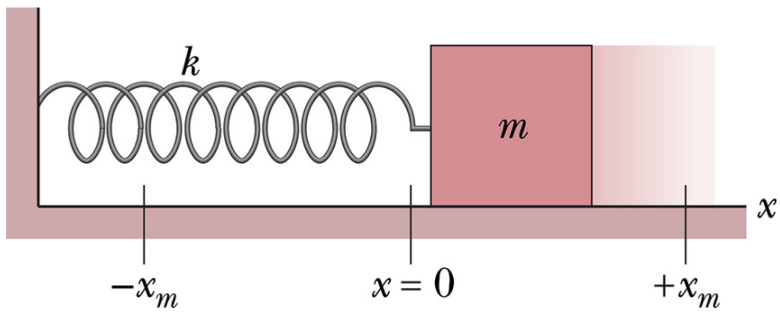
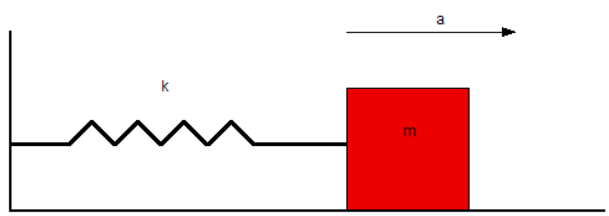
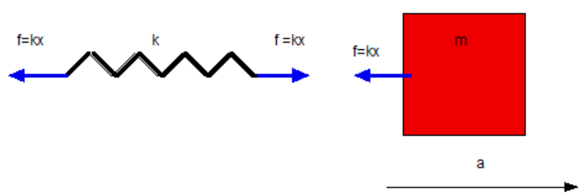
Lecture Note
Chapter 15

Harmonic Oscillation

1 Simple harmonics
1.1 Equation of motion







We consider an equation of motion given by

$$m\frac{d^2x}{dt^2} = F = -kx$$
$$\frac{d^2x}{dt^2} = -\omega^2 x$$

where

$$\omega = \sqrt{\frac{k}{m}}$$

$$\begin{aligned} x &= A_1\cos(\omega t) + B_1\sin(\omega t) \\ \frac{dx}{dt} &= -A_1\omega\sin(\omega t) + B_1\omega\cos(\omega t) \\ \frac{d^2x}{dt^2} &= -A_1\omega^2\cos(\omega t) - B_1\omega^2\sin(\omega t) \\ &= -\omega^2 x \end{aligned}$$

We note that x can be rewritten as

$$x = A_1\cos(\omega t) + B_1\sin(\omega t) = A\cos(\omega t + \phi)$$

where A, w and ϕ are constants, independent of time. The quantity A is called the **amplitude of the motion** and is the maximum displacement of the mass. The time-varying quantity ($\omega t + \phi$) is called the **phase of the motion** and ϕ is called the **phase constant**. The phase constant is determined by the initial conditions.

The **angular frequency** ω is a characteristic of the system, and does not depend on the initial conditions. The unit of angular frequency is rad/s.

The **period** T of the motion is defined as the time required to complete-one oscillation. Therefore, the displacement $x(t)$ must return to its initial value after one period

$$x(t) = x(t+T)$$

This is equivalent to

$$A\cos(\omega t + \phi) = A\cos(\omega t + \omega T + \phi)$$

Using the relation

$$\cos(\alpha) = \cos(\alpha + 2\pi)$$

it is immediately clear that

$$\omega T = 2\pi$$

The number of oscillations carried out per second is called the **frequency of the oscillation**. The symbol for frequency is f and its unit is the Hertz (Hz):

1 Hz = 1 oscillation per second = 1 s^{-1}

The period T and the frequency f are related as follows

$$T = \frac{1}{f}$$

1.2 Energy conservation law

$$m\frac{dx}{dt}\frac{d^2x}{dt^2} = -kx\frac{dx}{dt}$$

or

$$\frac{d}{dt}[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2] = -\frac{d}{dt}(\frac{1}{2}kx^2)$$

or

$$\frac{d}{dt}[\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 + \frac{1}{2}kx^2] = 0$$

or

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kA^2 \qquad \text{(energy-conservation law)}$$

since $v = 0$ at $x = A$.

or

$$E = K + U = \frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}mv_{\max}{}^2 \qquad \text{(energy conservation law)}$$

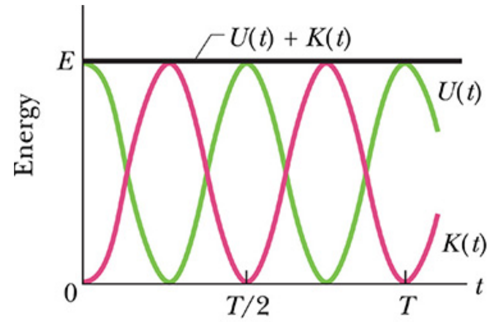
since $x = x_{\max}$ at $v = 0$.

((**Note**))
The amplitude A can be estimated from the initial condition ($x = x_0$, $v = v_0$ at $t = t_0$).

$$A = \sqrt{x_0{}^2 + \frac{m}{k}v_0{}^2} = \sqrt{x_0{}^2 + \frac{v_0{}^2}{\omega^2}}$$

Now we consider

$$K = \frac{1}{2}k(A^2 - x^2)$$



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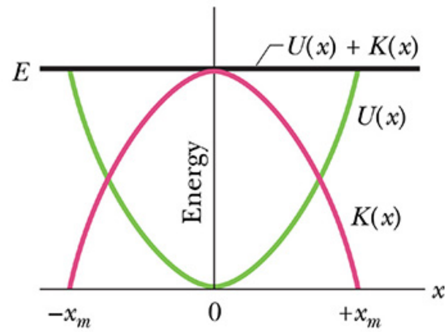


(b)

((Note)) Derivation of the equation of motion from the energy conservation law

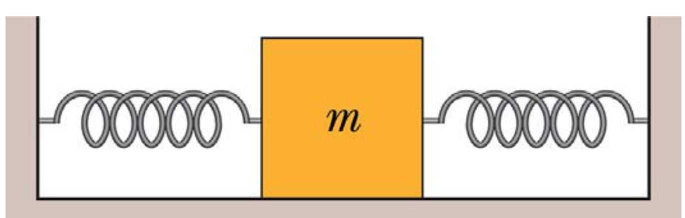
$$E = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2$$

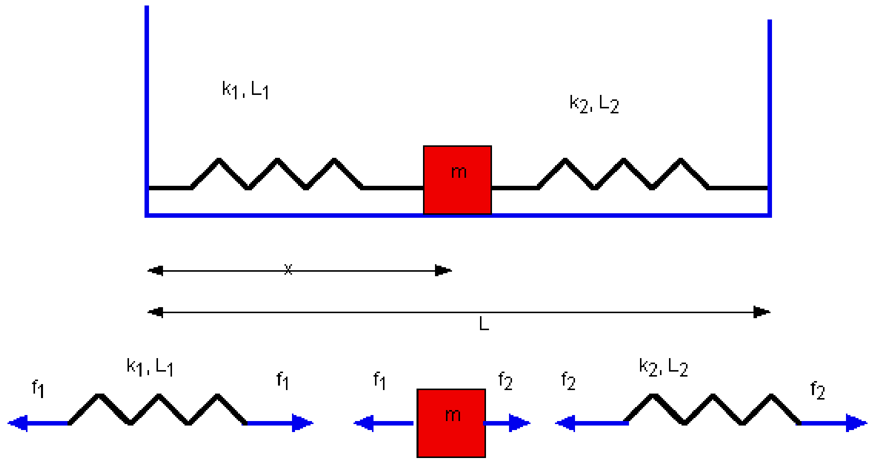
Taking the derivative of this with respect to *t*, we have

$$\frac{dE}{dt} = 0 = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = 0$$
$$m\ddot{x} + kx = 0$$

2 Simple harmonics

2.1 Configuration-1





$$f_1 = k_1(x - L_1)$$
$$f_2 = k_2(L - x - L_2)$$
$$m\ddot{x} = f_2 - f_1$$

or

$$\begin{aligned} m\ddot{x} &= k_2(L - x - L_2) - k_1(x - L_1) \\ &= -(k_1 + k_2)x + k_1 L_1 + k_2 L - k_2 L_2 \end{aligned}$$

or

$$\ddot{x} = -\frac{(k_1 + k_2)}{m}x + \frac{k_1 L_1 + k_2 L - k_2 L_2}{m}$$

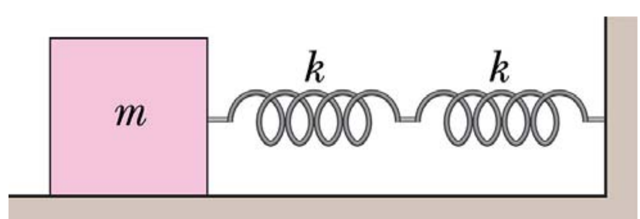
The resultant spring constant k is

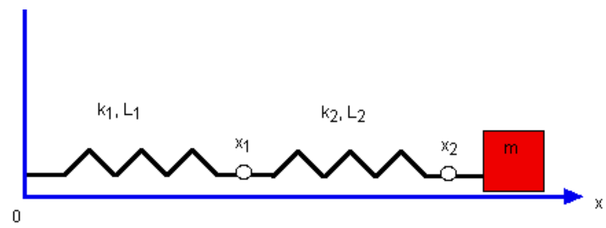
$$k = k_1 + k_2$$

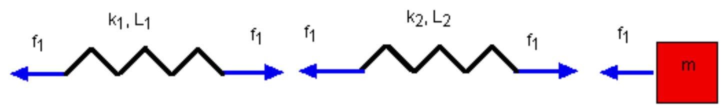
2.2 Configuration-2

Simple harmonics

Problem 15-24* (SP-15) (10-th edition)**







$$f_1 = k_1(x_1 - L_1)$$
$$f_1 = k_2(x_2 - x_1 - L_2)$$
$$m\ddot{x}_2 = -f_1$$

or

$$(k_1 + k_2)x_1 - k_2 x_2 = k_1 L_1 - k_2 L_2$$
$$x_1 = \frac{k_2 x_2}{k_1 + k_2} + \frac{k_1 L_1 - k_2 L_2}{k_1 + k_2}$$

Then we have an equation of motion for the mass *m*,

$$\begin{aligned} m\ddot{x}_2 &= -k_1(x_1 - L_1) \\ &= -k_1\left(\frac{k_2 x_2}{k_1 + k_2} + \frac{k_1 L_1 - k_2 L_2}{k_1 + k_2}\right) + k_1 L_1 \\ &= -\frac{k_1 k_2 x_2}{k_1 + k_2} - \frac{k_1(k_1 L_1 - k_2 L_2)}{k_1 + k_2} + k_1 L_1 \\ &= -\frac{k_1 k_2}{k_1 + k_2}(x_2 - L_1 - L_2) \end{aligned}$$

The resultant spring constant *k* is

$$\boxed{k = \frac{k_1 k_2}{k_1 + k_2}}$$

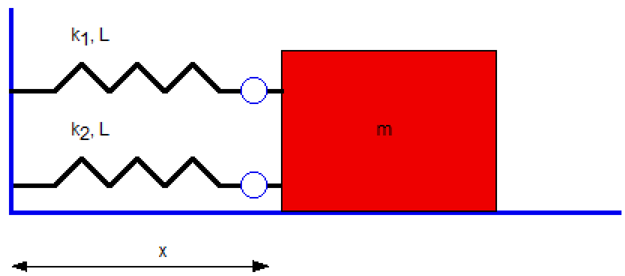
The solution of this differential equation is given by

