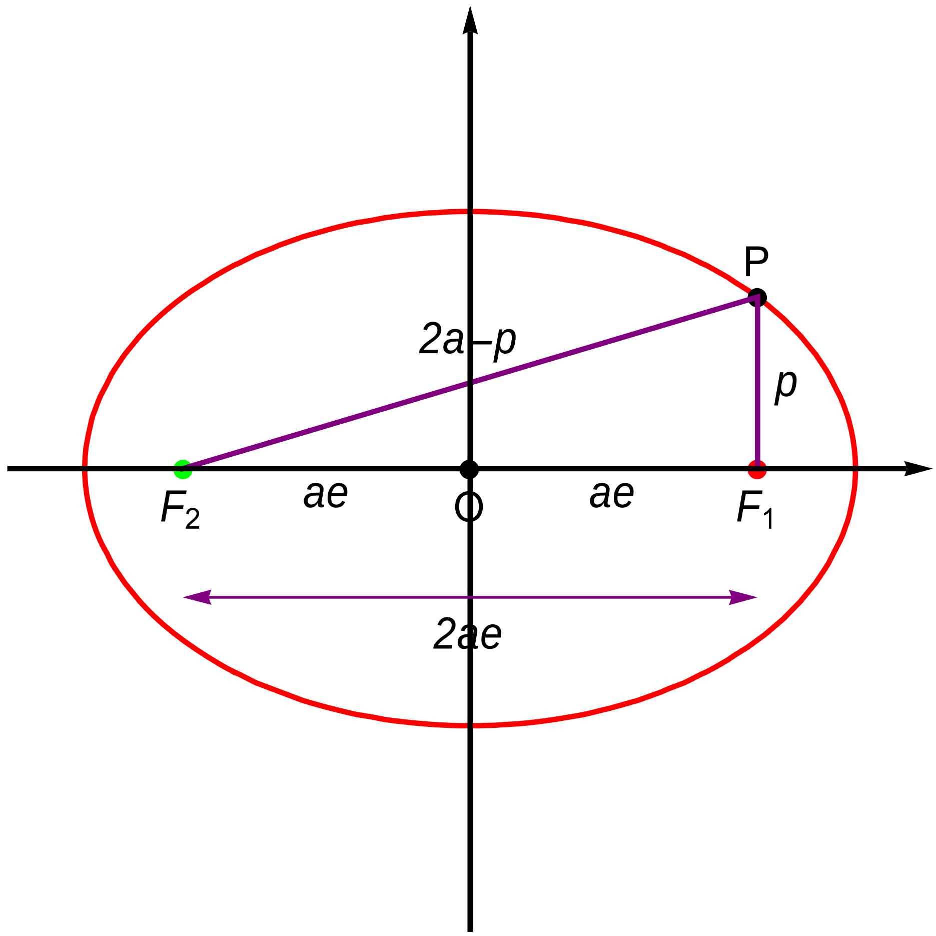
$$r_a = \frac{p}{1-e} = a(1+e)$$

Note that

$$\frac{1}{r_p} + \frac{1}{r_a} = \frac{1+e}{p} + \frac{1-e}{p} = \frac{2}{p}$$

((The semi latus rectum, *p*))

The chord of an ellipse which are perpendicular to the major axis and pass through the focal point F_1 is called the dr,i latus rectum of the ellipse. In this Fig. p is the length of PF_1. The value of p can be obtained from the Pythagorean theorem for the triangle ΔOF_1F_2.



In this figure, we get the relation,

$$(2a - p)^2 = p^2 + (2ae)^2$$

or

$$p = a(1 - e^2)$$

9 The angular momentum

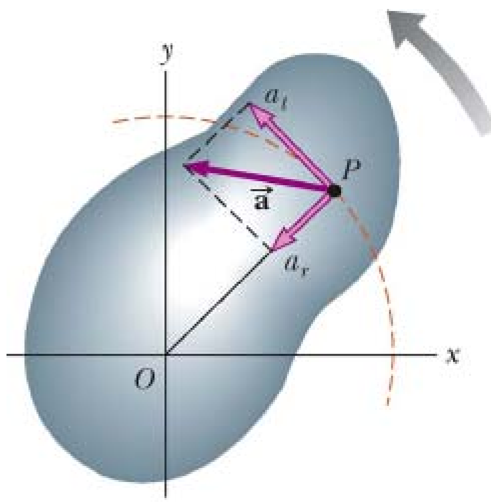
9.1 Central force problem

In general case

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta})$$
$$v_r = \dot{r}$$
$$v_\theta = r\dot{\theta}$$

Since the gravitational force is directed toward the origin (so called central field),

$$ma_r = -G\frac{Mm}{r^2}$$
$$ma_\theta = 0$$



In other words,

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) = 0$$
$$v_\theta = r\dot{\theta}$$

or

$$l = mr^2\dot{\theta} = mr^2\frac{v_\theta}{r} = mrv_\theta = cons\tan t$$

9.2 Angular momentum

The angular momentum is defined as

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p} = \boldsymbol{r} \times (m\boldsymbol{v}) = m(r\hat{r}) \times (v_r\hat{r} + v_\theta\hat{\theta}) = mrv_\theta\hat{k} = L_z\hat{k}$$

or

$$L_z = mv_\theta r$$

$$\frac{d\boldsymbol{L}}{dt} = \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) = \frac{d}{dt}(\boldsymbol{r} \times m\boldsymbol{v}) = \boldsymbol{r} \times \boldsymbol{F} = 0$$

since ***F*** is a central force (***r***//***F***), *L_z is a constant of motion.*

$$l = mrv_\theta = mr^2\dot{\theta}$$

((**Note-1**))
In general (Chapter 10), we have

$$\boldsymbol{v} = \dot{r}\hat{r} + r\dot{\theta}\hat{\theta}$$

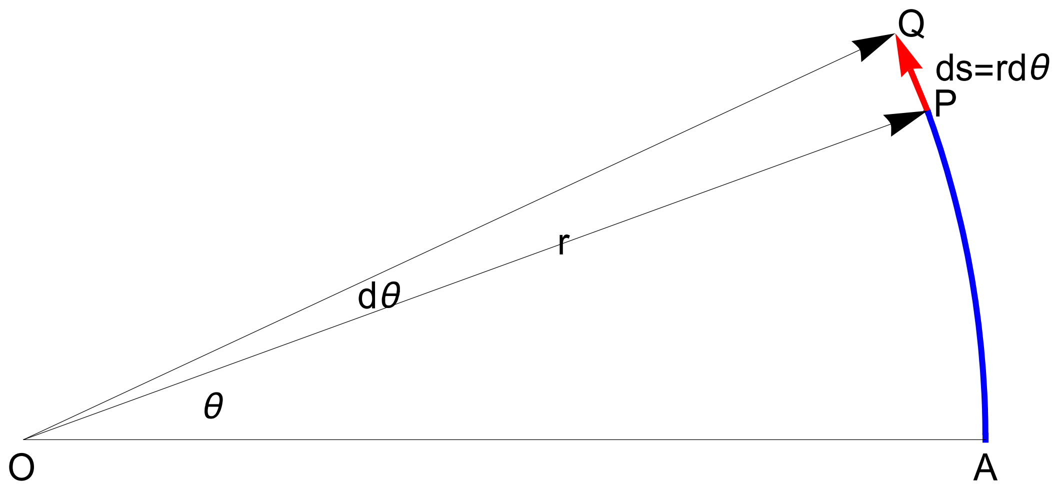
$$\boldsymbol{a} = (\ddot{r} - r\dot{\theta}^2)\hat{r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\hat{\theta}$$

((**Note-2**))
The velocity v_θ is given by

$$v_\theta = r\dot{\theta} = r\frac{l}{mr^2} = \frac{l}{mr}$$

9.3 Physical meaning

What is the physical meaning of the constant angular momentum? We now consider the d*A*/dt, where d*A* is the partial area of the ellipse.



$$ds = r d\theta \,, \qquad v_\theta = r\frac{d\theta}{dt}$$

dA (the area of the triangle ΔOPQ) is given by

$$dA = \frac{1}{2} r^2 d\theta$$

or

$$\frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2m} = \text{const}$$

since $l = mr^2\dot{\theta} = \text{const}$. The period T is evaluated as

$$T = \int dt = \frac{2m}{l}\int dA = \frac{2m}{l}\pi ab = \frac{2m}{l}\pi a^2 \sqrt{1-e^2} \ .$$

since $dt = \frac{2m}{l} dA$. Later we will show that

$$1 - e^2 = \frac{l^2}{mka}$$

where $k = GMm$. Using this relation we have the Kepler's third law,

$$T^2 = \frac{4m^2}{l^2}\pi^2 a^4(1-e^2) = \frac{4m^2}{l^2}\pi^2 a^4 \frac{l^2}{mka} = \frac{4\pi^2 m}{k} a^3$$

10. The effective potential

The total energy is a sum of the kinetic energy and the potential energy

$$E = \frac{1}{2}m\boldsymbol{v}^2 - \frac{k}{r} = \frac{1}{2}m({v_r}^2 + {v_\theta}^2) - \frac{k}{r}$$

or

$$E = \frac{1}{2}m(\dot{r}^2 + \frac{l^2}{m^2r^2}) - \frac{k}{r} = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} - \frac{k}{r} \qquad (1)$$

where $k = GMm$. The energy is dependent only on r (actually one dimensional problem).

$$U_{eff} = -\frac{k}{r} + \frac{l^2}{2mr^2} \qquad \text{(effective potential)}$$

The effective potential energy U_{eff} has a local minimum

$$U_{eff}^{\min} = -\frac{mk^2}{2l^2}$$

at

$$r_{\min} = \frac{l^2}{mk}.$$

Since E = constant, we have an equation of motion

$$\frac{dE}{dt} = m\dot{r}\ddot{r} + \frac{k}{r^2}\dot{r} - \frac{l^2}{mr^3}\dot{r}$$
$$= (m\ddot{r} + \frac{k}{r^2} - \frac{l^2}{mr^3})\dot{r} = 0$$

$$m\ddot{r} + \frac{k}{r^2} - \frac{l^2}{mr^3} = 0 \qquad \text{(equivalent 1D problem)}$$

Plot of the effective potential as a function of *r*

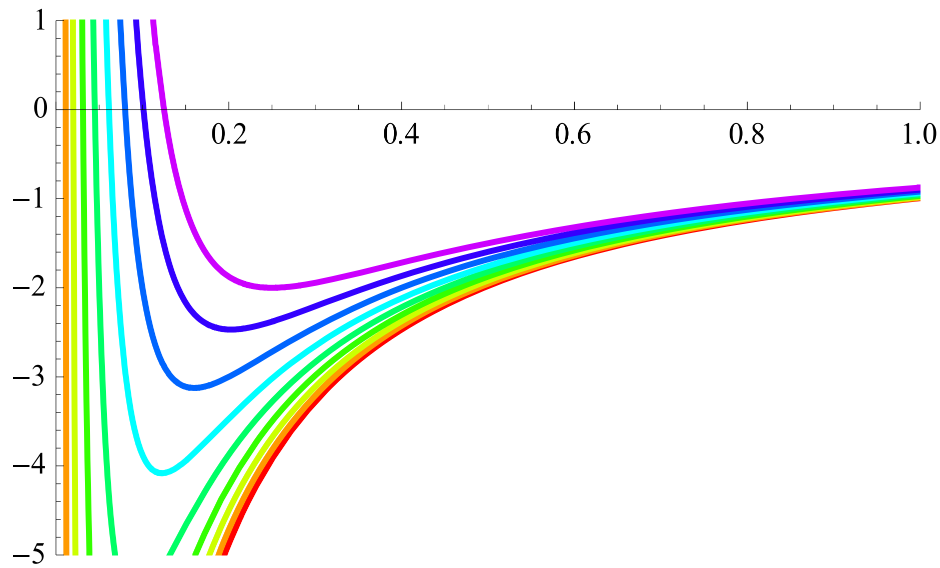


Figure The effective potential vs *r* with $l = 0.1 - 0.5$

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11. Perihelion and aphelion

When $\dot{r} = 0$ for the perihelion (nearest from the Sun) and the aphelion (farthest from Sun) r_p and r_a are the roots of Eq.(1).

$$r^2 + \frac{k}{E} r - \frac{l^2}{2mE} = 0$$

There are the relations between r_1 and r_2.

$$r_p + r_a = -\frac{k}{E} = \frac{k}{|E|} = 2a, \qquad r_p r_a = -\frac{l^2}{2mE} = \frac{l^2}{2m|E|}.$$

where

$$E = -|E| \qquad \text{(bound state; } E<0\text{)}$$

$$r_p = a(1-e), \qquad r_a = a(1+e)$$

From this we have

$$1-e^2 = \frac{r_p r_a}{a^2} = \frac{l^2}{a^2 2m|E|} = \frac{l^2}{2ma^2}\frac{2a}{k} = \frac{l^2}{mka}$$

For $e>1$ ($E>0$) hyperbola
For $0<e<1$ ($E<0$), ellipse
For $e = 0$ [$E = -mk^2/(2l^2)$], circle

((**Note**))

The eccentricity of the Earth's orbit is currently about 0.0167, meaning that the Earth's orbit is nearly circular, the semiminor axis is 98.6% of the semimajor axis. Over thousands of years, the eccentricity of the Earth's orbit varies from nearly 0.0034 to almost 0.058 as a result of gravitational attractions among the planets.

$$\frac{b}{a} = \sqrt{1-e^2} = 0.986 .$$

12 Kepler's Third Law

The period T is given by

$$T = \frac{2m}{l}\pi a^2\sqrt{1-e^2} = \frac{2m}{l}\pi a^2\sqrt{\frac{\ell^2}{mka}} = \frac{2\pi m a^{3/2}}{\sqrt{mk}}$$

or

$$T^2 = \frac{4\pi^2 m^2 a^3}{mk} = \frac{4\pi^2 m a^3}{k}$$

or

$$\frac{a^3}{T^2} = \frac{k}{4\pi^2 m} = \frac{GmM_{sun}}{4\pi^2 m} = \frac{GM_{sun}}{4\pi^2}$$

or

$$T^2 = \frac{4\pi^2}{GM_{sun}}a^3$$

or

$$[T(year)]^2 = [a(AU)]^3$$

13 Derivation of the Kepler's First Law

We start with

$$m\ddot{r} = mr\dot{\theta}^2 - \frac{k}{r^2}$$
$$mr^2\dot{\theta} = l = \text{constant}$$

Here we have

$$ldt = mr^2 d\theta$$

Note that r depends only on θ.

$$\frac{d}{dt} = \frac{d\theta}{dt}\frac{\partial}{\partial\theta} = \frac{l}{mr^2}\frac{d}{d\theta}$$

$$\frac{d}{dt}(\frac{d}{dt}) = \frac{l}{mr^2}\frac{d}{d\theta}(\frac{l}{mr^2}\frac{d}{d\theta})$$

or

$$\frac{l}{mr^2}\frac{d}{d\theta}(\frac{l}{mr^2}\frac{dr}{d\theta}) = \frac{l^2}{m^2r^3} - \frac{k}{mr^2}$$

We define u as $u = \frac{1}{r}$,

$$\frac{1}{r^2}\frac{dr}{d\theta} = -\frac{d}{d\theta}(\frac{1}{r}) = -\frac{du}{d\theta}$$

Then we have

$$\frac{l^2}{m^2r^2}\frac{d}{d\theta}(-\frac{du}{d\theta}) = \frac{l^2}{m^2r^3} - \frac{k}{mr^2}$$

or

$$\frac{d^2u}{d\theta^2}+u=\frac{mk}{l^2}$$

The solution of this equation is given by

$$u=\frac{1}{r}=\frac{mk}{l^2}(1+e\cos\theta)$$

where e is the eccentricity. Note that r (or u) is an even function of θ. There is no sine-term. Since $r_p=a(1-e)$ for $\theta=0$, and $r_a=a(1+e)$ for $\theta=\pi$, we get

$$\frac{1}{r_p}=\frac{mk}{l^2}(1+e)$$
$$\frac{1}{r_a}=\frac{mk}{l^2}(1-e)$$

or

$$\frac{1}{a(1-e)}=\frac{mk}{l^2}(1+e), \qquad \frac{1}{a(1+e)}=\frac{mk}{l^2}(1-e).$$

Then we have

$$p=a(1-e^2)=\frac{l^2}{mk}$$

leading to the expression

$$r=\frac{p}{1+e\cos\theta}$$

with

$$p=a(1-e^2)=\frac{l^2}{mk}$$

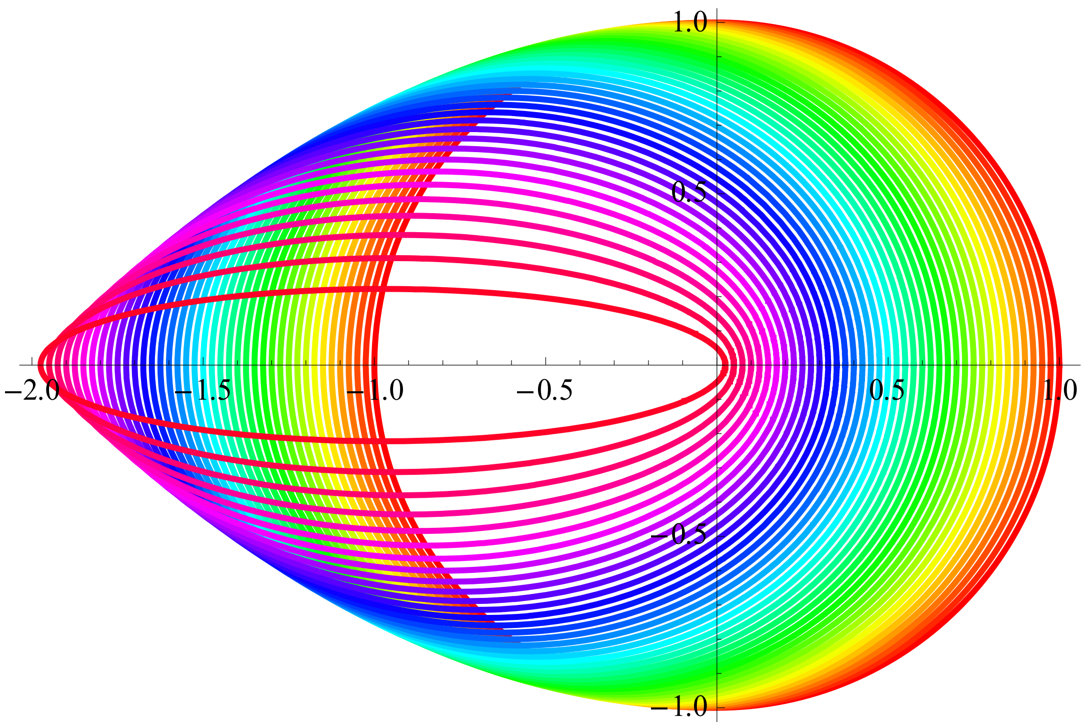


Fig. Ellipse orbits with various eccentricity e ($0<e<1$). The focus is located at the origin. The focus is located at the origin.

14. Black-hole

The escape velocity is the velocity at which a projectile (or particle) would have to be fired straight up so that it will eventually (infinitely far in the future) escape the gravity (come to rest at zero velocity infinitely far away). The escape velocity can be calculated from the energy equation:

$$E = \frac{1}{2}mv^2 - \frac{GmM}{r}$$

For escape, $v = 0$ at $r = \infty$, so therefore in such an orbit $E = 0$. Therefore, at the surface (or any radius r), the escape velocity is given by:

$$v^2 = \frac{2GM}{r}$$

Note that this velocity is higher than the (circular) orbital speed given by the centripetal velocity:

$$v^2 = \frac{GM}{r}$$

by a factor $\sqrt{2}$. If the speed of the Earth in its orbit is increased by more than the factor $\sqrt{2}$, then it would no longer be bound in orbit about the Sun and would be free to fly about the galaxy.

If a mass M is compressed to a radius

$$R_{SW} = \frac{2GM}{c^2}$$

or smaller, then the escape velocity at the radius R_{SW} will equal the speed of light. This radius is called the Schwarzschild Radius for the astrophysicist Karl Schwarzschild who calculated it soon after the publication of Einstein's theory in 1916.

An object with a radius equal to or less than the Schwarzschild Radius R_{SW} is called a black hole. Light, nor anything else, can ever escape the surface of such an object, and it will appear dark. Note that this calculation uses only Newton's theory for gravity. In fact, the possibility for the existence of "dark stars" was postulated as early as 1783.

The Schwarzschild radius for 1 M_{sun} is

$$R_{SW} = \frac{2GM_{sun}}{c^2} = 2.95 km$$

- if the Sun were to suddenly (and inexplicably) collapse to this radius it would become a black hole - though our orbit would remain unchanged since the gravitational force depends only on the mass and distance, not the size of the mass.

The effective radius of a black hole, the Schwarzschild radius, depends only on the mass itself, not on the actual density the mass has (beyond the fact that it must be within its own Schwarzschild radius. As you increase the mass, the radius of the black hole increases proportionally to the mass. Furthermore, since nothing can escape, even light, the mass and size of a black hole can only increase with time.

The spherical "surface" surrounding a black hole of mass M at distance of the Schwarzschild radius R_{SW} is called the event horizon. Once within the event horizon, matter (or radiation) is lost forever from contact with the universe outside the event horizon. The event horizon is the boundary between what we can know about and what we cannot at outside the horizon. Of course, someone unlucky to be inside the event horizon of the black hole can receive news of the outside world in a one-way information transfer.

15. Hodographic solution to the Kepler's problem

In order to see the detail of the following discussion, see the note in the web site, http://bingweb.binghamton.edu/~suzuki/GeneralPhysLN.html

15.1 Geometry of ellipse orbit

(a) Definition of ellipse

An ellipse is the curve that can be made, by taking one string and two tracks and putting a pencil here and going around. Or mathematically, it is the locus, such that the sum of the distance SQ and the distance FQ remains constant (see Fig.1), where S (the Sun) and F are the two fixed points. One may have heard another definition of an ellipse: these two points are called the foci, and this focus means that the light emitted from S will bounce to F from any point on the ellipse (ellipse optic theorem).

Suppose that the Earth undergoes an orbital motion of ellipse where the Sun (S) is one of the focus of the ellipse, and F is another focus. We consider the point Q on the ellipse orbit.

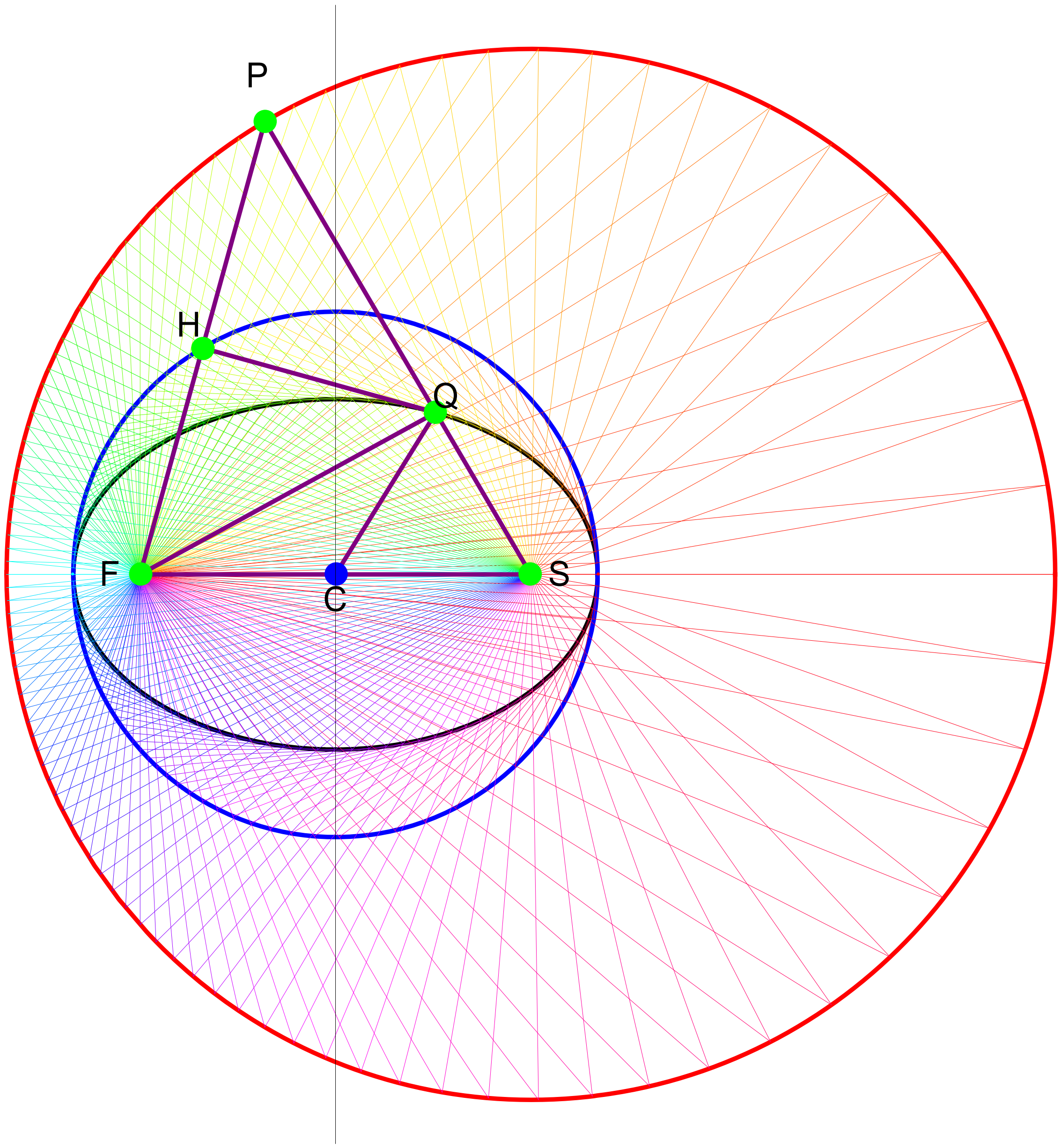


Fig. Hodograph diagram. Q (the Earth) is on the ellipse. F and S (the Sun) are foci. FQ + QS = 2 a. $\angle FQC = \angle SQC$ (ellipse optic theorem). The point P is on the circle (radius $2a$) centered at S. FP is proportional to the velocity at the point Q. The direction of the velocity is parallel to the tangential line at Q.