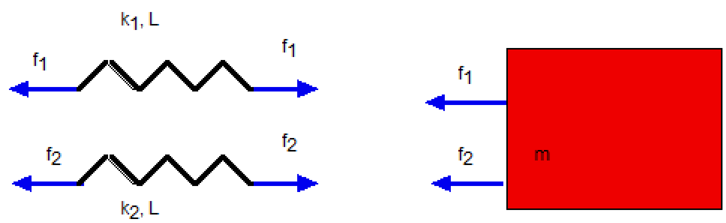
$$x_2 = L_1 + L_2 + A\cos(\omega t + \phi)$$

where

$$\omega = \sqrt{\left(\frac{k_1 + k_2}{k_1 + k_2}\right)\frac{1}{m}}$$

2.3 Configuration-3





We now consider the case shown above.

$$m\ddot{x} = -f_1 - f_2$$
$$f_1 = k_1(x - L)$$
$$f_2 = k_2(x - L)$$

or

$$\ddot{x} = -\frac{(k_1 + k_2)}{m}(x - L) = -\omega^2(x - L)$$

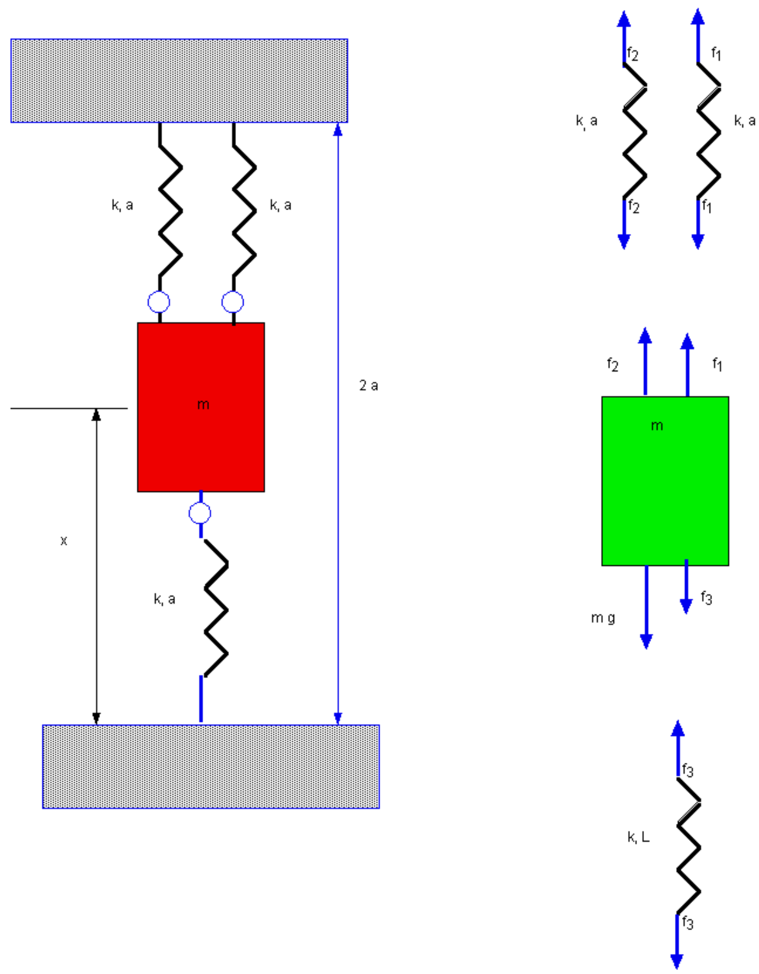
where

$$\omega = \sqrt{\frac{k_1 + k_2}{m}}$$

The period T is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{k_1 + k_2}}$$

2.4 Mass hanging from the ceiling (advanced topics)



Equation of motion

$$
\begin{aligned}
m\ddot{x} &= f_1 + f_2 - f_3 - mg \\
f_1 &= f_2 = k(2a - x - a) = k(a - x) \\
f_3 &= k(x - a)
\end{aligned}
$$

or

$$m\ddot{x} = 2k(a - x) - k(x - a) - mg = -3k(x - a) - mg$$

At equilibrium,

$$
\begin{aligned}
&-3k(x_0 - a) - mg = 0 \\
&x_0 = a - \frac{mg}{3k}
\end{aligned}
$$

$$\ddot{x} = -\frac{3k}{m}(x - x_0)$$

This indicates the simple harmonic oscillation with an angular frequency

$$\boxed{\omega = \sqrt{\frac{3k}{m}}}$$

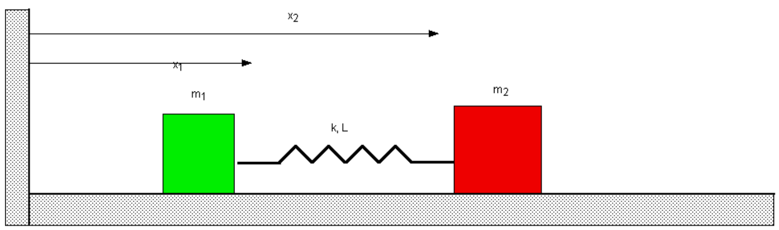
The solution for x is

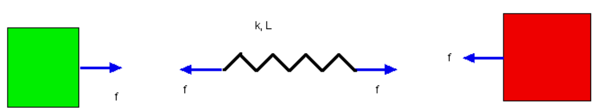
$$x = x_0 + A\cos(\omega t + \phi)$$

3. Two-body oscillations

3.1

We imagine the molecules to be represented by two particles of masses m_1 and m_2 connected by a spring force constant k, as shown below. Here we examine the motion of this system.





We apply the Newton's second law to the system with two masses m_1 and m_2.

$$m_1\ddot{x}_1 = f \qquad (1)$$

$$m_2\ddot{x}_2 = -f \qquad (2)$$

$$f = k(x_2 - x_1 - L)$$

From the calculation of m_2 x Eq(1) $-m_1$ x Eq(2), we have

$$m_1 m_2(\ddot{x}_1 - \ddot{x}_2) = k(m_1 + m_2)(x_2 - x_1 - L)$$

Here we introduce a new variable x defined by

$$x = x_1 - x_2 - L$$

and the reduced mass μ defined as

$$\mu = \frac{m_1 m_2}{m_1 + m_2}$$

Then we get

$$\ddot{x} = -\frac{k}{\mu}x = -\omega^2 x$$

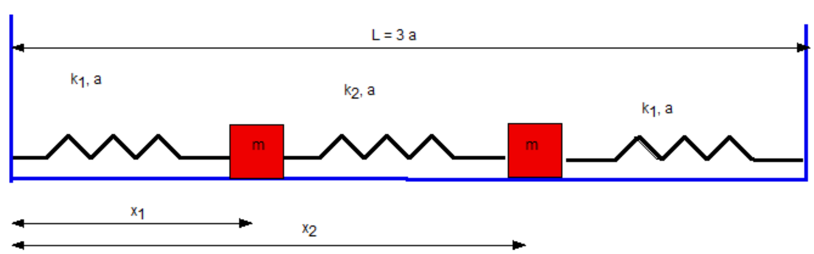
where

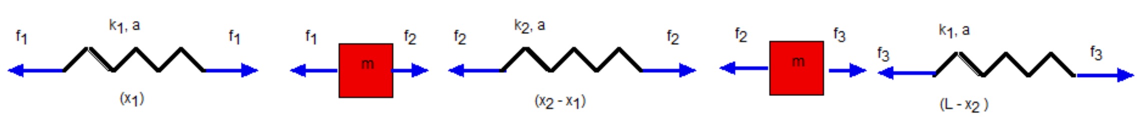
$$\omega = \sqrt{\frac{k}{\mu}}$$

This indicates that two particles connected (with a spring constant *k*) can be replaced by a single particle (with a spring constant *k*) with a mass equal to the reduced mass of the system.

3.2 Longitudinal oscillations of two coupled masses

We find the modes and their frequencies for the coupled springs and masses sliding on a frictionless surface. At equilibrium the springs are relaxed.





We set up equations of motion for mass m_1 (= m) and m_2 (= m).

$$
\begin{aligned}
m_1\ddot{x}_1 &= f_2 - f_1 \\
m_2\ddot{x}_2 &= f_3 - f_2 \\
f_1 &= k_1(x_1 - a) \\
f_2 &= k_2(x_2 - x_1 - a) \\
f_3 &= k_1(3a - x_2)
\end{aligned}
$$

or

$$
\begin{aligned}
m_1\ddot{x}_1 &= k_2(x_2 - x_1 - a) - k_1(x_1 - a) = k_2x_2 - (k_1 + k_2)x_1 + (k_1 - k_2)a \\
m_2\ddot{x}_2 &= k_1(3a - x_2) - k_2(x_2 - x_1 - a) = -(k_1 + k_2)x_2 + k_2x_1 + (3k_1 + k_2)a
\end{aligned}
$$

First we find the solutions of $\ddot{x}_1 = \ddot{x}_2 = 0$

$$
\begin{aligned}
&k_2x_2^{\,0} - (k_1 + k_2)x_1^{\,0} + (k_1 - k_2)a = 0 \\
&-(k_1 + k_2)x_2^{\,0} + k_2x_1^{\,0} + (3k_1 + k_2)a = 0
\end{aligned}
$$

From these equations, we have

$$
\begin{aligned}
x_1^{\,0} &= \frac{a(k_1 + 3k_2)}{k_1 + 2k_2} \\
x_2^{\,0} &= \frac{a(3k_1 + 5k_2)}{k_1 + 2k_2}
\end{aligned}
$$

For convenience, we introduce new variables y_1 and y_2 defined by

$$
\begin{aligned}
y_1 &= x_1 - x_1^{\,0} \\
y_2 &= x_2 - x_2^{\,0}
\end{aligned}
$$

Then

$$
\begin{aligned}
m_1\ddot{y}_1 &= k_2y_2 - (k_1 + k_2)y_1 \\
m_2\ddot{y}_2 &= -(k_1 + k_2)y_2 + k_2y_1
\end{aligned}
$$

We assume that

$$
\begin{aligned}
\ddot{y}_1 &= -\omega^2 y_1 \\
\ddot{y}_2 &= -\omega^2 y_2
\end{aligned}
$$

which leads to

$$-m_1\omega^2 y_1 = k_2 y_2 - (k_1 + k_2) y_1$$
$$-m_2\omega^2 y_2 = -(k_1 + k_2) y_2 + k_2 y_1$$

or

$$[m_1\omega^2 - (k1 + k_2)]y_1 + k_2 y_2 = 0$$
$$k_2 y_1 + [m_2\omega^2 - (k_1 + k_2)]y_2 = 0$$

For the nontrivial solutions of y_1 and y_2, we have the condition

$$\det = \begin{vmatrix} m_1\omega^2 - (k_1 + k_2) & k_2 \\ k_2 & m_2\omega^2 - (k_1 + k_2) \end{vmatrix} = 0$$

When $m_1 = m_2 = m$, we have two mode frequencies

$$\omega_1{}^2 = \frac{k_1}{m}$$
$$\omega_2{}^2 = \frac{k_1 + 2k_2}{m}$$

((**Physical meaning**))
We start from the differential equations

$$m\ddot{y}_1 = k_2 y_2 - (k_1 + k_2) y_1$$
$$m\ddot{y}_2 = -(k_1 + k_2) y_2 + k_2 y_1$$