From the property of the ellipse, we have

$$FQ + SQ = 2a,$$

$$OF = OS = ae,$$

where a is the semi-major axis and e is the eccentricity; $0<e<1$. When $FQ = QP$, we have

$$SP = 2a.$$

The point P is located at the circle with radius $2a$ centered at the focal point S.

(b) Ellipse optic theorem

First we demonstrate the equivalence of these two definitions for ellipse. The light is reflected as though the surface were a plane tangent to the actual curve. We know that the law of reflection for the light from a plane is that the angle of incidence and reflection are the same. In other words, the angles made with the two lines FQ and SQ are equal, that that line is then tangent to the ellipse.

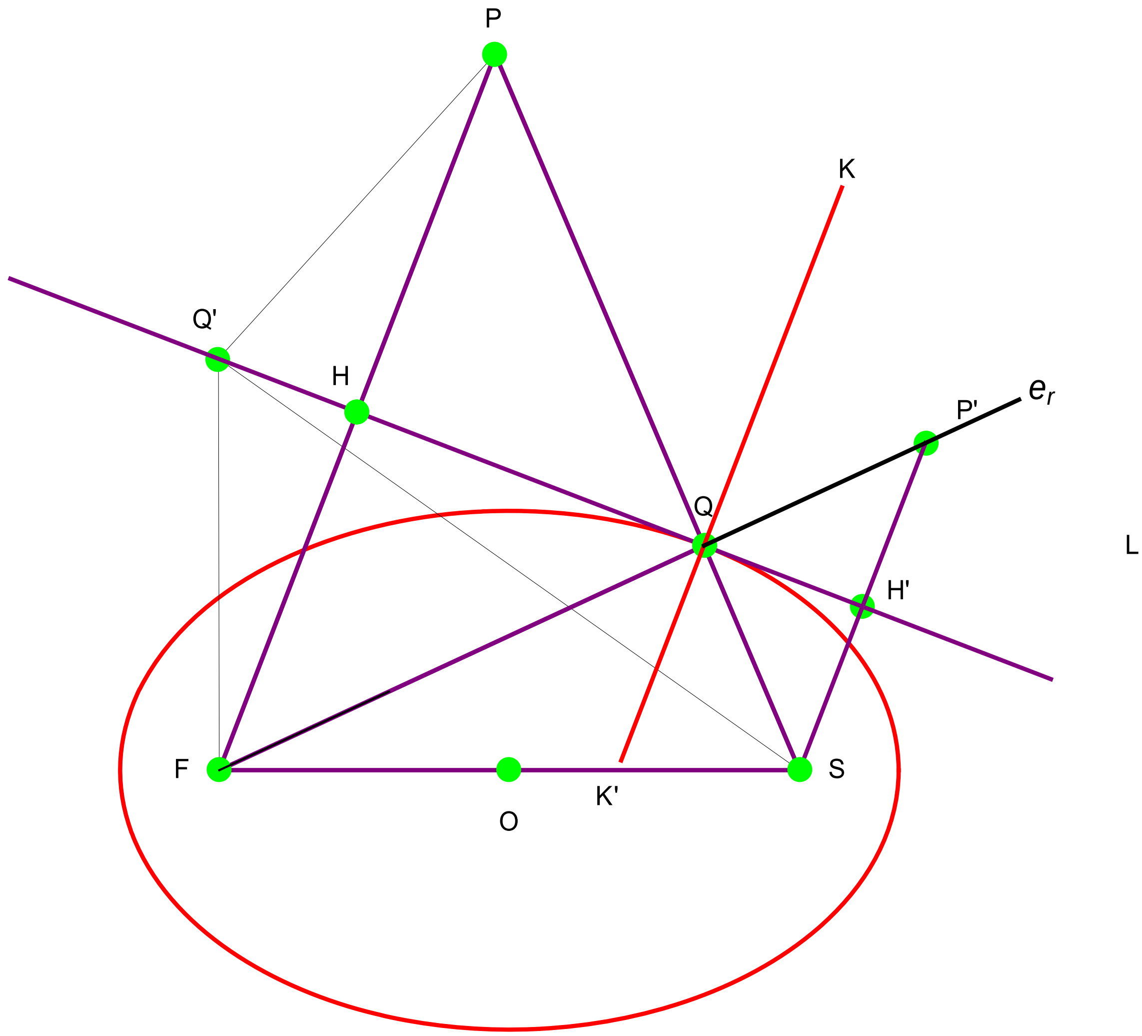


Fig. Q (the Earth) on the ellipse with foci S (the Sun) and F. The green circle (radius a) centered at the origin O. FS = 2*ae*.

((**Proof**))

First we extend the perpendicular from F to the tangential line at the point Q, the same distance on the other side, to obtain P, the image of F; Now connect the point Q to P. Two right triangles are exactly the same (see Fig.2). Thus we have

$$\angle FQH = \angle PQH, \qquad PQ = FQ. \qquad \text{(ellipse optic theorem)}$$

So we get

$$FQ + QS = PQ + QS = SP = 2a \ ,$$

Suppose that we takes any other point on the tangent, Q'. We take the sum of distances,

$$FQ' + Q'S = PQ' + Q'S \ ,$$

where

$$FQ' = PQ' \ .$$

It is clear that the inequality

$$PQ' + Q'S > SP = 2a \ ,$$

in the triangle $\Delta PQ'S$ .In other words, for any point on the tangential line, the sum of the distances from Q' to F and from Q' to F is greater than it is for a point Q on the ellipse.

15.2 The velocity on the ellipse orbit ((J.C. Maxwell))

Here we show the discussion on the velocity, which was given by J.C. Maxwell (see Fig.3). The physics given by Maxwell is very clear for me. We consider the ellipse A_0QP_0 with foci F and S (S standing for the Sun, A_0 the aphelion, and P_0 the perihelion). Let Q be any point on the ellipse, and draw SP through Q, such that SP = A_0P_0 = $2a$. In order to avoid the confusion, we use A_0 and P_0 for the aphelion and perihelion. Draw a line from F to P. It remains to be shown that PF is perpendicular to, and proportional to, the velocity at point Q, and that the locus of P is a circle.

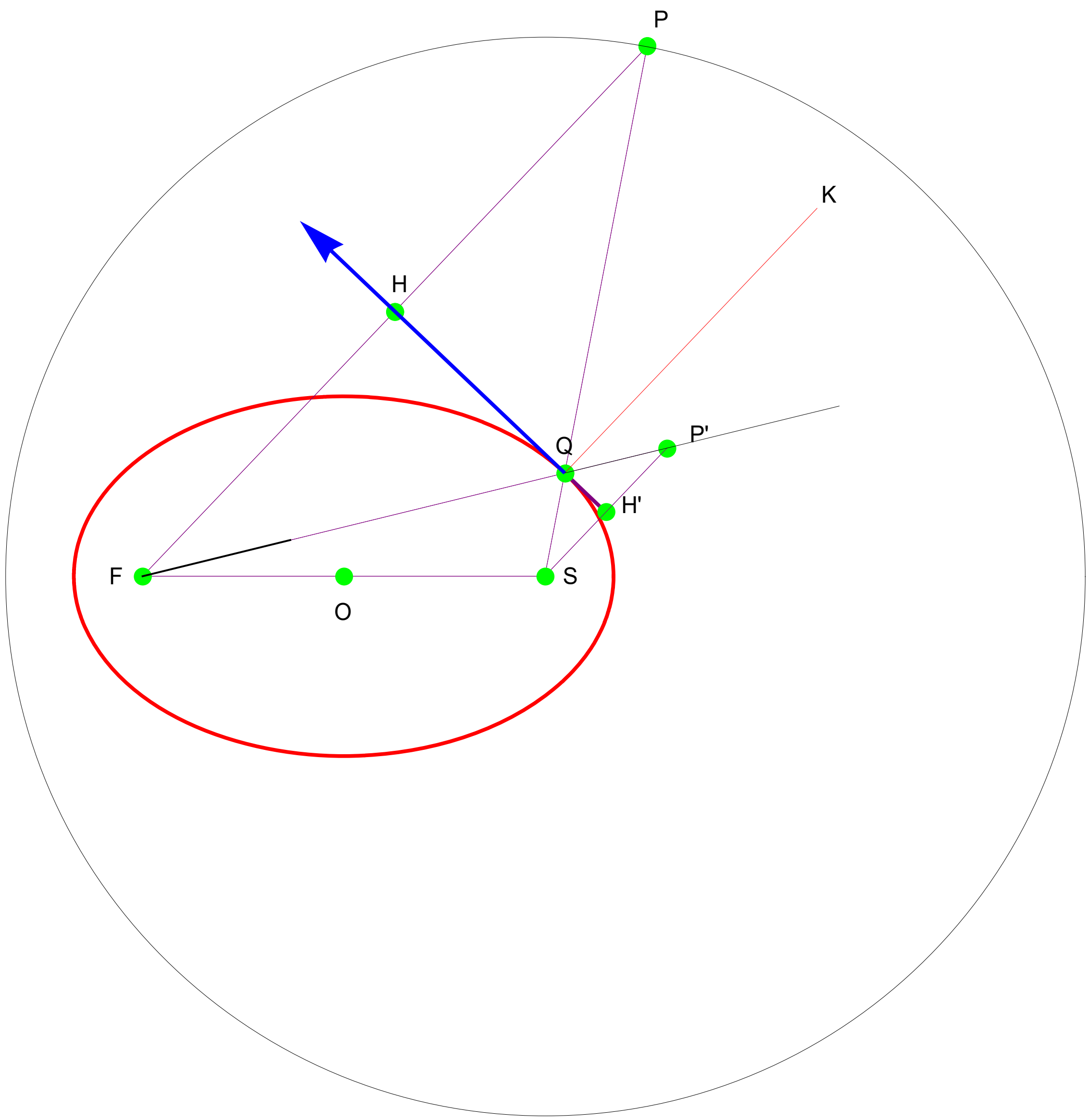


Fig. The direction of the velocity vector. The magnitude of the velocity is proportional to the distance FP.

((**Proof**))

In the ellipse, PF is perpendicular to the velocity at the point Q. Draw a tangent from Q to intersect PF at H. Then by the ellipse optical theorem,

$$\angle SQH' = \angle FQH,$$

and

$$\angle FQH = \angle PQH .$$

We also have

$$PQ = PS - QS = 2a - QS = FQ .$$

where

$PS = 2a$.   ($2a$: the distance between the perihelion and aphelion)

So HQ is perpendicular to PF. Then the direction of PF is perpendicular to the tangent, and hence the velocity at Q.

In the ellipse, PF is proportional to the velocity at Q. Draw a perpendicular line from S to the tangent to intersect the tangent at H'. Let $\boldsymbol{v}$ be the velocity at Q, of the magnitude v. By the conservation of angular momentum, we get

$$mvH'S = l ,$$

where l is a constant. Using the geometrical theorem

$$HF \cdot H'S = b^2 ,$$

we get

$$\frac{1}{H'S} = \frac{mv}{l} = \frac{HF}{b^2} ,$$

or

$$HF = \frac{mb^2}{l} v = \frac{1}{2} PF ,$$

so PF is proportional to v.

Since SP is always equal to the major axis, it follows that the locus of P is a circle, with common origin of the velocity vectors at F; this circle is the hodograph turned through 90° because PF is perpendicular to $\boldsymbol{v}$. Then we have

$$v = \frac{l}{mb^2} HF , \qquad \text{or} \qquad v = \frac{l}{2mb^2} PF$$

$\frac{l}{mb^2}$ is the scaling factor and the unit is [1/s].

Note that this scaling factor. We consider the aphelion and the perihelion

(i) At the aphelion

$$l = mv_A r_A,$$

$$v_A = \alpha a(1-e), \qquad r_A = a(1+e)$$

Then we have

$$l = m\alpha a^2(1+e)(1-e) = m\alpha a^2(1-e^2) = \alpha m b^2.$$

The scaling factor is obtained as

$$\alpha = \frac{l}{mb^2}.$$

(ii) At the perihelion,

$$l = mv_P r_P,$$

$$v_p = \alpha a(1+e), \qquad r_p = a(1-e).$$

Then we have

$$l = m\alpha a^2(1-e)(1+e) = m\alpha a^2(1-e^2) = m\alpha b^2,$$

or

$$\alpha = \frac{l}{mb^2} = \frac{l}{ma^2(1-e^2)} = \frac{l}{map},$$

where b is the minor axis distance,

$$b = a\sqrt{1-e^2},$$

and p is the semi-latus rectum;

$$p = a(1 - e^2).$$

15.3. Centripetal acceleration

First we discuss the velocity vectors at the point Q_1 and Q_2 on the same ellipse, where Q_1 and Q_2 are very close. The velocity at the point Q_1 is proportional to the length FP_1.

$$v_1 = \frac{l}{2map}\left|\overrightarrow{FP_1}\right|.$$

The velocity is directed along the tangential line at the point Q_1. The rotation of the vector $\overrightarrow{FP_1}$ around the point F by $\pi/2$ in a counterclockwise leads to the direction of the velocity. For this rotation we use the geometrical rotation operator $\Re(z, \frac{\pi}{2})$.

$$\boldsymbol{v}_1 = \frac{l}{2map}\Re(z, \frac{\pi}{2})\overrightarrow{FP_1}.$$

When the particle rotates from Q_1 to $Q_{2'}$ on the ellipse during the time Δt, the instantaneous acceleration $\boldsymbol{a}$ is

$$\boldsymbol{a} = \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v}_2 - \boldsymbol{v}_1}{\Delta t} = \frac{l}{2map_0\Delta t}\Re(z, \frac{\pi}{2})[\overrightarrow{FP_2} - \overrightarrow{FP_1}] = \frac{l}{2map_0\Delta t}\Re(z, \frac{\pi}{2})[\overrightarrow{P_1P_2}].$$

where

$$\boldsymbol{v}_2 = \frac{l}{2map_0}\Re(z, \frac{\pi}{2})\overrightarrow{FP_2}.$$

In the limit where the point P_2 is very close to P_1. the vector $\overrightarrow{P_1P_2}$ is perpendicular to the vector $\overrightarrow{SP_1}$. Then the acceraltion is directed toward the Sun (one of the focus in the ellipsoid).

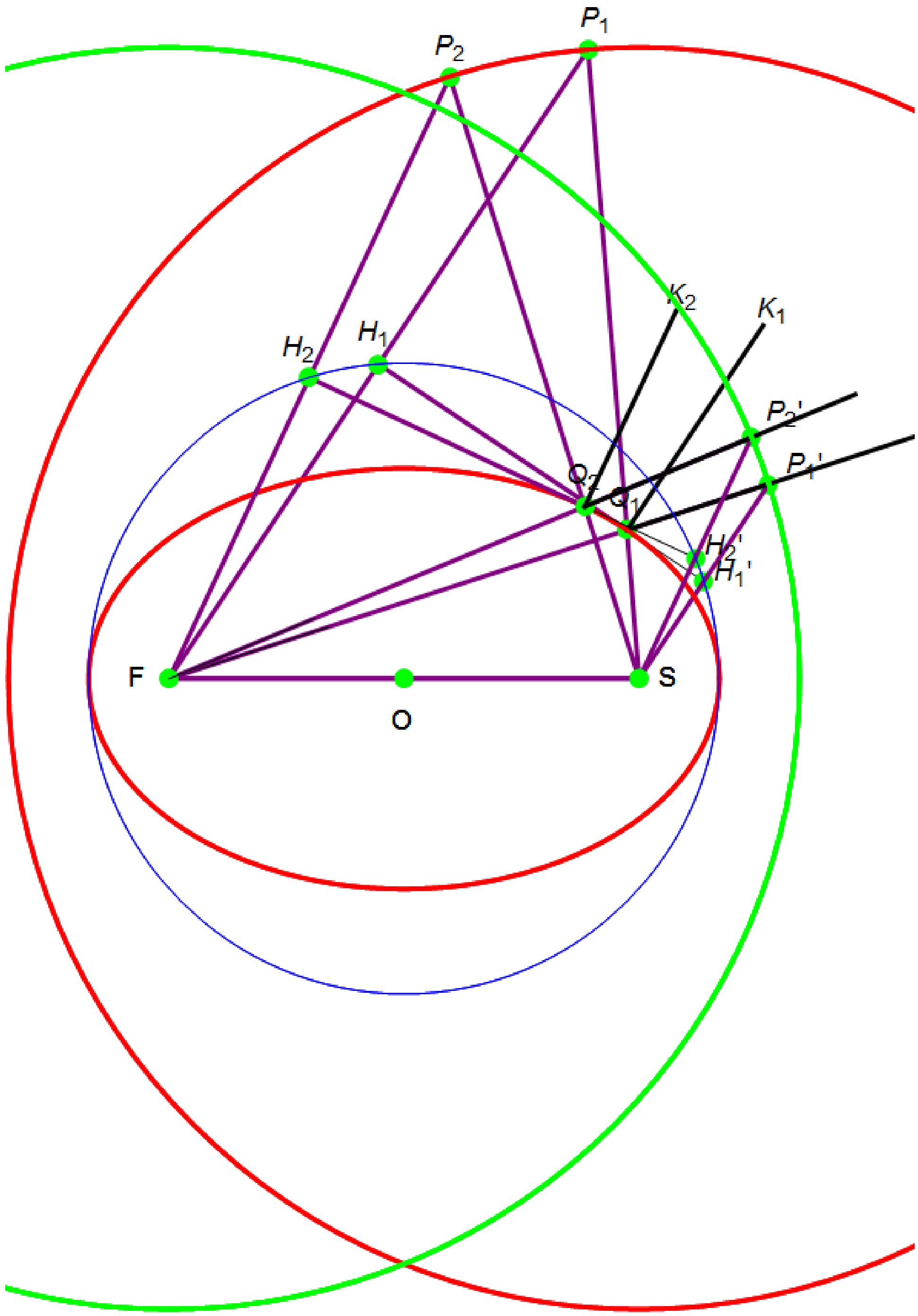


Fig. The points P_1 and P_2 are on the circle (radius 2a) centered at S. The point P_1' and P_2' are on the circle (radius 2a) centered at F. The points H_1, H_2, H_1', and H_2' are on the circle (radius a) centered at the origin O. $K_1Q_1 \perp H_1Q_1$. $K_2Q_2 \perp H_2Q_2$.

15.4. Centripetal acceleration: central-force problem

If PP' is the arc described in unit of time, then PP' represents the acceleration, and since P P' is on a circle whose center is S, the distance of arc PP' will be a measure of angular velocity,

$$\text{PP'} = 2a\Delta\theta = 2a\omega\Delta t = \frac{2al}{mr^2}\Delta t,$$

where ω is the angular velocity at the point Q,

$$\omega = \frac{l}{mr^2}.$$

Note that the angular momentum l is conserved and is given by

$$l = mr^2\frac{d\theta}{dt} = mr^2\omega.$$

The acceleration $\boldsymbol{a}$ is obtained as

$$\begin{aligned}
\boldsymbol{a} &= \frac{\Delta\boldsymbol{v}}{\Delta t} \\
&= \frac{l}{2map_0\Delta t}\Re(z,\frac{\pi}{2})[\overrightarrow{PP'}] \\
&= \frac{l}{2map_0\Delta t}\Re(z,\frac{\pi}{2})[\frac{2al}{mr^2}\Delta t\boldsymbol{e}_\theta] \\
&= \frac{l}{map_0}\frac{al}{mr^2}\Re(z,\frac{\pi}{2})[\boldsymbol{e}_\theta]
\end{aligned}$$

or

$$\boldsymbol{F} = m\boldsymbol{a} = -\frac{l^2}{mpr^2}\boldsymbol{e}_r,$$

where $\Re(z,\frac{\pi}{2})$ is the geometrical rotation operator (counter-clockwise rotation of the system around the z axis by .

$$\Re(z,\frac{\pi}{2})[\boldsymbol{e}_\theta] = -\boldsymbol{e}_r,$$

and

$$p = a(1 - e^2).$$

The acceleration is inversely as the square of the distance SQ. Hence the acceleration of the planet is in the direction of the Sun, and is inversely as the square of the distance from the Sun.

This, therefore, is the law according to which the attraction of the Sun on a planet varies as the planet moves in the orbit and alters its distance from the Sun.

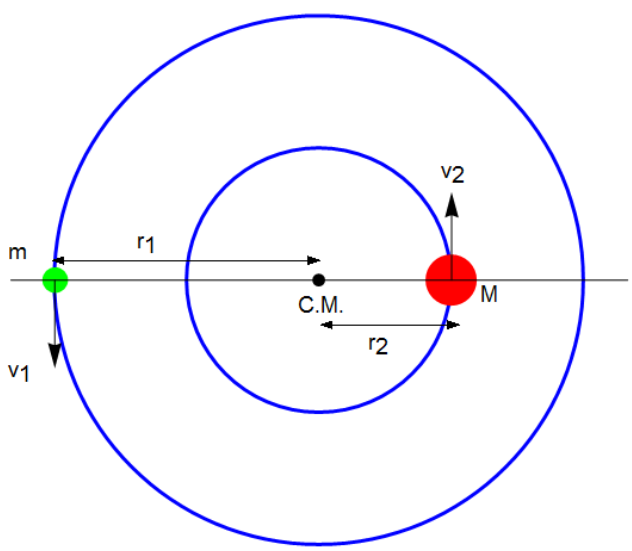
16. Advanced problems

16.1 13-57 Serway

Two stars of masses M and m, separated by a distance d, resolve in circular orbits about their center of mass. Show that each star has a period given by

$$T^2 = \frac{4\pi^2 d^3}{G(M+m)}$$

Proceed by applying Newton's second law to each star. Note that the center-of-mass condition requires that $Mr_2 = mr_1$, where $r_1 + r_2 = d$.



((**Solution**)

The origin is the center-of-mass of the two stars.

$$0 = Mr_2 + m(-r_1)$$
$$Mr_2 = mr_1$$

where $d = r_1 + r_2$.

For each mass,

$$mr_1\omega^2 = \frac{GmM}{d^2}, \quad Mr_2\omega^2 = \frac{GmM}{d^2} \qquad \text{or} \qquad r_1\omega^2 = \frac{GM}{d^2}, \quad r_2\omega^2 = \frac{Gm}{d^2}$$

or

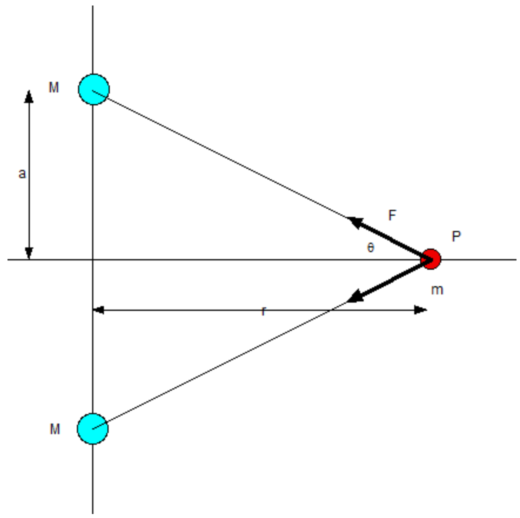
$$(r_1 + r_2)\omega^2 = \frac{G(M+m)}{d^2}$$
$$\omega^2 = \frac{G(M+m)}{d^3}$$

Since $T = \dfrac{2\pi}{\omega}$, we have

$$\frac{4\pi^2}{T^2} = \frac{G(M+m)}{d^3}$$
$$T^2 = \frac{4\pi^2 d^3}{G(M+m)}$$

16.2. Advanced problem-2
Serway 13-23

Compute the vector gravitational field at point P on the perpendicular bisector of the line, joining two objects of equal mass separated by a distance $2a$. (b) Explain physically why the field should approach zero as $r \to 0$. (c) Prove mathematically why the magnitude of the field should approach $2GM/r^2$ as $r \to \infty$. (e) Prove mathematically that the answer to part (a) behaves correctly in this limit.



((**Solution**))

(a)

$$F_x = -2F\cos\theta = -2\frac{GMm}{(x^2+a^2)}\frac{x}{(x^2+a^2)^{1/2}} = -\frac{2GMmx}{(x^2+a^2)^{3/2}}$$

(b) The gravitational force g_x is defined by

$$g_x = \frac{F_x}{m} = -\frac{2GMx}{(x^2+a^2)^{3/2}}$$

In the limit of $x >> a$,

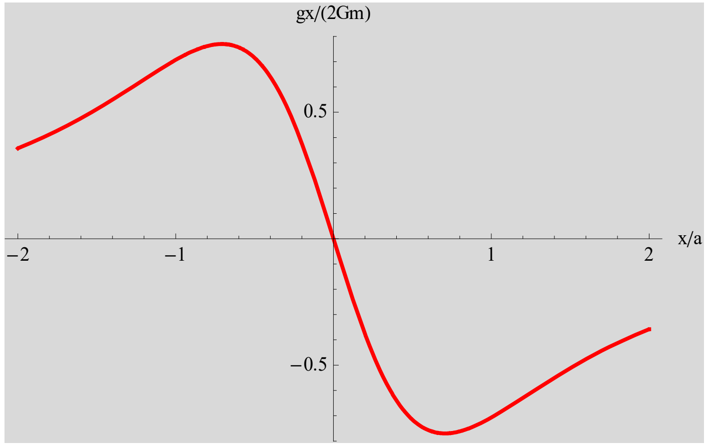
$$g_x = -\frac{2GM}{x^2}$$

(d)

$$\frac{g_x}{2GM} = f(x) = -\frac{x}{(x^2+a^2)^{3/2}}$$

$$f'(x) = -\frac{2(x^2 - \frac{a^2}{2})}{(x^2+a^2)^{5/2}}$$

Then *f*(*x*) has a local maximum at $x = \frac{a}{\sqrt{2}}$ and a local minimum at $x = -\frac{a}{\sqrt{2}}$.