These equations can be rewritten as

$$\frac{d^2}{dt^2}(y_1 + y_2) = -\frac{k_1}{m}(y_1 + y_2) = -\omega_1{}^2 (y_1 + y_2)$$
$$\frac{d^2}{dt^2}(y_1 - y_2) = -\frac{(k_1 + 2k_2)}{m}(y_1 - y_2) = -\omega_2{}^2 (y_1 - y_2)$$

or

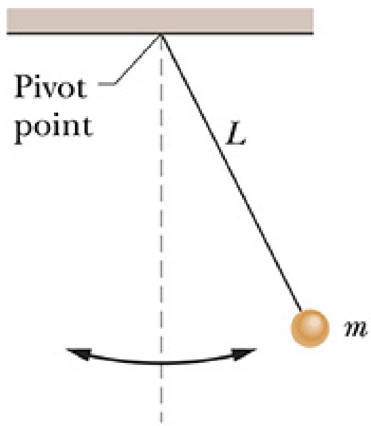
$$y_1 + y_2 = A_1 \cos(\omega_1 t + \phi_1)$$
$$y_1 - y_2 = A_2 \cos(\omega_2 t + \phi_2)$$

Finally we get

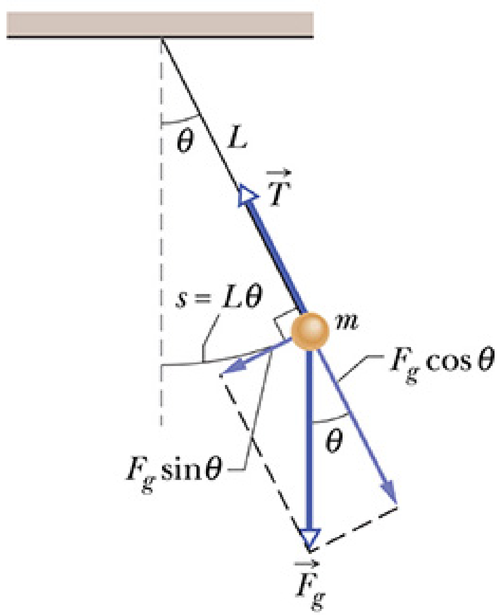
$$y_1 = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) + \frac{A_2}{2}\cos(\omega_2 t + \phi_2)$$
$$y_2 = \frac{A_1}{2}\cos(\omega_1 t + \phi_1) - \frac{A_2}{2}\cos(\omega_2 t + \phi_2)$$

In conclusion, there are two modes, since there are two degrees of freedom.

4 Simple pendulum



(a)



(b)

$$m\frac{d^2 s}{dt^2} = -mg\sin\theta$$
$$s = L\theta$$

or

$$\boxed{\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta = -\omega_0^{\ 2}\sin\theta}$$

In the limit of $\theta \rightarrow 0$,

$$\frac{d^2\theta}{dt^2} \approx -\omega_0^{\ 2}\theta$$

or

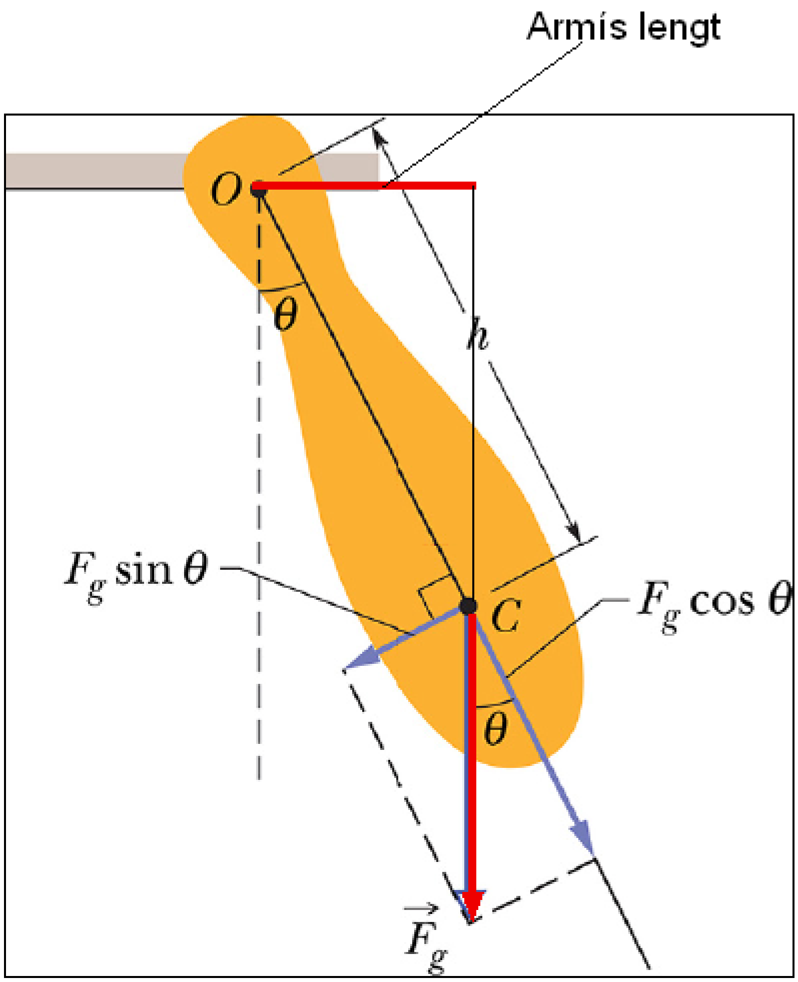
$$\theta = A_0 \cos(\omega_0 t + \phi)$$

In the large θ, we have a nonlinear differential equation. We can solve the problem numerically using Mathematica (see the Section 12).

((Note)) A different derivation of the equation of motion for the simple pendulum is presented in Section 12.

5 Physical Pendulum

5.1



In the real world pendulums are far from simple. In general, the mass of the pendulum is not concentrated in one point, but will be distributed. Figure shows a physical pendulum. The physical pendulum is suspended through point O. The effect of the force of gravity can be replaced by the effect of a single force, whose magnitude is m g, acting on the center of gravity of the pendulum (which is equal to the center of mass if the gravitational acceleration is constant). The resulting torque (with respect to O) is given by

$$\tau = -mgh\sin\theta$$

where h is the distance between the rotation axis and the center of gravity. In the limit of small angles ($\theta \approx 0$), this torque can be rewritten as

$$\tau = -mgh\theta$$

The angular acceleration α of the pendulum is related to the torque τ and the rotational inertia I

$$\tau = I\alpha$$

We therefore conclude that

$$\alpha = \frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\theta$$

This is again the equation for harmonic motion with an angular frequency given by

$$\omega^2 = \frac{mgh}{I}$$

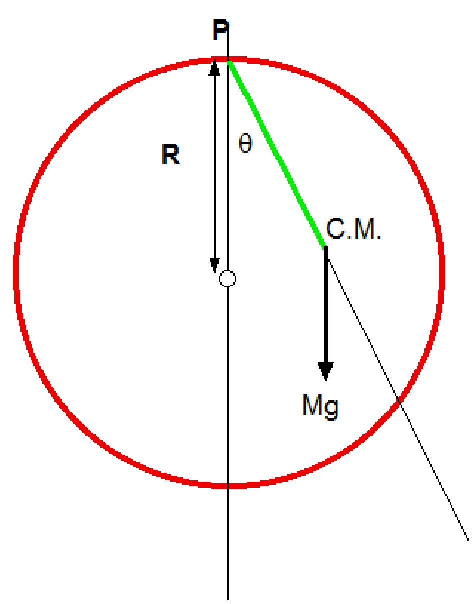
and a period equal to

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{mgh}}$$

Note that the simple pendulum is a special case of the physical pendulum: $h = L$ and $I =$ m L^2. The period of the oscillation is then given by

$$T = 2\pi\sqrt{\frac{I}{mgh}} = 2\pi\sqrt{\frac{L}{g}}$$

5.2 Ring (hula hoop)



$$I\ddot{\theta} = \tau = -RMg\sin\theta$$

where R is the distance between the pivot point and the center of mass. I is the moment of inertia around the pivot point P,

$$I = I_{cm} + MR^2 = MR^2 + MR^2 = 2MR^2 \qquad \text{(parallel-axis theorem)}$$

In the limit of small angle θ, the motion undergoes a simple harmonic oscillation,

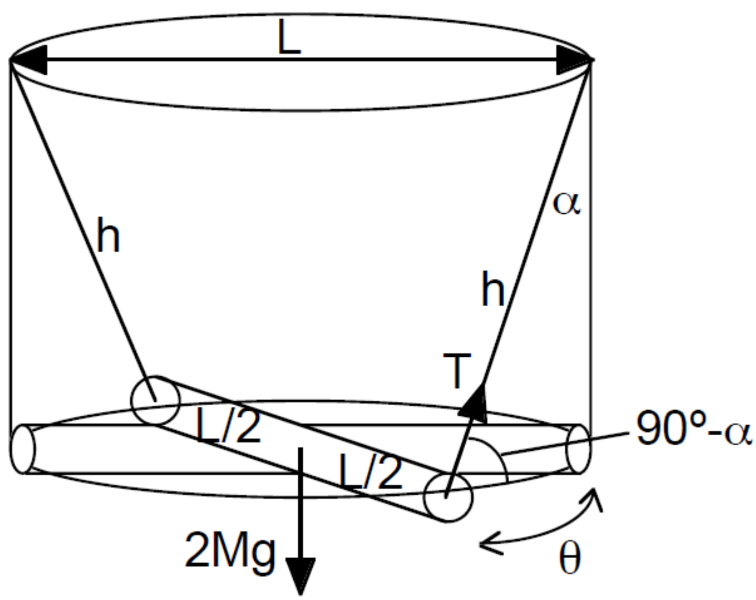
$$\ddot{\theta} = -\omega^2\theta$$

Here we have

$$\omega = \sqrt{\frac{RMg}{I}}\ .$$

$$T = 2\pi\sqrt{\frac{I}{MgR}} = 2\pi\sqrt{\frac{2MR^2}{MgR}} = 2\pi\sqrt{\frac{2R}{g}}\ .$$

5.3 Bifilar pendulum



For simplicity, the extended mass is described by a simple rod in this figure. We have

$$h\alpha = \frac{L}{2}\theta, \qquad \text{and} \qquad T = Mg$$

where $\alpha = 0$ or $\theta = 0$. The equation of motion for the bifilar pendulum is described by

$$\begin{aligned} I\ddot{\theta} &= -2T\cos\left(\frac{\pi}{2} - \alpha\right)\frac{L}{2} \\ &= -2Mg\frac{L}{2}\sin\alpha \\ &= -MgL\sin\frac{L\theta}{2h} \\ &= -MgL^2\frac{\theta}{2h} \end{aligned}$$

or

$$I\ddot{\theta} = -\omega^2\theta \qquad \text{(simple harmonic oscillation)}$$

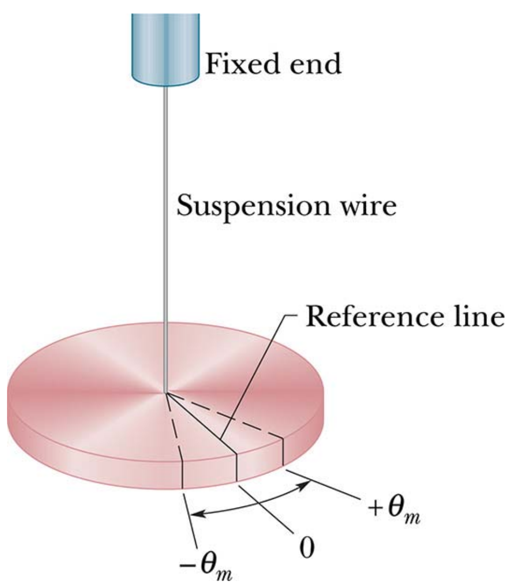
where

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{MgL^2}{2Ih}}\,.$$

The moment of inertia I can be derived as

$$I = \frac{MgL^2T^2}{8\pi^2 h}\,.$$

6 The torsion pendulum



The operation of a torsion pendulum is associated with twisting a suspension wire. The motion described by the torsion pendulum is called **angular simple harmonic motion**. The restoring torque is given by

$$\tau = -\kappa\theta$$

where κ is a torque constant that depends on the properties of the suspension wire (its length, diameter and material). This equation is essentially a torsional equivalent to Hooke's law. For a given torque we can calculate the angular acceleration α

$$\tau = I\alpha$$

or

$$\alpha = \frac{d^2\theta}{dt^2} = \frac{\tau}{I} = -\frac{\kappa}{I}\theta$$

where I is the moment of inertia of the disk (about a perpendicular axis through its centre). Comparing this equation with the relation between the linear acceleration and the linear displacement of an object, we conclude that

$$\omega^2 = \frac{\kappa}{I}$$

The period of the torsion pendulum is given by

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{I}{\kappa}}$$

7 Damped oscillation

So far we have discussed systems in which the force is proportional to the displacement, but pointed in an opposite direction. In these cases, the motion of the system can be described by simple harmonic motion. However, if we include the friction force, the motion will not be simple harmonic anymore. The system will still oscillate, but its amplitude will slowly decrease over time.

We now consider the simple harmonics with a damping constant b,

$$m\ddot{x}(t) + b\dot{x}(t) + kx(t) = 0$$

with the initial conditions

$$\dot{x}(0) = v_0 \text{ and } x(0) = x_0$$

The differential equation can be written as

$$\ddot{x}(t)+\frac{b}{m}\dot{x}(t)+\frac{k}{m}x(t)=0$$

We introduce

$$\omega_0=\sqrt{\frac{k}{m}}, \qquad \beta=\frac{b}{2m}.$$

Then we get

$$\ddot{x}(t)+2\beta\dot{x}(t)+\omega_0{}^2x(t)=0$$

The solution of this differential equation is classed into three types,

(1)	underdamping:	$\beta^2-\omega_0^2<0$
(2)	critical damping	$\beta^2-\omega_0^2=0$
(3)	overdamping	$\beta^2-\omega_0^2>0$

The solution for the overdamping is given by

$$x(t)=e^{-\beta t}[C_1\cos(\omega_1 t)+C_2\sin(\omega_1 t)]$$

with

$$\omega_1=\sqrt{\omega_0{}^2-\beta^2}$$

From the initial condition, we have

$$C_1=x_0$$

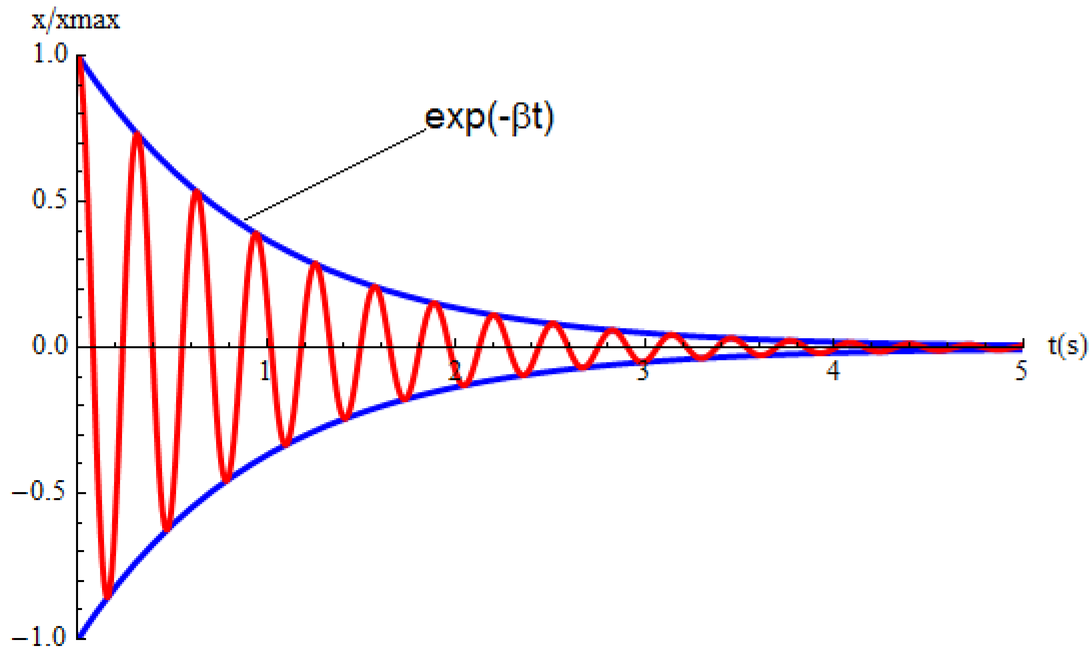
$$C_2=\frac{v_0+\beta x_0}{\omega_1}$$

The final form is given by

$$\begin{aligned}x(t)&=e^{-\beta t}[x_0\cos(\omega_1 t)+\left(\frac{v_0+\beta x_0}{\omega_1}\right)\sin(\omega_1 t)]\\&=e^{-\beta t}x_{\max}\cos(\omega_1 t+\phi)\end{aligned}$$

where ϕ is the phase factor and x_{max} is the amplitude. The period T_1 is

$$T_1 = \frac{2\pi}{\omega_1} = \frac{2\pi}{\sqrt{{\omega_0}^2 - \beta^2}}$$



((**Mathematica**))

second-order differential equation for a simple harmonics with damping

```
Clear["Global`*"];
eq1 = {x''[t] + 2 β x'[t] + ω0^2 x[t] == 0, x'[0] == v0, x[0] == x0};

eq2 = DSolve[eq1, x[t], t] // Simplify
```

$$\left\{\left\{x[t] \to \frac{1}{2\sqrt{\beta^2-\omega 0^2}} e^{-t\left(\beta+\sqrt{\beta^2-\omega 0^2}\right)} \left(\left(-1+e^{2t\sqrt{\beta^2-\omega 0^2}}\right) v0 + x0\left(\left(-1+e^{2t\sqrt{\beta^2-\omega 0^2}}\right)\beta + \left(1+e^{2t\sqrt{\beta^2-\omega 0^2}}\right)\sqrt{\beta^2-\omega 0^2}\right)\right)\right\}\right\}$$

```
x[t_] = Simplify[x[t] /. eq2[[1]] /. {β^2 - ω0^2 → -ω1^2}, ω1 > 0];

x1[t_] = Exp[β t] x[t] // ExpToTrig // Simplify
```

$$\frac{x0\,\omega 1\,\mathrm{Cos}[t\,\omega 1] + (v0 + x0\,\beta)\,\mathrm{Sin}[t\,\omega 1]}{\omega 1}$$

```
x11[t_] = Exp[-β t] x1[t]
```

$$\frac{e^{-t\beta}\left(x0\,\omega 1\,\mathrm{Cos}[t\,\omega 1] + (v0 + x0\,\beta)\,\mathrm{Sin}[t\,\omega 1]\right)}{\omega 1}$$

```
Limit[x11[t] /. ω1 → Sqrt[ω0^2 - β^2], β → ω0] // Simplify
```

$$e^{-t\,\omega 0}\,(x0 + t\,(v0 + x0\,\omega 0))$$

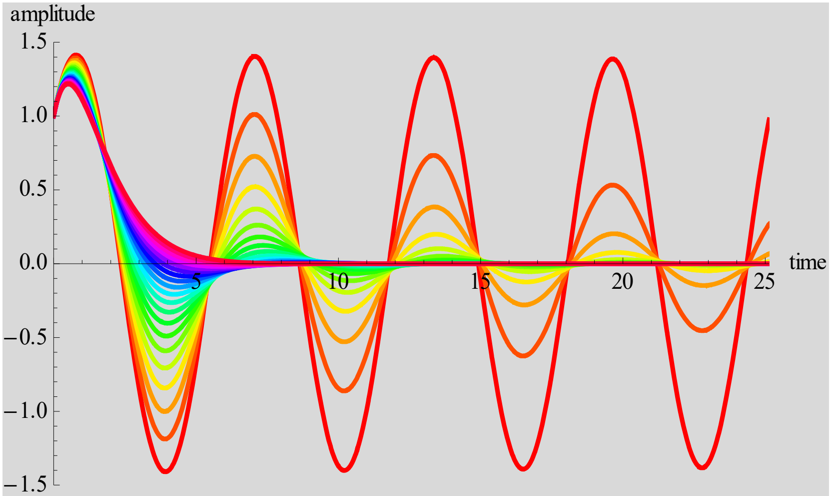
```
v11[t_] = D[x11[t], t] // Simplify
```

$$\frac{e^{-t\beta}\left(v0\,\omega1\,\mathrm{Cos}[t\,\omega1] - \left(v0\,\beta + x0\left(\beta^2+\omega1^2\right)\right)\mathrm{Sin}[t\,\omega1]\right)}{\omega1}$$

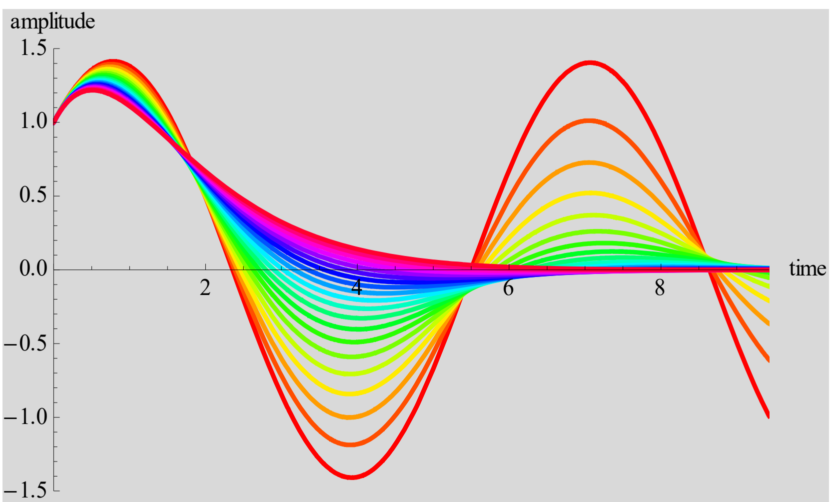
```
Limit[v11[t] /. ω1 → Sqrt[ω0^2 - β^2], β → ω0] // Simplify
```

$$e^{-t\,\omega0}\left(v0 - t\,v0\,\omega0 - t\,x0\,\omega0^2\right)$$

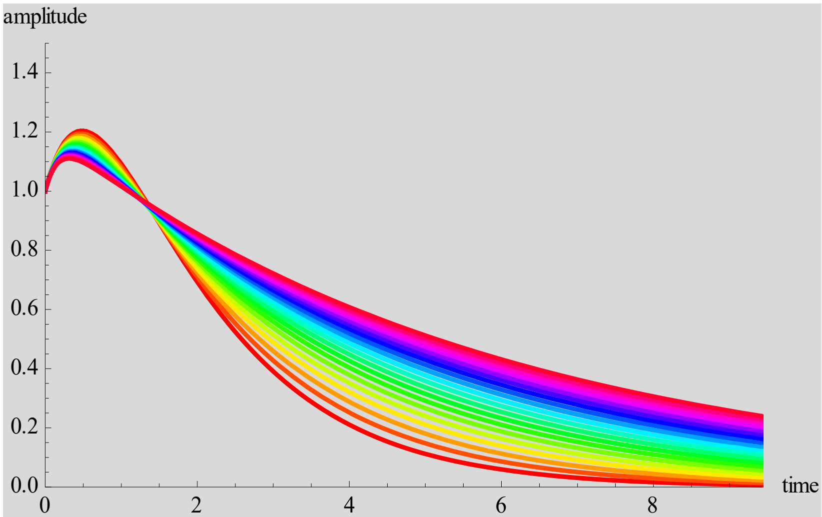
```
x2[t_] = x11[t] /. ω1 → Sqrt[ω0^2 - β^2] // Simplify;

v2[t_] = v11[t] /. ω1 → Sqrt[ω0^2 - β^2] // Simplify;

rule1 = {ω0 → 1, x0 → 1, v0 → 1};

x3[t_, β_] = x2[t] /. rule1 // Simplify;

v3[t_, β_] = D[x3[t, β], t] /. rule1 // Simplify;

Plot[Evaluate[Table[x3[t, β], {β, 0.001, 1, 0.05}]],
 {t, 0, 8 π}, PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 20}],
 AxesLabel → {"time", "amplitude"},
 PlotRange → {{0, 8 π}, {-1.5, 1.5}}, Background → LightGray]
```