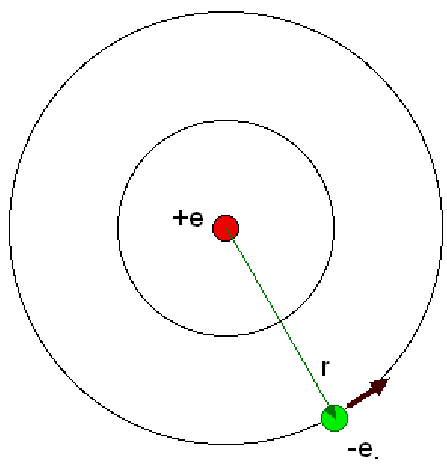


17 Bohr model

Niels Bohr

Niels Henrik David Bohr in Danish; October 7, 1885 – November 18, 1962) was a Danish physicist who made fundamental contributions to understanding atomic structure and quantum mechanics, for which he received the Nobel Prize in Physics in 1922. Bohr mentored and collaborated with many of the top physicists of the century at his institute in Copenhagen. He was also part of the team of physicists working on the Manhattan Project. Bohr married Margrethe Nørlund in 1912, and one of their sons, Aage Niels Bohr, grew up to be an important physicist who, like his father, received the Nobel prize, in 1975. Bohr has been described as one of the most influential physicists of the 20th century.

We now consider the Bohr model shown in this figure. The system consists of a proton and an electron. These two particles are coupled with an attractive Coulomb interaction.



The total energy is a sum of kinetic energy and potential energy (CGS units are used here)

$$E = \frac{1}{2}mv^2 - \frac{e^2}{r}$$

$$m\frac{v^2}{r} = \frac{e^2}{r^2}, \qquad mv^2 r = e^2 \tag{1}$$

or

$$E = \frac{1}{2}\frac{e^2}{r} - \frac{e^2}{r} = -\frac{e^2}{2r}$$

Note that in SI units, the energy is given by

$$E = \frac{1}{2}mv^2 - \frac{e^2}{4\pi\varepsilon_0 r}.$$

The de Broglie relation:

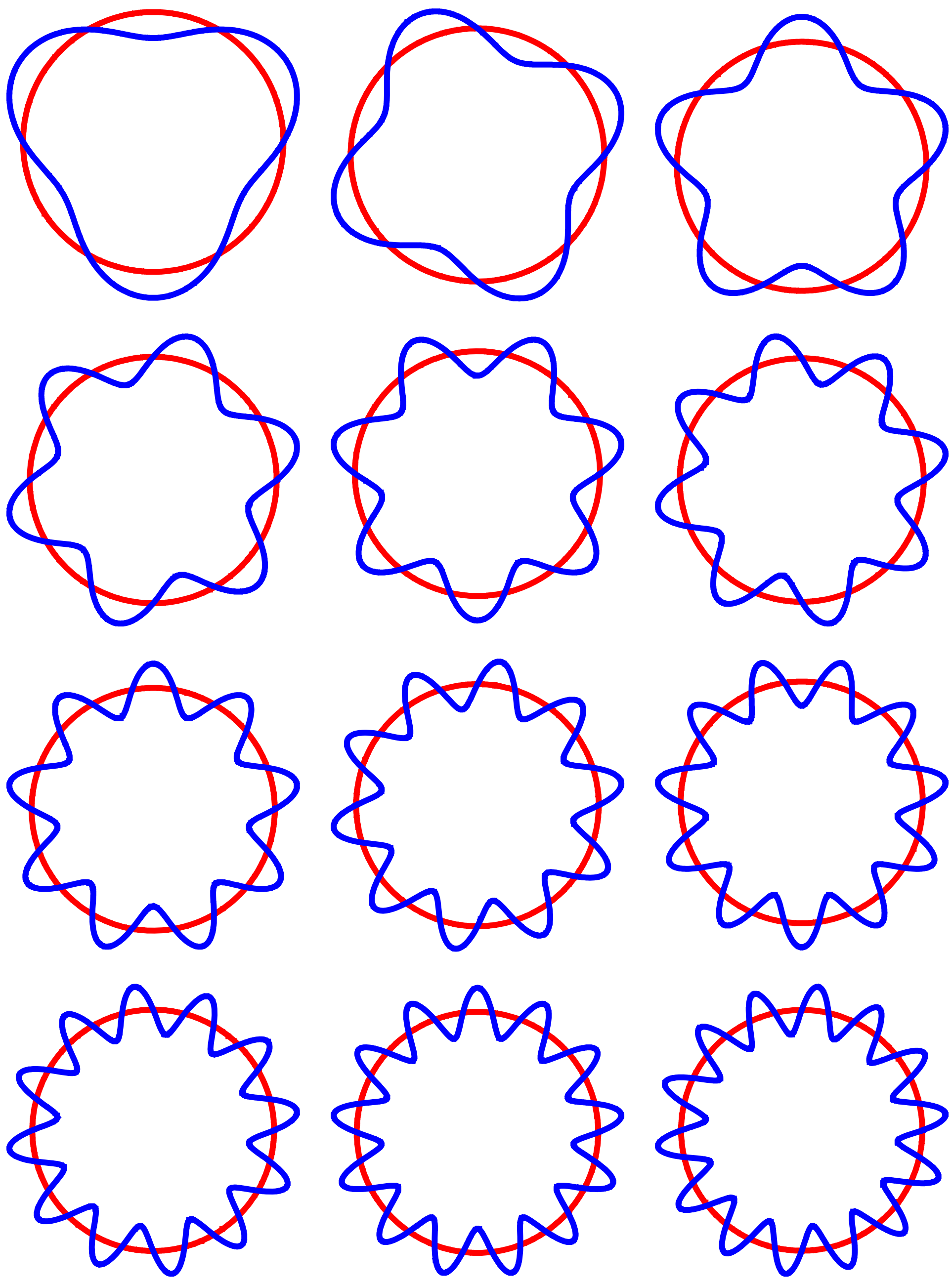


Fig. Acceptable wave on the ring (circular orbit). The circumference should be equal to the integer *n* (=1, 2, 3,…) times the de Broglie wavelength λ. The picture of fitting the de Broglie waves onto a circle makes clear the reason why the orbital angular momentum is quantized. Only integral numbers of wavelengths can be fitted. Otherwise, there would be destructive interference between waves on successive cycles of the ring.

$$2\pi r = n\lambda$$

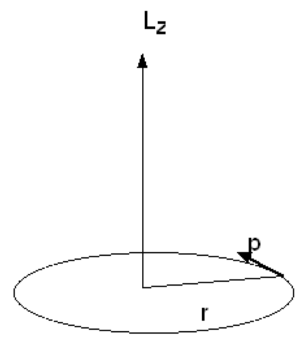
where n is integer.

de Broglie relation

$$p = \frac{h}{\lambda}$$

$$p(2\pi r) = \frac{h}{\lambda} 2\pi r = nh$$

The angular momentum L_z:



$$L_z = pr = \frac{nh}{2\pi} = n\hbar \quad \text{or} \quad mvr = n\hbar \tag{2}$$

The angular momentum is quantized.

From Eqs.(1) and (2),

$$\frac{mv^2 r}{mvr} = \frac{e^2}{n\hbar}, \quad \text{or} \quad v = \frac{e^2}{n\hbar}$$

$$m\left(\frac{e^2}{n\hbar}\right)^2 r = e^2, \quad \text{or} \quad r = \frac{n^2\hbar^2}{me^2}$$

Then the total energy is obtained by

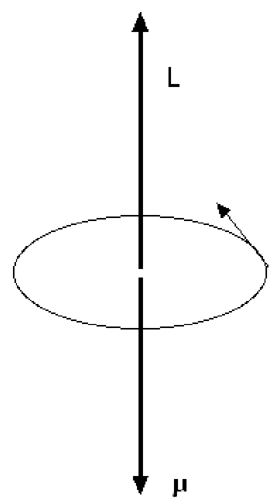
$$E_n = -\frac{e^2}{2\left(\frac{n^2\hbar^2}{me^2}\right)} = -\frac{me^4}{2\hbar^2 n^2} = -\frac{R}{n^2}$$

where R=13.6 eV (Rydberg constant).

The energy is quantized. The ground state is a state with $n = 1$.
Note that the magnetic moment μ due to the orbital motion is also quantized.

$$\mu = \frac{IA}{c}$$

where A is the area: $A = \pi r^2$. I is the current: $I = \frac{e}{T} = \frac{e}{(2\pi r/v)} = \frac{ev}{2\pi r}$. The direction of the current is opposite to the direction of velocity of electron because the charge is negative. We assume that the electron has a charge $-e$ (e<0).



$$\mu_z = \frac{IA}{c} = \frac{ev}{2\pi rc}\pi r^2 = \frac{evr}{2c} = \frac{emvr}{2mc} = \frac{e}{2mc}L_z = \frac{e\hbar}{2mc}\frac{L_z}{\hbar}$$

The Bohr magneton μ_B is defined as

$$\mu_B = \frac{e\hbar}{2mc} = 9.27410 \times 10^{-21} emu$$

where emu = erg/G

The spin magnetic moment is given by

$$\mu_S = \mu_B \frac{2S_z}{\hbar}$$

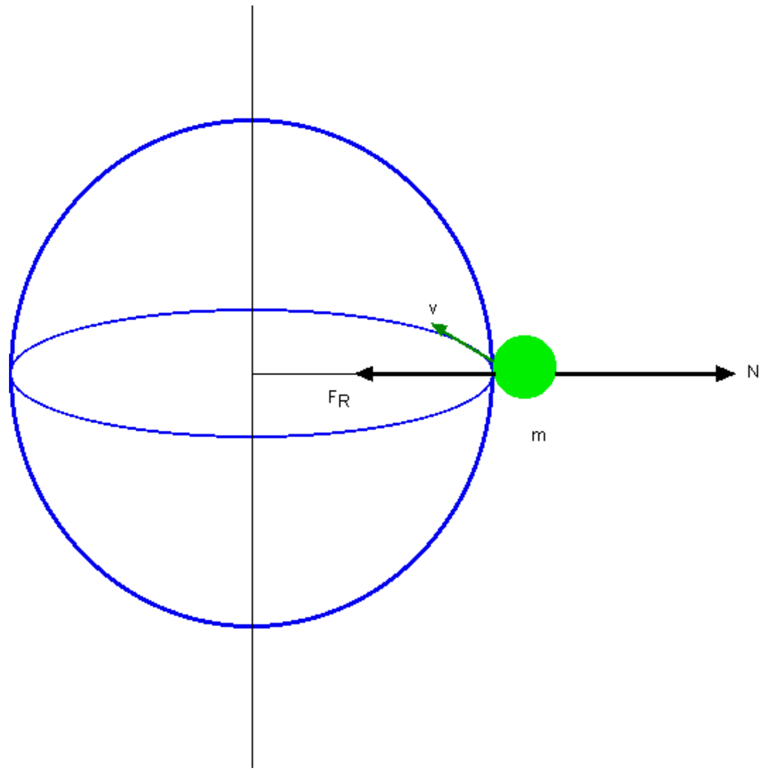
18. HW and SP

18.1

Problem 13-28 (SP-13) (8-th edition)**
Problem 13-26 (SP-13) (8-th edition)**

Consider a pulsar, a collapsed star of extremely high density, with a mass M equal to that of the Sun (1.98 x 10^{30} kg), a radius R of 12 km, a rotational period T of 0.041 s. By what percentage does the free-fall accelerating g differ from the gravitational acceleration a_g at the equater of this spherical star?