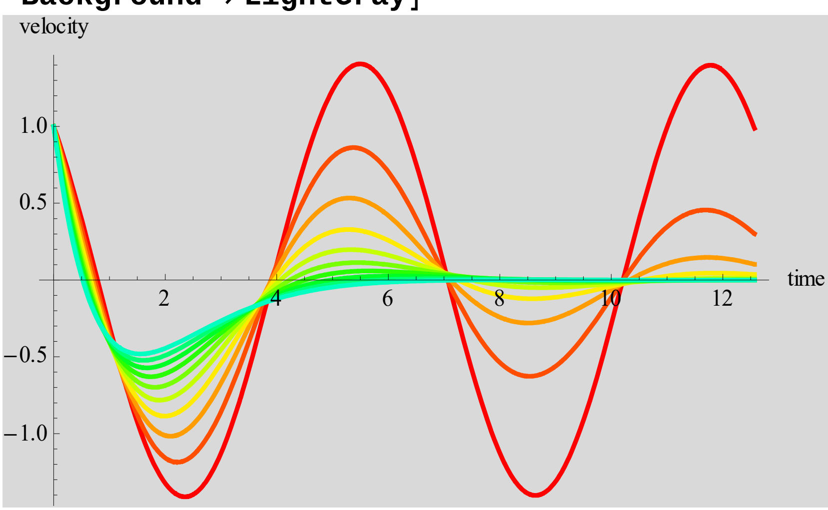

```
Plot[Evaluate[Table[x3[t, β], {β, 0.001, 1, 0.05}]],
 {t, 0, 3 π}, PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 20}],
 AxesLabel → {"time", "amplitude"},
 PlotRange → {{0, 3 π}, {-1.5, 1.5}}, Background → LightGray]
```



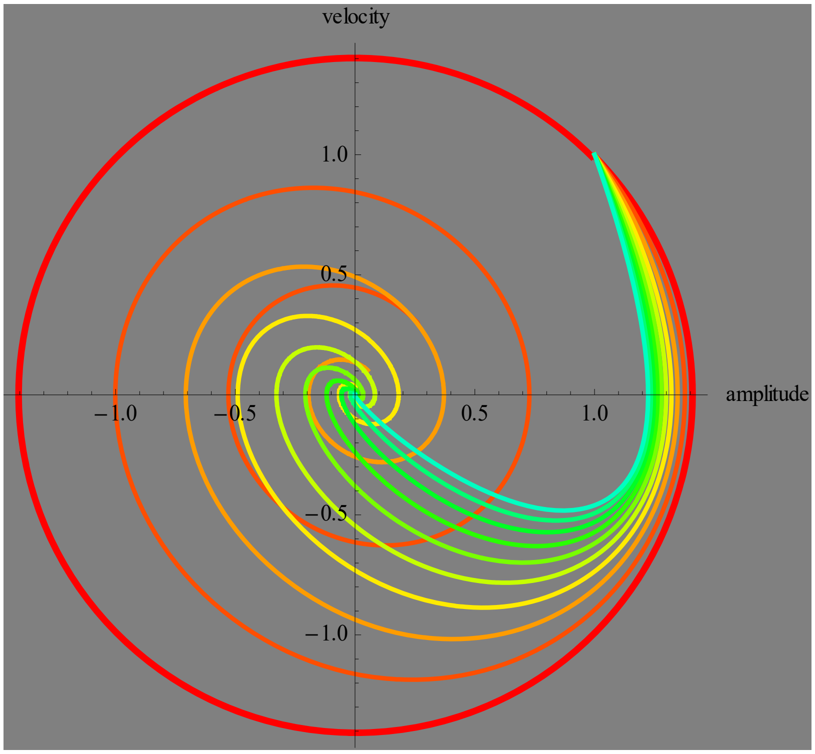
```
Plot[Evaluate[Table[x3[t, β], {β, 1.1, 3, 0.1}]], {t, 0, 3 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 20}],
 AxesLabel → {"time", "amplitude"},
 PlotRange → {{0, 3 π}, {0, 1.5}}, Background → LightGray]
```



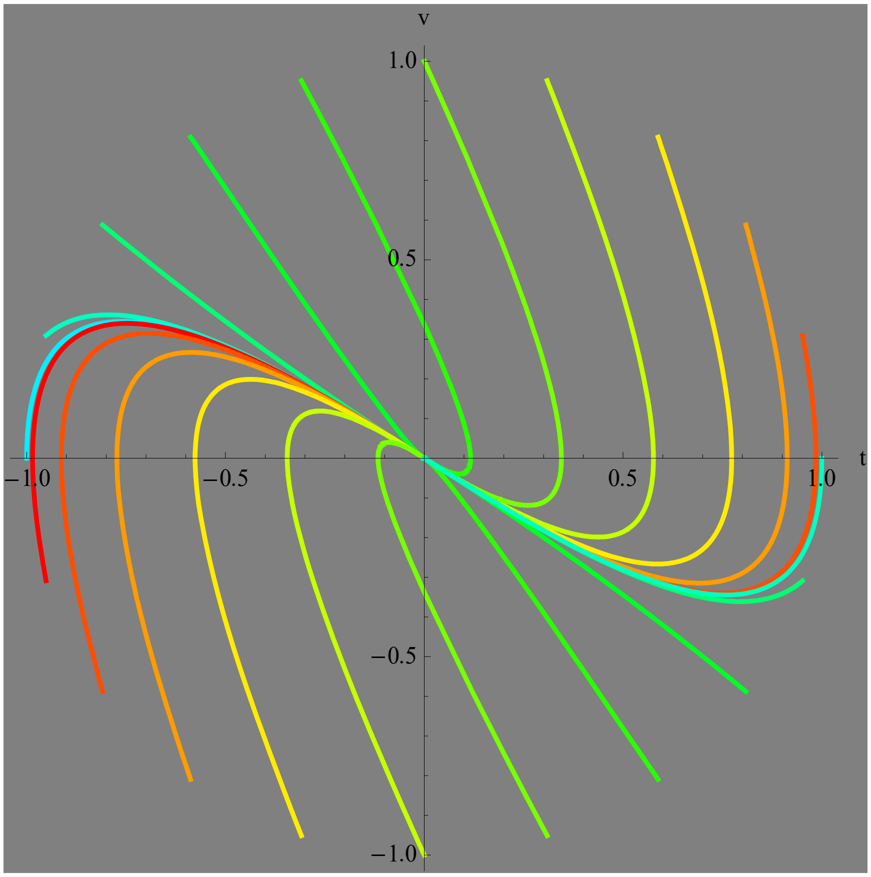

```
Plot[Evaluate[Table[v3[t, β], {β, 0.001, 1, 0.1}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"time", "velocity"}, Prolog → AbsoluteThickness[2],
 Background → LightGray]
```




```
ParametricPlot[
 Evaluate[Table[{x3[t, β], v3[t, β]}, {β, 0.001, 1, 0.1}]],
 {t, 0, 4 π}, PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"amplitude", "velocity"}, Background → Gray,
 AspectRatio → 1]
```

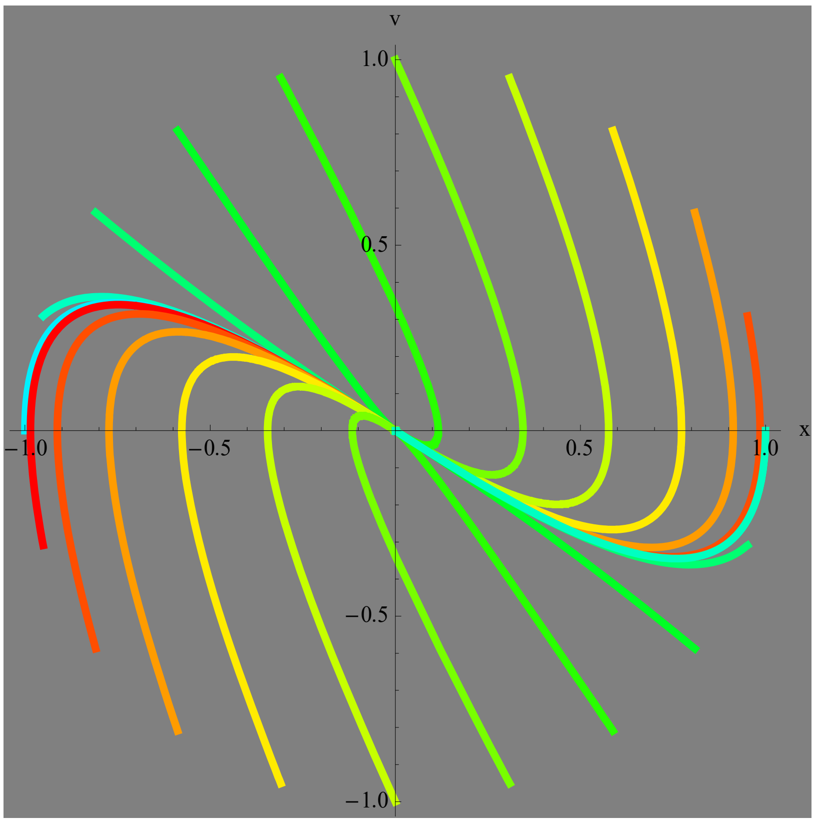



```
ParametricPlot[
 Evaluate[
  Table[{x2[t], v2[t]} /. {β → 1.1, ω0 → 1, x0 → Cos[θ], v0 → Sin[θ]},
   {θ, 0, 2 π, π / 10}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"t", "v"}, Background → Gray, PlotRange → All]
```

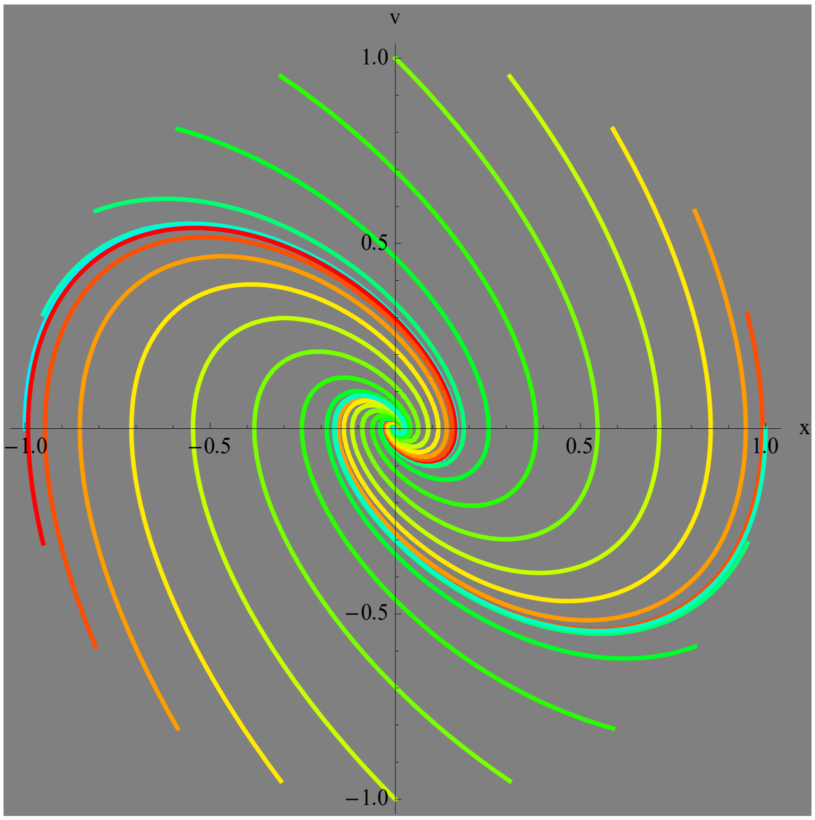


v
t
1.0
0.5
−0.5
−1.0
−1.0
−0.5
0.5
1.0

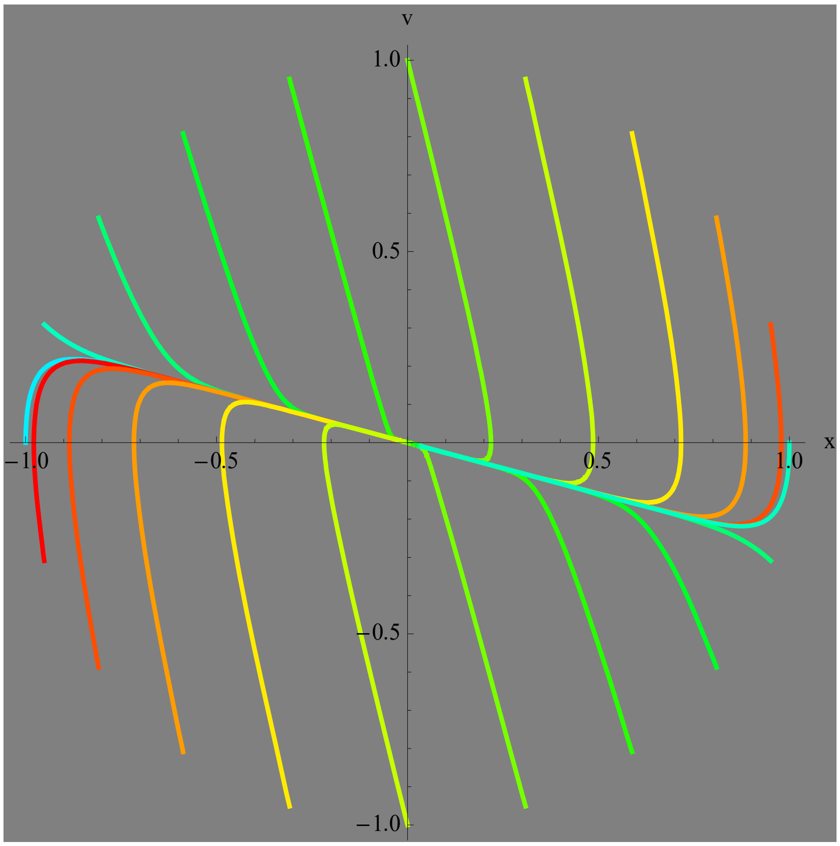
```
ParametricPlot[
 Evaluate[
  Table[{x2[t], v2[t]} /. {β → 1.1, ω0 → 1, x0 → Cos[θ], v0 → Sin[θ]},
   {θ, 0, 2 π, π / 10}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thickness[0.01]}, {i, 0, 10}],
 AxesLabel → {"x", "v"}, Prolog → AbsoluteThickness[2],
 Background → Gray, PlotRange → All]
```



```
ParametricPlot[
 Evaluate[
  Table[{x2[t], v2[t]} /. {β → 0.5, ω0 → 1, x0 → Cos[θ], v0 → Sin[θ]},
   {θ, 0, 2 π, π / 10}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"x", "v"}, Background → Gray, PlotRange → All]
```



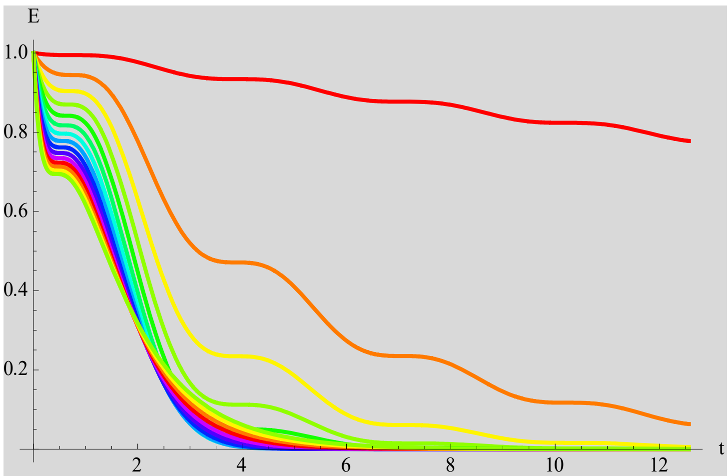
```
ParametricPlot[
 Evaluate[
  Table[{x2[t], v2[t]} /. {β → 2, ω0 → 1, x0 → Cos[θ], v0 → Sin[θ]},
   {θ, 0, 2 π, π / 10}]], {t, 0, 4 π},
 PlotStyle → Table[{Hue[0.051 i], Thick}, {i, 0, 10}],
 AxesLabel → {"x", "v"}, Background → Gray, PlotRange → All]
```



Total energy which cahnges with time

```
E1 = 1/2 m v3[t, β]^2 + 1/2 k x3[t, β]^2 /. {k → 1, m → 1};
```

```
Plot[Evaluate[Table[E1 /. rule1, {β, 0.01, 1.5, 0.1}]],
 {t, 0, 4 π}, PlotStyle → Table[{Hue[0.08 i], Thick}, {i, 0, 10}],
 AxesLabel → {"t", "E"}, Background → LightGray, PlotRange → All]
```



Overdamping, phase space

$$\texttt{ratio[t\_] :=} \frac{\texttt{v2[t]}}{\texttt{x2[t]}} \;/.\; \left\{\sqrt{-\beta^2 + \omega0^2} \to i\, p\right\} \;//\; \texttt{TrigToExp} \;//\; \texttt{FullSimplify}$$

```
ratio[t]
```

$$\frac{p\, v0\, \mathrm{Cosh}[p\, t] - \left(v0\, \beta + x0\, \omega0^2\right) \mathrm{Sinh}[p\, t]}{p\, x0\, \mathrm{Cosh}[p\, t] + (v0 + x0\, \beta)\, \mathrm{Sinh}[p\, t]}$$

$$\texttt{A1 = Limit[ratio[t], t} \to \infty\texttt{, Assumptions} \to \{p > 0\}] \;/.\; p \to \sqrt{\beta^2 - \omega0^2}$$

$$\frac{-v0\, \beta - x0\, \omega0^2 + v0 \sqrt{\beta^2 - \omega0^2}}{v0 + x0 \left(\beta + \sqrt{\beta^2 - \omega0^2}\right)}$$

```
Solve[a == A1 /. {v0 → a x0}, a]
```

$$\left\{\left\{a \to -\beta - \sqrt{\beta^2 - \omega0^2}\right\}, \left\{a \to -\beta + \sqrt{\beta^2 - \omega0^2}\right\}\right\}$$

8 Forced oscillation (steady state solution)

$$x''(t) + 2\beta x'(t) + \omega_0{}^2 x(t) = \xi_0 \cos(\omega t)$$

We assume that $x(t)$ can be given by

$$x(t) = \mathrm{Re}[Ae^{i\omega t}]$$

Re denotes a real part. A is in general a complex number. i (= $\sqrt{-1}$) is a pure imaginary

((Note))
Euler's equation

$$e^{i\theta} = \cos\theta + i\sin\theta$$

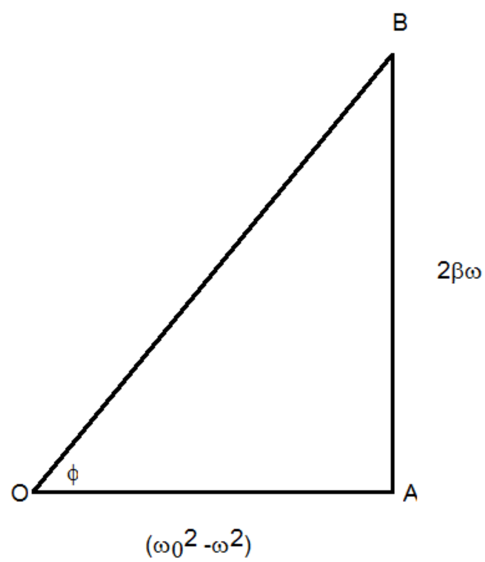
R.P. Feynman: This is the most remarkable formula. This is our jewel (22-10 volume-1, Feynman's lecture on physics.)

Then we have

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega_0{}^2 x(t) = \mathrm{Re}[(-\omega^2 + 2\beta i\omega + \omega_0{}^2)Ae^{i\omega t}] = \mathrm{Re}[\xi_0 e^{i\omega t}]$$

or

$$A = \frac{\xi_0}{(-\omega^2 + \omega_0{}^2) + 2\beta i\omega} = \frac{\xi_0}{\sqrt{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2}} e^{-i\phi}$$



Then x is obtained as

$$x(t) = \mathrm{Re}[Ae^{i\omega t}] = \mathrm{Re}[\frac{\xi_0}{\sqrt{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2}} e^{i(\omega t - \phi)}]$$

$$= \frac{\xi_0}{\sqrt{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2}} \cos(\omega t - \phi)$$

or

$$\left|\frac{A}{\xi_0}\right| = \frac{1}{\sqrt{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2}} = \frac{1}{\omega_0{}^2\sqrt{(\frac{\omega^2}{\omega_0{}^2} - 1)^2 + 4\frac{\beta^2}{\omega_0{}^2}\frac{\omega^2}{\omega_0{}^2}}}$$

or

$$Y = \left|\frac{A}{\xi_0}\right|\omega_0{}^2 = \frac{1}{\sqrt{(\frac{\omega^2}{\omega_0{}^2} - 1)^2 + 4\frac{\beta^2}{\omega_0{}^2}\frac{\omega^2}{\omega_0{}^2}}} = \frac{1}{\sqrt{(x^2 - 1)^2 + 4\varsigma^2 x^2}}$$

Now we calculate the value of Y as a function of x when a parameter ζ is changed.