((**Solution**))
M = 1.98 x 10^{30} kg
R = 12 km
T = 0.41 s



N: Normal force

$$F_R - N = m\frac{v^2}{R}$$

$$N = F_R - m\frac{v^2}{R}$$

where

$$F_R = \frac{mMG}{R^2}$$

The value of *g* (free-fall acceleration) is defined as

$$N = mg = F_R - m\frac{v^2}{R} = m(\frac{MG}{R^2} - \frac{v^2}{R})$$

$$g = \frac{MG}{R^2} - \frac{v^2}{R}$$

$$= \frac{MG}{R^2} - R\omega^2$$

$$= \frac{MG}{R^2} - R\left(\frac{2\pi}{T}\right)^2 = 9.17256\times 10^{11} m/s^2$$

The gravitational acceleration g_0

$$g_0 = \frac{MG}{R^2} = 9.17538\times 10^{11} m/s^2$$

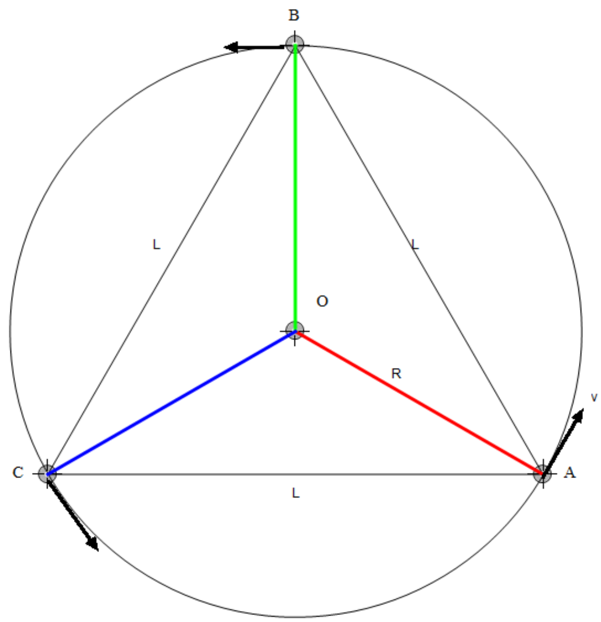
$$\frac{\Delta g}{g_0} = \frac{g_0 - g}{g_0} = 0.031\%$$

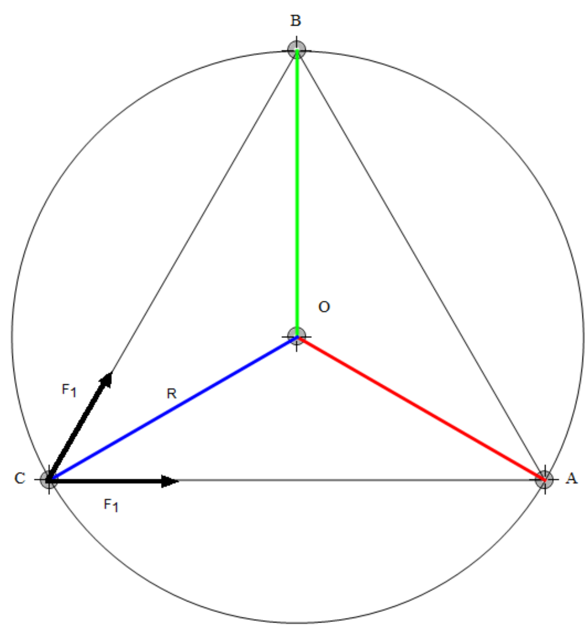
18.2

Problem 13-59 (SP-13)* (10-th edition)**

Three identical stars of mass *M* from an equilateral triangle that rotates around the triangle's center as the stars move in a common circle about that center. The triangle has edge length *L*. What is the speed of stars?

((**Solution**))





(a) $L = 2R\cos 30° = \sqrt{3}R$

(b)

$$F_{net} = 2F_1\cos 30° = 2\frac{M^2G}{(\sqrt{3}R)^2}\frac{\sqrt{3}}{2} = \frac{M^2G}{\sqrt{3}R^2} = M\frac{v^2}{R}$$

or

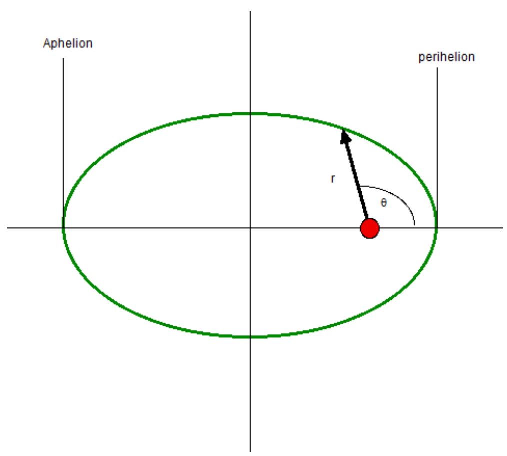
$$v = \sqrt{\frac{MG}{\sqrt{3}R}} = \sqrt{\frac{GM}{L}}$$

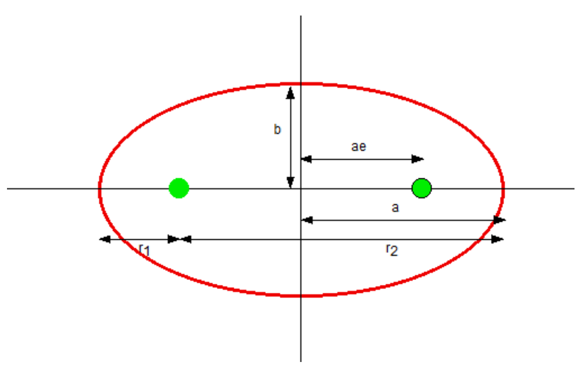
18.3

Problem 13-82 (HW-13) (10-th edition)

A satellite is in elliptical orbit with a period of 8.00 x 10^4 s about a planet of mass 7.00 x 10^{24} kg. At aphelion, at radius 4.4 x 10^7 m, the satellite's angular speed is 7.158 x 10^{-5} rad/s. What is its angular speed at perihelion?

((**Solution**))





T = 8.0 x 10^4 s, M = 7.00 x 10^{24} kg

$r_a = a(1+e) = 4.5\times10^7 m$ for aphelion (farthest)

$\omega_a = 7.158\times10^{-5} rad/s$

$r_p = a(1-e)$ for the perihelion (nearest)

where e is the essentricity.

What is the value of ω_p?

Kepler's third law

$$\frac{a^3}{T^2} = \frac{GM}{4\pi^2}$$

Since $\ell = mr^2\omega = const$ (angular momentum conservation), we have

$$\ell = mr_a{}^2\omega_a = mr_p{}^2\omega_p$$

19 Link

Ring of Saturn
http://en.wikipedia.org/wiki/Rings_of_Saturn

Gauss' law for gravity
http://en.wikipedia.org/wiki/Gauss%27s_law_for_gravity

Escape velocity
http://en.wikipedia.org/wiki/Escape_velocity

Kepler's law of planetary motion
http://en.wikipedia.org/wiki/Kepler%27s_laws_of_planetary_motion

Kepler's law
http://hyperphysics.phy-astr.gsu.edu/hbase/kepler.html

Black hole
http://en.wikipedia.org/wiki/Black_hole

Bohr model
http://en.wikipedia.org/wiki/Bohr_model

de Broglie relation
http://en.wikipedia.org/wiki/De_Broglie_hypothesis

The Bohr model of the atom
http://www.upscale.utoronto.ca/GeneralInterest/Harrison/BohrModel/BohrModel.html

Orbital magnetic moment
http://hyperphysics.phy-astr.gsu.edu/Hbase/quantum/orbmag.html

Spin magnetic moment
http://en.wikipedia.org/wiki/Spin_magnetic_moment

Lecture Note (University of Rochester)
http://teacher.pas.rochester.edu/phy121/LectureNotes/Contents.html

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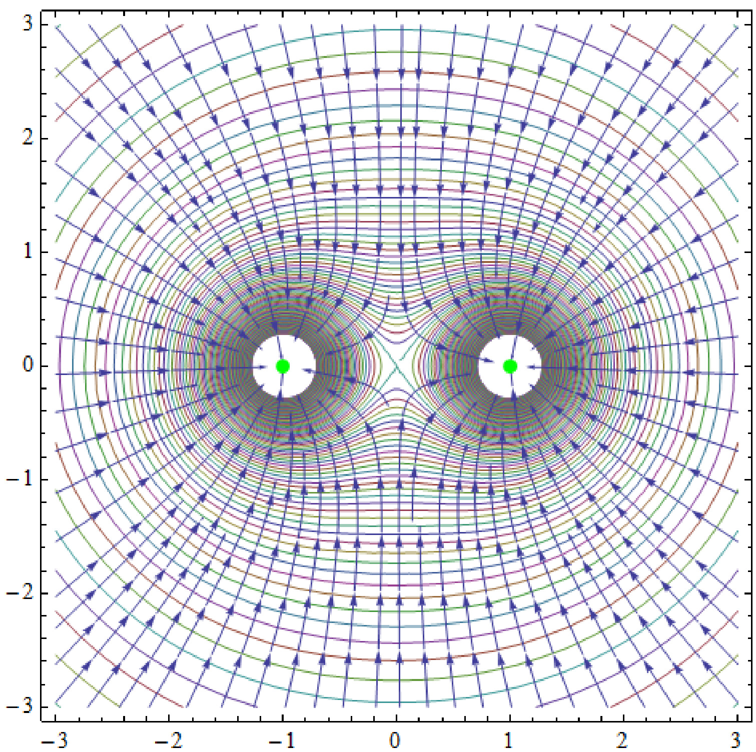
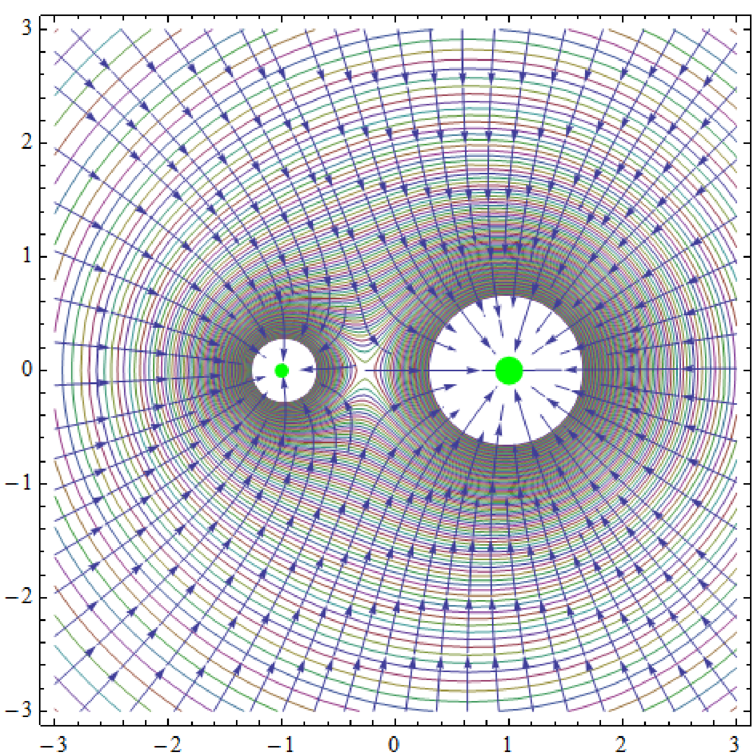
Youtube:
Carl Sagan: Kepler's law
http://www.youtube.com/watch?v=XFqM0lreJYw

Appendix-1 Contour map of equipotential surface

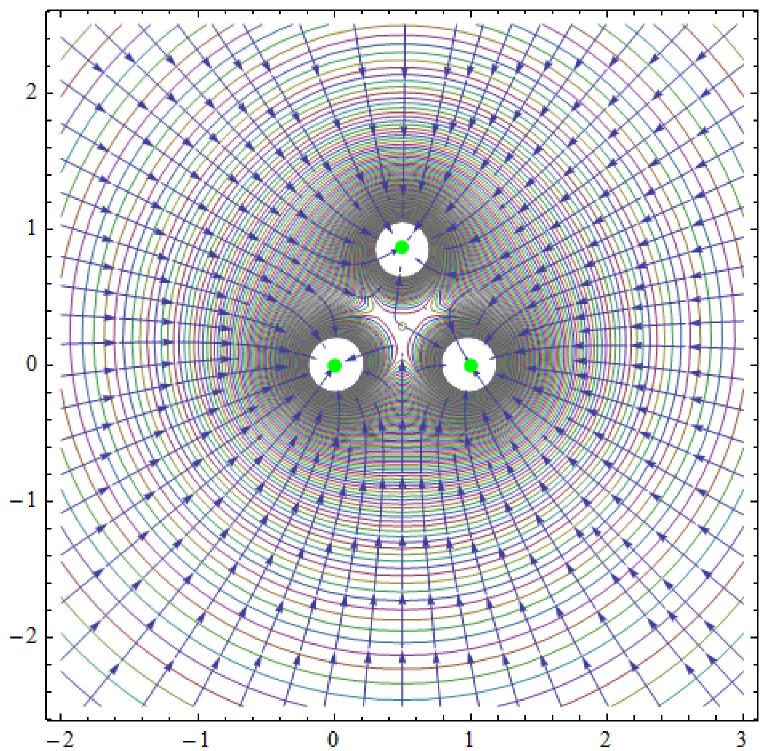
(1) Contour map of equipotential surface between two equal masses at (-*a*,0) and the point (*a*, 0)