$$x = \frac{\omega}{\omega_0}$$

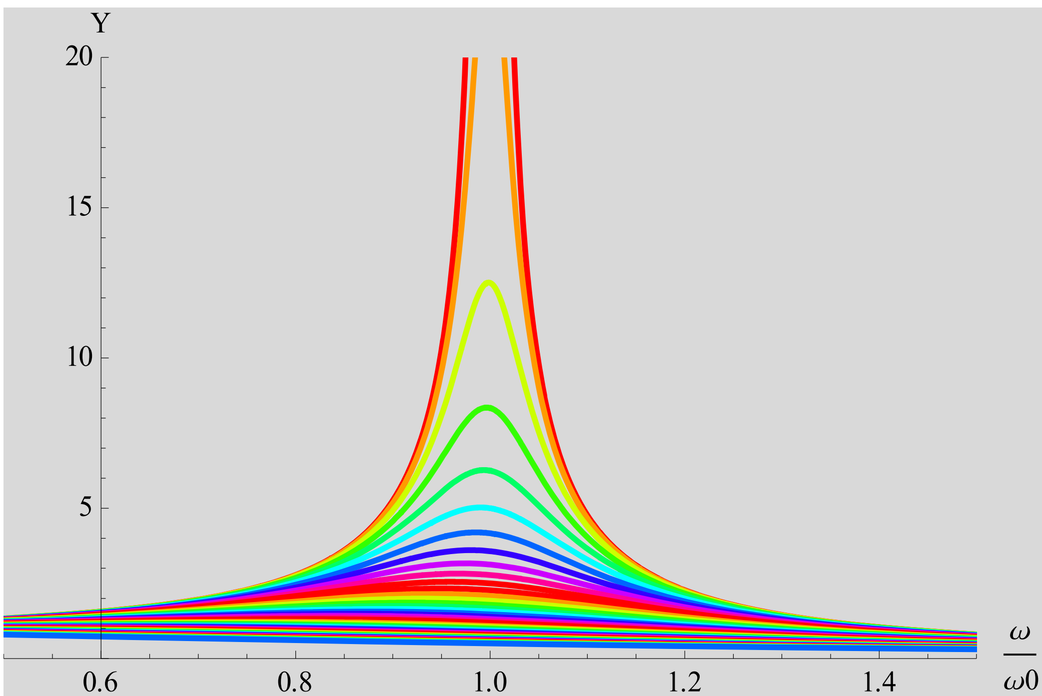
$$\varsigma = \frac{\beta}{\omega_0}$$

((**Mathematica**))

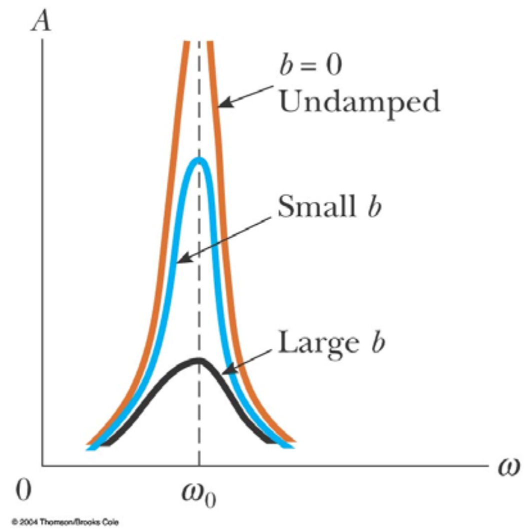
$$Y = \frac{1}{\sqrt{\left(x^2 - 1\right)^2 + 4\,\zeta^2\,x^2}}$$

$$\frac{1}{\sqrt{\left(-1 + x^2\right)^2 + 4\,x^2\,\zeta^2}}$$

```
Plot[Evaluate[Table[Y, {ζ, 0, 1, 0.02}]], {x, 0.5, 1.5},
 PlotRange → {{0.5, 1.5}, {0, 20}},
 PlotStyle → Table[{Hue[0.1 i], Thick}, {i, 0, 10}],
 Background → LightGray, AxesLabel → {"ω/ω0", "Y"}]
```



Y vs $\dfrac{\omega}{\omega_0}$ where $\zeta\left(=\dfrac{\beta}{\omega_0}\right)$ is changed as a parameter. $\zeta = 0 - 1.0$.



((**Note**)) **Simple explanation for the resonance**

We consider the special case (b = 0, or $\beta = 0$).

$$\ddot{x}(t) + \omega_0{}^2 x(t) = \xi_0 \cos \omega t$$

We assume that the solution of $x(t)$ is given by

$$x(t) = A_0 \cos \omega t$$

Then we have

$$-A_0\omega^2 + A_0\omega_0{}^2 = \xi_0$$

or

$$A_0 = \frac{\xi_0}{\omega_0{}^2 - \omega^2}$$

We note that

$$|A_0| = \left|\frac{\xi_0}{\omega_0{}^2 - \omega^2}\right|$$

becomes divergent as ω approaches ω_0.

9. Energy consideration in the forced oscillation

We start from

$$\ddot{x}(t)+2\beta\dot{x}(t)+\omega_0{}^2x(t)=\xi_0\cos(\omega t).$$

Multiplying $\dot{x}(t)$ on both sides, we have

$$\dot{x}'(t)\ddot{x}(t)+2\beta[\dot{x}(t)]^2+\omega_0{}^2x(t)\dot{x}(t)=\xi_0\cos(\omega t)\dot{x}(t),$$

or

$$\frac{d}{dt}\{\frac{1}{2}[\dot{x}(t)]^2+\frac{\omega_0{}^2}{2}[x(t)]^2\}+2\beta[\dot{x}(t)]^2=\xi_0\cos(\omega t)\dot{x}(t).$$

This equation can be rewritten as

$$\{\frac{1}{2}[\dot{x}(t)]^2+\frac{\omega_0{}^2}{2}[x(t)]^2-\frac{1}{2}[\dot{x}(t=0)]^2-\frac{\omega_0{}^2}{2}[x(t=0)]^2+\int_0^t 2\beta[\dot{x}(t_1)]^2dt_1=\int_0^t\xi_0\cos(\omega t_1)\dot{x}(t_1)dt_1$$

Here we introduce the instantaneous energy $\varepsilon(t)$ which is defined by

$$\varepsilon(t)=\frac{1}{2}[\dot{x}(t)]^2+\frac{\omega_0{}^2}{2}[x(t)]^2.$$

We take an average of the above equation over a one period *T*,

$$\frac{1}{T}\{\varepsilon(t=T)-\varepsilon(t=0)\}+\frac{1}{T}\int_0^T 2\beta[\dot{x}(t_1)]^2dt_1-\frac{1}{T}\int_0^T\xi_0\cos(\omega t_1)\dot{x}(t_1)dt_1=0.$$

We now calculate the second and third terms using our steady-state solution

$$x(t)=\mathrm{Re}[Ae^{i\omega t}]$$
$$\dot{x}(t)=\mathrm{Re}[A(i\omega)e^{i\omega t}],$$

with

$$A=\frac{\xi_0}{(-\omega^2+\omega_0{}^2)+2\beta i\omega}=\frac{\xi_0[(-\omega^2+\omega_0{}^2)-2\beta i\omega]}{(-\omega^2+\omega_0{}^2)^2+4\beta^2\omega^2}=\chi'-i\chi''$$

and

$$iA = \frac{i\xi_0}{(-\omega^2 + \omega_0{}^2) + 2\beta i\omega} = \frac{\xi_0[i(-\omega^2 + \omega_0{}^2) + 2\beta\omega]}{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2}$$

where χ' and χ'' are the real part and imaginary part of A.

The calculation of the second term:

$$\begin{aligned}
\frac{1}{T}\int_0^T 2\beta[x'(t_1)]^2 dt_1 &= \frac{2\beta}{T}\int_0^T \frac{[A(i\omega)e^{i\omega t_1} + A^*(-i\omega)e^{-i\omega t_1}][A(i\omega)e^{i\omega t_1} + A^*(-i\omega)e^{-i\omega t_1}]}{4} dt_1 \\
&= \frac{2\beta}{4T}[2A(i\omega)A^*(-i\omega)] \\
&= \frac{\beta}{T}|A|^2\omega^2 \\
&= \frac{\beta}{2\pi}|A|^2\omega^3 \\
&= \frac{\beta}{2\pi}\xi_0{}^2 \frac{\omega^3}{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2} \\
&= \frac{\xi_0\omega^2}{4\pi}\chi''
\end{aligned}$$

The calculation of the third term:

$$\begin{aligned}
\frac{1}{T}\int_0^T \xi_0\cos(\omega t_1)x'(t_1)dt_1 &= \frac{\xi_0}{T}\int_0^T \frac{(e^{i\omega t_1} + e^{-i\omega t_1})[A(i\omega)e^{i\omega t_1} + A^*(-i\omega)e^{-i\omega t_1}]}{4} dt_1 \\
&= \frac{\xi_0}{4T}[A(i\omega) + A^*(-i\omega)] \\
&= \frac{\xi_0}{2T}\mathrm{Re}[A(i\omega)] \\
&= \frac{\xi_0\omega}{2T}\frac{\xi_0 2\beta\omega}{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2} \\
&= \frac{\beta}{2\pi}\xi_0{}^2 \frac{\omega^3}{(-\omega^2 + \omega_0{}^2)^2 + 4\beta^2\omega^2} \\
&= \frac{\xi_0\omega^2}{4\pi}\chi''
\end{aligned}$$

Then it is found that the second term is equal to the third term. These terms are proportional to χ'' (imaginary part of A). The energy absorbed by the system from the external force is dissipated through the resistive damping. Then we have

$$\varepsilon(t = T) = \varepsilon(t = 0).$$

The sum of the kinetic energy and the potential energy is a periodic function of t with a period of T.

10 Example

10.1 Example-1:

Problem 15-26*(SP-15) (10-th edition)**

In Fig., two blocks (m = 1.8 kg and M = 10 kg) and a spring (k = 200 N/m) are arranged on a horizontal, frictionless surface. The coefficient of static friction between the two blocks is 0.40. What amplitude of simple harmonic motion of the spring–blocks system puts the smaller block on the verge of slipping over the larger block?

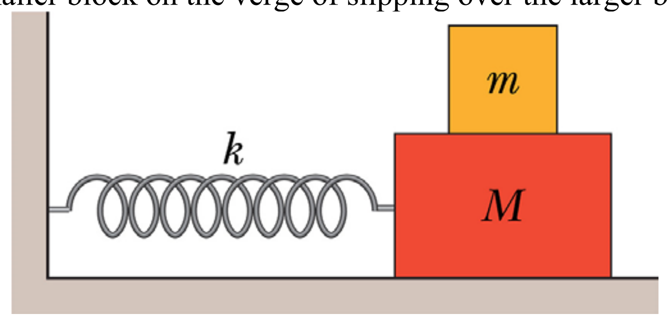
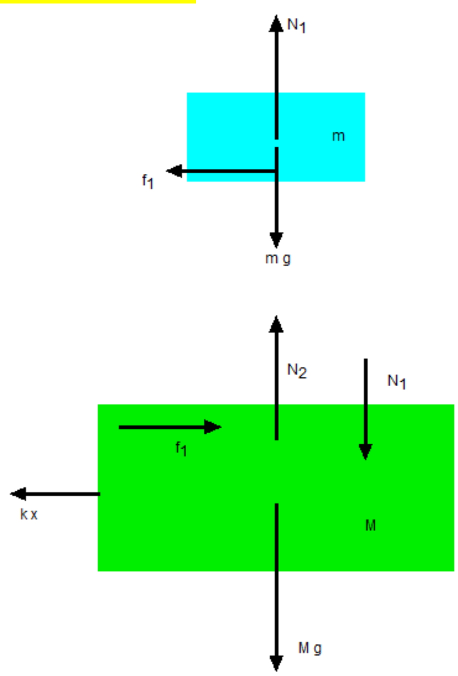


FIGURE 15-34 Problem 24.