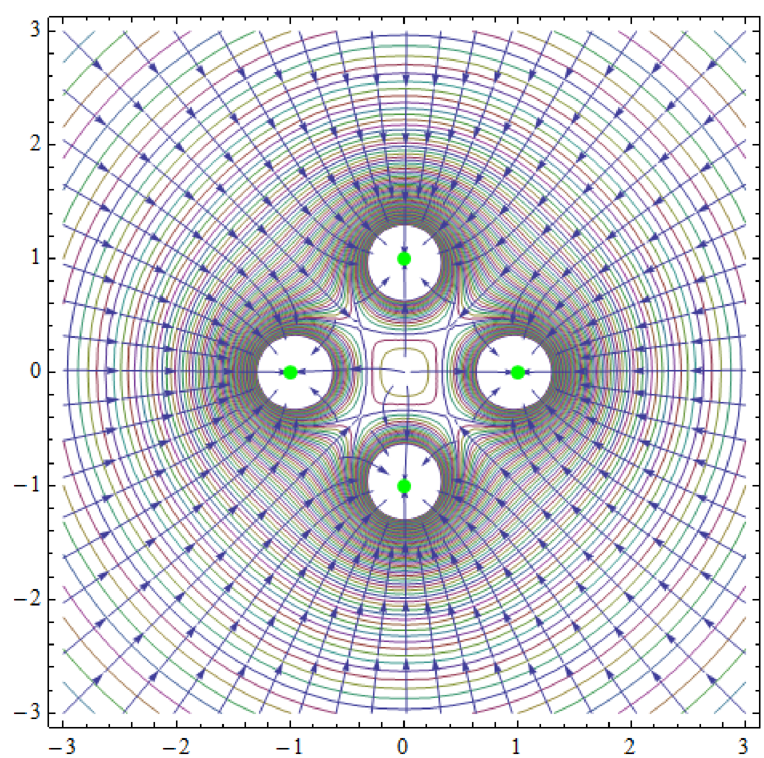


(2) Contour map of equipotential surfaces between mass M at $(-a,0)$ and $3M$ at the point (a, 0).



(3) Contour map of equipotential surfaces between there identical masses (M) at the origin, the point (a, 0), and the point (a/2, $\sqrt{3}a/2$)





APPENDIX II Terminology

(a) Eccentricity *e*

From Medieval Latin *eccentricus*, derived from Greek *ekkentros* "out of the center", from *ek*-, *ex*- "out of" + *kentron* "center". Eccentric first appeared in English in 1551, with the definition "a circle in which the earth, sun. etc. deviates from its center." Five years later, in 1556, an adjective form of the word was added.

(b) Semi latus rectum *p*

The chord through a focus parallel to the conic section directrix of a conic section is called the latus rectum, and half this length is called the semi latus rectum (Coxeter 1969). "Semi latus rectum" is a compound of the Latin *semi*-, meaning half, *latus*, meaning 'side,' and *rectum*, meaning 'straight.'

(c) Perihelion

The perihelion is the point in the orbit of a planet, asteroid or comet where it is nearest to the sun. The word perihelion stems from the Greek words "peri" (meaning "near") and "helios" (meaning "sun").

(d) Aphelion

Derivative terms are used to identify the body being orbited. The most common are perigee and apogee, referring to orbits around the Earth (Greek γῆ, *gê*, "earth"), and perihelion and aphelion, referring to orbits around the Sun (Greek ἥλιος, *hēlios*, "sun").

APPENDIX-III Method of Lagrangian (advanced topics)

Mathematica program for the Kepler’s problem. I use the method of Lagrangian for this Kepler’s problem. See the advanced textbook of classical mechanics such as H. Goldstein, Classical Mechanics).

Method of Lagrangian

```
Clear["Global`*"]; << "VariationalMethods`"
```

Lagrange equation

$$L = \frac{1}{2} m \left(r'[t]^2 + r[t]^2 \phi'[t]^2\right) - V[r[t]];$$

```
eq1 = EulerEquations[L, {r[t], ϕ[t]}, t];
```

Note that the central force is expressed by
f(r)= - V'(r)

$$eq11 = f[r[t]] + m\, r[t]\, \phi'[t]^2 - m\, r''[t] == 0;$$

```
eq2 = EulerEquations[L, {r[t], ϕ[t]}, t]
```

$$\{-V'[r[t]] + m\, r[t]\, \phi'[t]^2 - m\, r''[t] == 0,\ -m\, r[t]\, (2\, r'[t]\, \phi'[t] + r[t]\, \phi''[t]) == 0\}$$

FirstIntegral[ϕ] = constant = l Angular momentum conservation
FirstIntegral[t] = constant = E Energy conservation

```
eq3 = FirstIntegrals[L, {r[t], ϕ[t]}, t] // Simplify
```

$$\{\text{FirstIntegral}[\phi] \to -m\, r[t]^2\, \phi'[t],$$
$$\text{FirstIntegral}[t] \to \frac{1}{2}\left(2\, V[r[t]] + m\left(r'[t]^2 + r[t]^2\, \phi'[t]^2\right)\right)\}$$

$$eq4 = eq11\ /.\ \left\{\phi'[t] \to \frac{l}{m\, r[t]^2}\right\}\ //\ \text{Simplify}$$

$$f[r[t]] + \frac{l^2}{m\, r[t]^3} == m\, r''[t]$$

$$m \frac{d^2}{dt^2} r - \frac{l^2}{m r^3} = f(r)$$

$$m \frac{l}{m r^2} \frac{d}{d\phi} (\frac{l}{m r^2} \frac{d}{d\phi}) r - \frac{l^2}{m r^3} = f(r)$$

$$\frac{l}{r^2} \frac{d}{d\phi} \left(\frac{l}{m r^2} \frac{d}{d\phi}\right) r - \frac{l^2}{m r^3} = f(r)$$

We notice that

$$\frac{d}{d\phi} \frac{1}{r} = - \frac{1}{r^2} \frac{dr}{d\phi}$$

We put

$$u = \frac{1}{r}$$

Then we have

$$\frac{l^2 u^2}{m} \frac{d}{d\phi} (- \frac{du}{d\phi}) - \frac{l^2}{m} u^3 = f(\frac{r}{u})$$

or

$$\frac{l^2 u^2}{m} [\frac{d^2 u}{d\phi^2} + u] = - f(\frac{1}{u})$$

or

$$\frac{d^2 u}{d\phi^2} + u = - \frac{m}{l^2} \frac{1}{u^2} f\left(\frac{1}{u}\right)$$

■

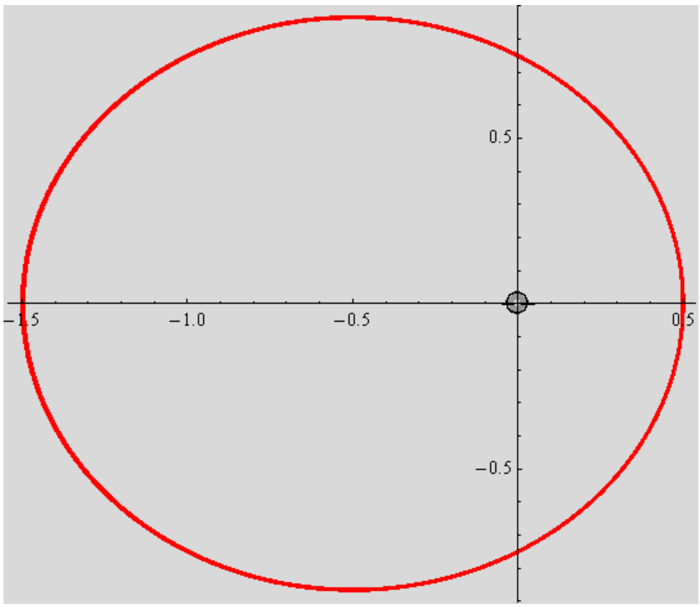
Solution of Kepler's problem

```
Clear["Global`*"];

eq1 = u''[ϕ] + u[ϕ] == -m/(l^2 u[ϕ]^2) f[1/u[ϕ]];

forceRule = {V → (-k/# &), f[1/u[ϕ]] → -V'[r], r → 1/u[ϕ]};

eq2 = eq1 //. forceRule // ExpandAll;

eq3 = DSolve[{eq2, u'[0] == 0}, u[ϕ], ϕ] // Simplify // Flatten;

cRule = {C[1] → e k m/l^2};

usol = eq3 /. cRule // Simplify;

eq31 = eq3 /. cRule // FullSimplify;

elRule = Solve[k m/l^2 == 1/(a (1 - e^2)), l][[2]] // Simplify;

u1[ϕ_] = u[ϕ] /. eq31[[1]] /. elRule[[1]];

x[ϕ_] := Cos[ϕ] / u1[ϕ]; y[ϕ_] := Sin[ϕ] / u1[ϕ];
```

0<e<1: ellipse   (e=0.5)

```
values = {a → 1, e → 0.5};
```

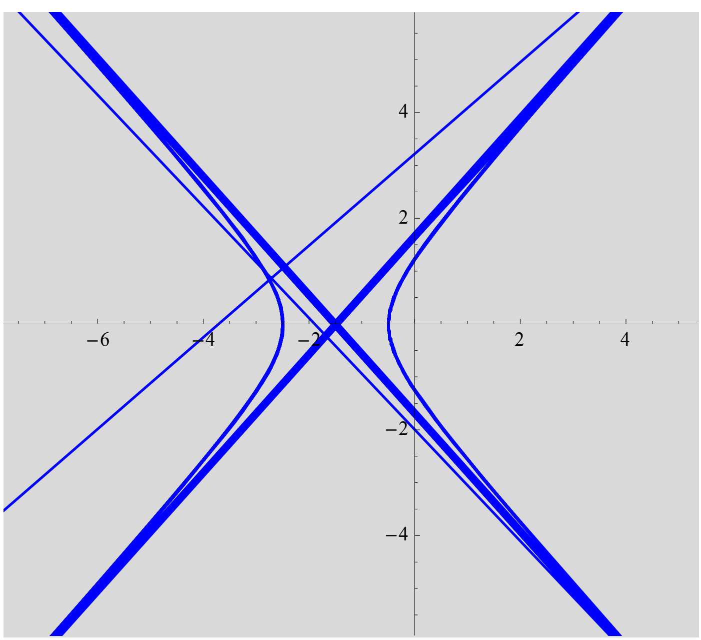
```
ParametricPlot[Evaluate[{x[ϕ], y[ϕ]} /. values], {ϕ, 0, 100 π},
 PlotStyle → {Red, Thick}, Background → LightGray,
 Epilog → {Blue, Thick, Locator[{0, 0}]}]
```



e >1: hyperbola (e =1.5)

```
values = {a → 1, e → 1.5};
```

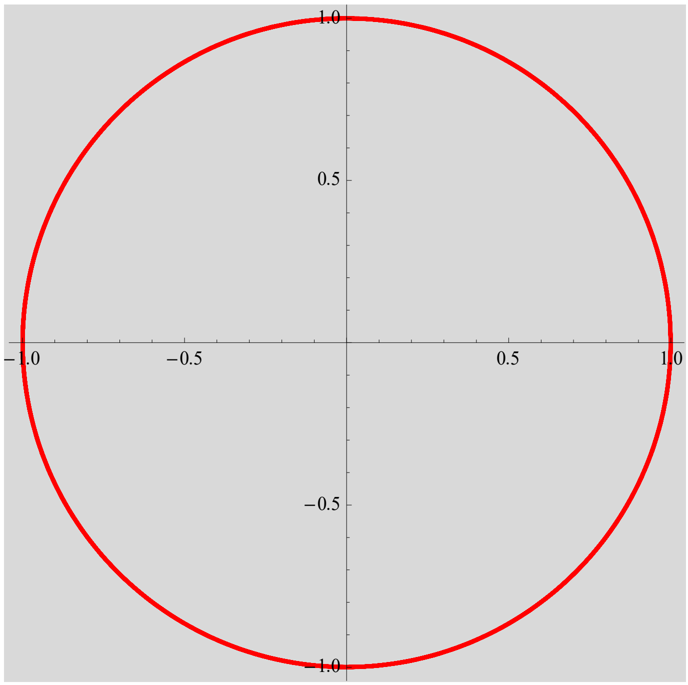
```
ParametricPlot[Evaluate[{x[ϕ], y[ϕ]} /. values], {ϕ, 0, 100 π},
 PlotStyle → {Red, Blue}, Background → LightGray]
```



e = 0: circle

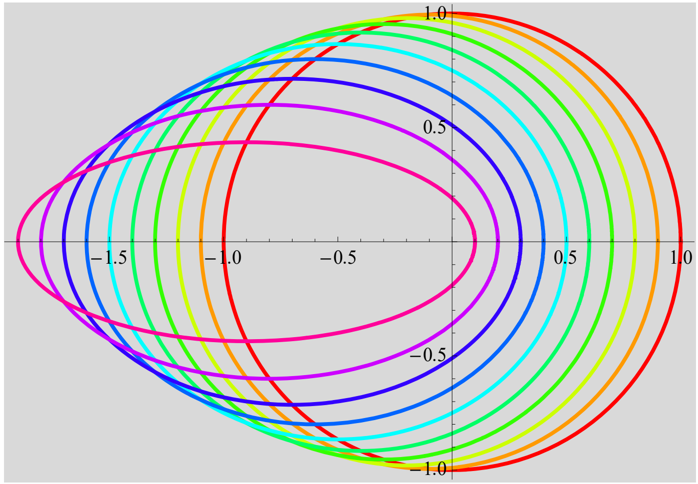
```
values = {a → 1, e → 0.0};
```

```
ParametricPlot[Evaluate[{x[ϕ], y[ϕ]} /. values], {ϕ, 0, 100 π},
 PlotStyle → {Red, Thick}, Background → LightGray]
```



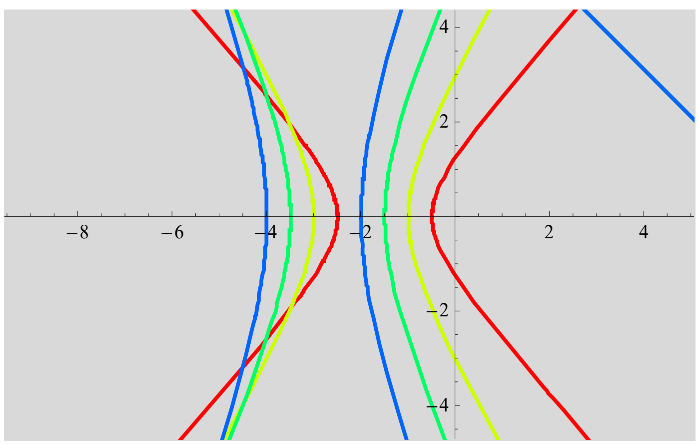
PolarPlot
e = 0, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9

```
PolarPlot[Evaluate[Table[1 / u1[ϕ] /. {a → 1}, {e, 0, 0.9, 0.1}]], {ϕ, 0, 2 π},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}], PlotStyle → {Red, Thick},
 Background → LightGray]
```



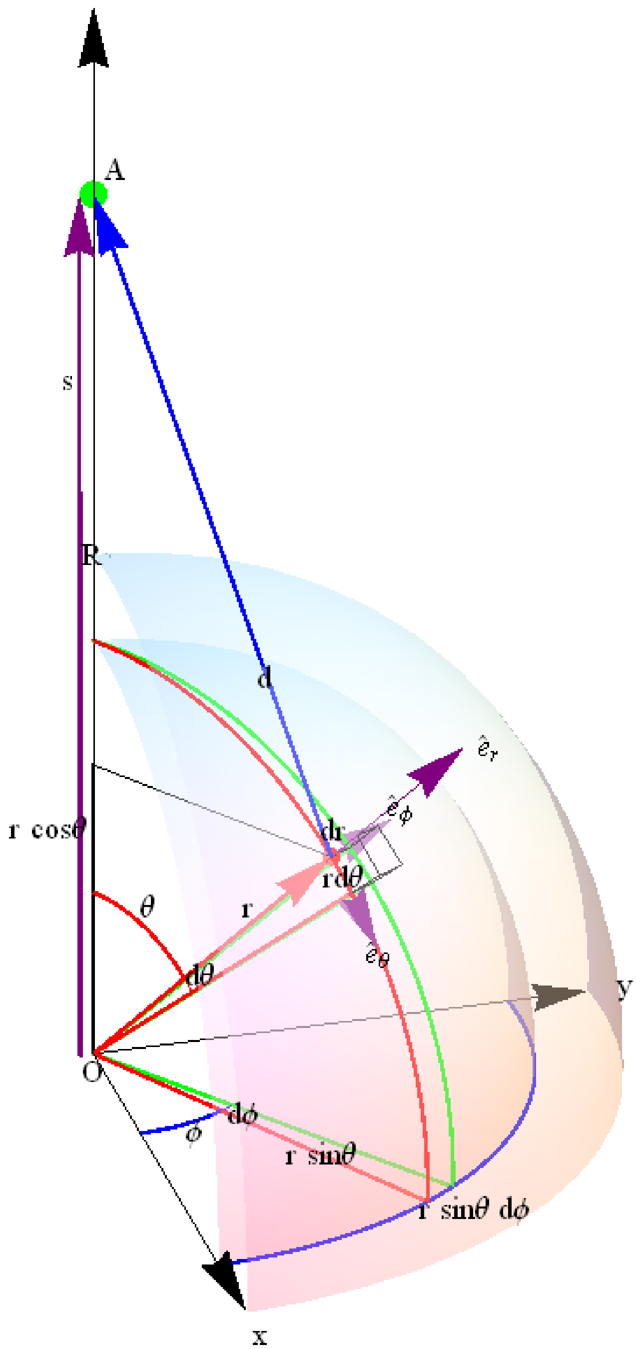
PolarPlot
e = 1.5, 2, 2.5, 3

```
PolarPlot[Evaluate[Table[1 / u1[ϕ] /. {a → 1}, {e, 1.5, 3, 0.5}]], {ϕ, 0, 2 π},
 PlotStyle → Table[{Hue[0.2 i], Thick}, {i, 0, 5}], PlotStyle → {Red, Thick},
 Background → LightGray]
```



APPENDIX-IV

Derivation of the expression of the potential energy



We now calculate the potential energy at the point A outside the sphere (with the mass *M* and the radius R). We assume that the density of the sphere is uniform inside the sphere. Then we have

$$dV = -\frac{Gm}{\sqrt{r^2 + s^2 - 2rs\cos\theta}}\rho r^2 dr \sin\theta d\theta d\phi ,$$

where s is the distance between the point A and the center of the sphere. The potential energy at the point A is obtained as

$$\begin{aligned} V(s) &= -\rho G m \int_0^R r^2 dr \int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} \int_0^{2\pi} d\phi \\ &= -2\pi\rho G m \int_0^R r^2 dr \int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} \end{aligned}$$

where ρ is the density of sphere with the total mass M and radius R. Using Mathematica we can calculate the integral

$$\int_0^R r^2 dr \int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} = \frac{2R^3}{3s}$$

for $s>R$. Thus the potential energy at the point A outside the sphere, is obtained as

$$\begin{aligned} V(s) &= -\rho G m \int_0^R r^2 dr \int_0^\pi \frac{\sin\theta d\theta}{\sqrt{r^2+s^2-2rs\cos\theta}} \int_0^{2\pi} d\phi \\ &= -2\pi\rho G m \frac{2R^3}{3s} \\ &= -\frac{GMm}{s} \end{aligned}$$

((**Mathematica**))

```
Clear["Global`*"]; f1 = r^2 Sin[θ] / Sqrt[r^2 + s^2 - 2 r s Cos[θ]];
h1 = Integrate[f1, {θ, 0, π}] //
  Simplify[#, s > r > 0] &;
h2 = Integrate[h1, {r, 0, R}] //
  Simplify[#, {R > 0, s > R}] &
```

$$\frac{2 R^3}{3 s}$$

APPENDIX V Proof of Kepler's laws

(a) Conserved angular momentum

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p},$$

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = \boldsymbol{r} \times \boldsymbol{F}.$$

For the central field, $\boldsymbol{r} \,//\, \boldsymbol{F}$ leading to

$$\boldsymbol{\tau} = \frac{d\boldsymbol{L}}{dt} = 0,$$

which means that the angular momentum $\boldsymbol{L}$ is conserved. We assume that $\boldsymbol{L}$ is directed along the z axis.

Since

$$\boldsymbol{r} \cdot \boldsymbol{L} = \boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{p}) = 0$$

the motion occurs in the x y plane.

$$\boldsymbol{L} = r\boldsymbol{e}_r \times (\mu\dot{r}\,\boldsymbol{e}_r + r\dot{\theta}\,\boldsymbol{e}_\theta) = \mu r^2 \dot{\theta}\, e_z$$

or

$$L_z = \mu r^2 \dot{\theta} = l = \text{constant}$$

where μ is the reduced mass,

$$\frac{1}{\mu} = \frac{1}{m} + \frac{1}{M}.$$

(b) Kepler's second law

The rate which a line from the sum to a planet sweeps out area is

$$\frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta} = \frac{l}{2\mu}$$

which is constant. This is the Kepler's second law.

(c) The period *T*

The total area of the ellipse orbit A is given by

$$A = \pi ab = \pi a^2\sqrt{1-e^2}$$

The period T is

$$T = \frac{A}{\frac{dA}{dt}} = \frac{\pi a^2\sqrt{1-e^2}}{\frac{l}{2\mu}} = \frac{2\mu}{l}\pi a^2\sqrt{1-e^2}$$

(d) The semi latus rectum

From the geometry of the ellipse, we have

$$p = a(1-e^2)$$

(e) The semi minor axis

From the geometry of the ellipse, we have

$$b = a\sqrt{1-e^2}$$

(f) The energy conservation

The total energy is given by

$$E = -|E| = \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\theta}^2) - \frac{k}{r}$$

where k is given by

$$k = mMG$$

Using the relation

$$\dot{\theta} = \frac{l}{\mu r^2},$$

the total energy can be rewritten as

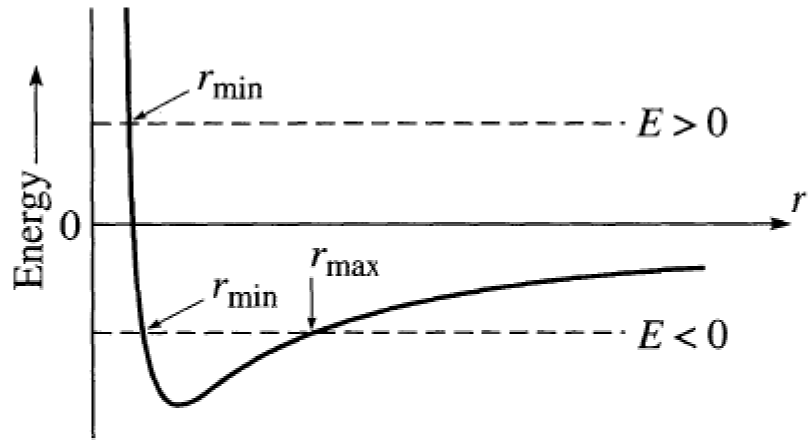
$$E = -|E| = \frac{1}{2}\mu\dot{r}^2 + \frac{1}{2}\frac{l^2}{\mu r^2} - \frac{k}{r} = \frac{1}{2}\mu\dot{r}^2 + U_{eff}(r)$$

Note that E is negative for the bound state (such as ellipse orbit).

(g) The effective potential

The effective potential is defined by

$$U_{eff}(r) = \frac{1}{2}\frac{l^2}{\mu r^2} - \frac{k}{r}$$



(h) Determination of r_p and r_a

When $\dot{r} = 0$,

$$-|E| = \frac{l^2}{2\mu r^2} - \frac{k}{r}$$

or

$$|E| r^2 - kr + \frac{l^2}{2\mu} = 0.$$

The solution of this quadratic equation is given by

$r_a = a(1+e)$ (aphelion),

$r_p = a(1-e)$ (perihelion)

The sum and product of r_a and r_p are obtained as

$$r_a + r_p = 2a = \frac{k}{|E|}, \qquad r_a r_p = a^2(1-e^2) = \frac{l^2}{2\mu|E|}$$

or

$$a^2(1-e^2)=\frac{l^2a}{\mu k}, \qquad (1-e^2)=\frac{l^2}{\mu ka}$$

The semi latus

$$p=a(1-e^2)=\frac{l^2}{\mu k}$$

(i) total energy (bound state)

$$|E|=\frac{k}{2a}=\frac{\mu k^2}{2l^2}(1-e^2)=\frac{k}{2p}(1-e^2)$$

or

$$E=-\frac{k}{2a}=-\frac{\mu k^2}{2l^2}(1-e^2)=-\frac{k}{2p}(1-e^2)$$

The circular orbit corresponds to the case of $e = 0$.

$$r_a=\frac{p}{1-e}, \qquad r_p=\frac{p}{1+e}, \qquad a=\frac{p}{1-e^2}.$$

(i) Kepler's third law

$$T=\frac{2\mu\pi a^2}{l}\sqrt{1-e^2}=\frac{2\mu\pi a^2}{l}\sqrt{\frac{l^2}{\mu ka}}=2\pi a^{3/2}\sqrt{\frac{\mu}{k}}$$

or

$$T^2=4\pi^2a^3\frac{\mu}{k}=4\pi^2a^3\frac{\mu}{mMG}=\frac{4\pi^2a^3}{(M+m)G}\approx\frac{4\pi^2a^3}{MG}$$

where

$$\mu=\frac{mM}{m+M}.$$

This is the Kepler's third law.