$m = 1.8$ kg, $M = 10$ kg, $k = 200$ N/m



For the small mass,

$$m\ddot{x} = -f_1$$
$$N_1 = mg \qquad (1)$$
$$f_1 \le \mu_s N_1 = \mu_s mg$$

For the large mass,

$$M\ddot{x} = f_1 - kx$$
$$N_2 = N_1 + Mg = (M+m)g \qquad (2)$$

From Eqs.(1) and (2),

$$(M+m)\ddot{x} = -kx$$
$$\ddot{x} = -\omega^2 x$$

with

$$\omega^2 = \frac{k}{M+m}$$

The solution of the differential equation is

$$x = A\cos(\omega t + \phi)$$

with $\omega = 4.11 rad/s$

$$N_1 = mg$$
$$f_1 = -m\ddot{x} = \frac{mkx}{M+m} \le \mu_s mg$$

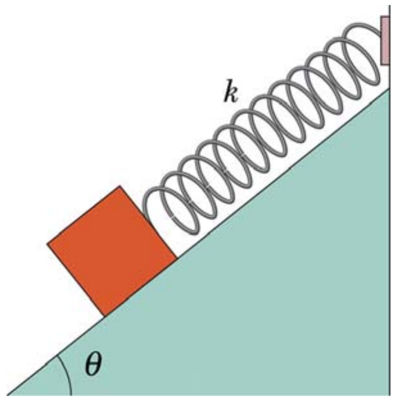
or

$$x \le \frac{\mu_s g}{k}(M+m) = 0.23m$$

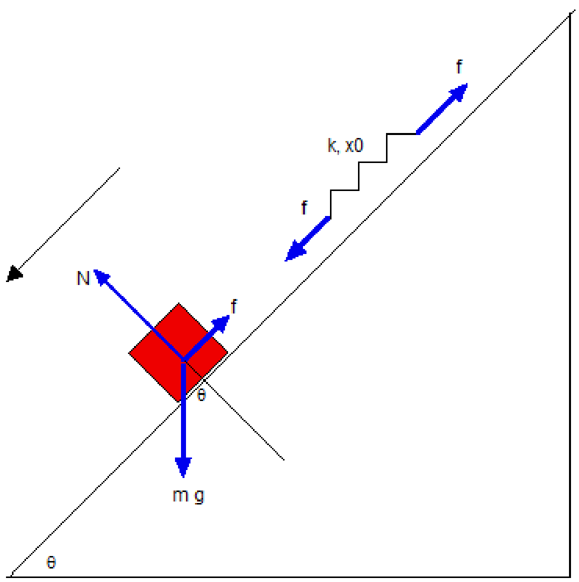
10.2 Example-2

Problem 15-25* (10-th edition)**

In Fig., a block weighing 14.0 N, which can slide without friction on an incline at angle θ = 40°, is connected to the top of the incline by a massless spring of un-stretched length 0.450 m and spring constant 120 N/m. (a) How far from the top of the incline is the block's equilibrium point? (b) If the block is pulled slightly down the incline and released, what is the period of the resulting oscillations?



$mg = 14.0$ N, $\theta = 40°$, $x_0 = 0.450$ m, $k = 120$ N/m



(a)

$$mg\sin\theta = f$$
$$f = k(x_1 - x_0)$$

or

$$x_1 = x_0 + \frac{mg}{k}\sin\theta = 0.525m$$

(b)

$$m\ddot{x} = mg\sin\theta - f$$
$$f = k(x - x_0)$$

or

$$\ddot{x} = -\frac{k}{m}(x - x_0 - \frac{mg}{k}\sin\theta) = -\frac{k}{m}(x - x_1)$$

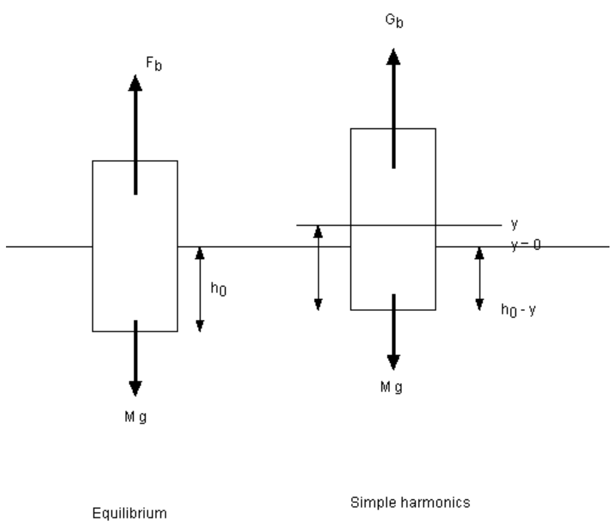
The solution of this equation is

$$x = x_1 + A\cos(\omega t + \phi)$$

with $\omega = \sqrt{\frac{k}{m}}$

The period T is $T = \frac{2\pi}{\omega} = 0.686s$

10.3 Example-3 Simple harmonics in fluid mechanics



In equilibrium

$$F_b = A\rho g h_0 = Mg$$

or

$$h_0 = \frac{M}{A\rho}$$

Simple harmonics

$$M\ddot{y} = G_b - Mg = (h_0 - y)\rho Ag - Mg$$

where G_b is the buoyant force.

$$M\ddot{y} = (h_0 - y)\rho Ag - Mg = -y\rho Ag$$

$$\ddot{y} = -\frac{\rho Ag}{M}y = -\omega^2 y \qquad \text{(simple harmonics)}$$

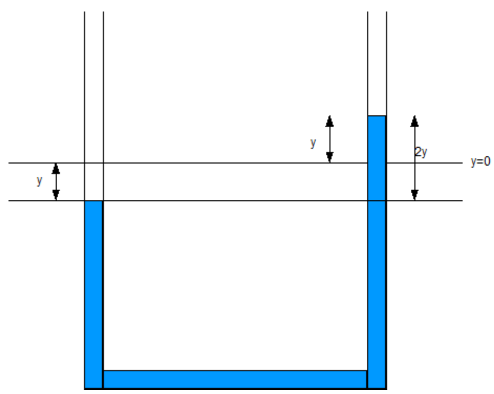
where

$$\omega = \sqrt{\frac{\rho Ag}{M}}$$

$$T = 2\pi\sqrt{\frac{M}{\rho Ag}}$$

10.4 Simple harmonics in fluid mechanics

V: volume of liquid with the density ρ
$m = \rho V$: mass of liquid
$L = V/A$: total length of the liquid column



$$m\ddot{y} = F = -(2y\rho g)A$$

$$m = \rho V = \rho LA$$

or

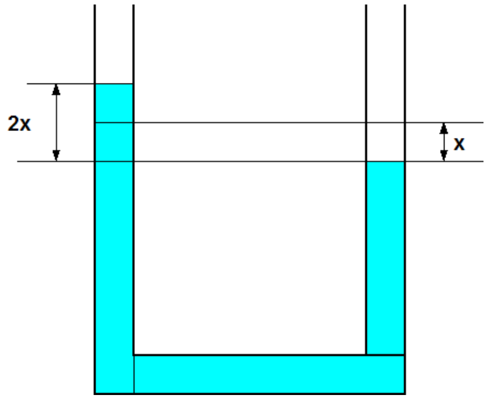
$$\ddot{y} = -\frac{(2y\rho g)A}{\rho LA} = -\frac{2g}{L}y = -\omega^2 y$$

where

$$\omega = \sqrt{\frac{2g}{L}}, \qquad T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{2g}}$$

((**Example**)) **Oscillation of liquid in a U-tube**

The mass m (= 9 kg) of mercury is poured into a glass U-tube as shown in Fig. The tube's inner diameter is 1.2 cm and the mercury oscillates freely up and down about its position of equilibrium (x = 0). Compute (a) the effective spring constant k for the oscillation, and (b) the period of oscillation. The density of mercury is ρ = 13.6 x 10^3 kg/m^3. Ignore frictional and surface tension effects.



((**Solution**))

r = 0.6 cm. ρ = 13.6 x 10^3 kg/m^3.
m = 9.0 kg.

When the mercury is displace x from its equilibrium position, the restoring force is the weight of the unbalanced column of mercury with height 2x. The restoring force F is described by

$$F = -\rho A(2x)g,$$

where A is the area of U-tube ($A = \pi r^2$). Then the mercury undergoes a simple harmonic oscillation,

$$m\ddot{x} = F = -\rho A(2x)g = -kx,$$

where

$$k = \frac{2\rho A g}{m} = \frac{2\pi r^2 \rho g}{m} = 30 \text{ N/m.}$$

This equation can be rewritten as

$$\ddot{x} = -\omega^2 x$$

where w is the angular frequency,

$$\omega = \sqrt{\frac{k}{m}}\,.$$

The period of the oscillation, T, is

$$T = 2\pi\sqrt{\frac{m}{k}} = 3.4 \text{ s.}$$

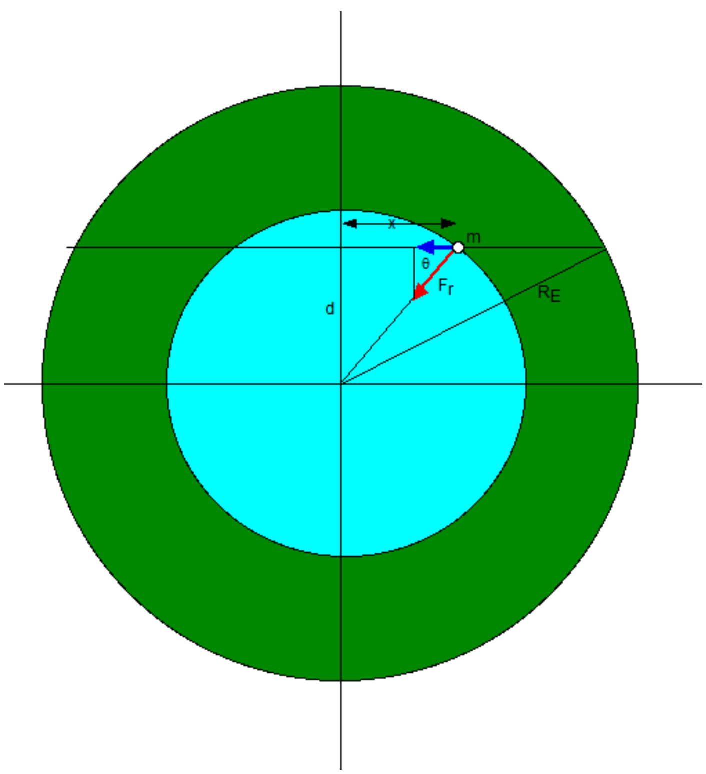
10.5 Example-4 (Serway 12-75)

Imagine that a hole is drilled through the center of the Earth to the other side. An object of mass *m* at a distance *r* from the center of the Earth is pulled toward the center of the Earth only by the mass within the sphere of radius *r* (the reddish region in Fig.).

(a) Write Newton's second law of gravitation for an object at the distance r from the center of the Earth, and show that the force on it is of Hooke's law form $F = -kr$, where the effective force constant is $k = (4/3)\pi\rho G m$. Here r is the density of the Earth assumed uniform, and *G* is the gravitational constant.

(b) Show that a sack of mail dropped into the hole will execute simple harmonic motion if it moves without friction.

(c) When will it arrive at the other side of the Earth.



$$M_E = \frac{4\pi}{3}\rho R^3$$

$$M_r = \frac{4\pi}{3}\rho r^3$$

$$\frac{M_r}{M_E} = \frac{r^3}{R^3}$$

The force is directed toward the center.

$$\boldsymbol{F} = -\frac{GmM_r}{r^2}\hat{r} = -\frac{GmM_E}{r^2}\frac{r^3}{{R_E}^3}\hat{r} = -\frac{GmM_E}{{R_E}^3}r\hat{r} = F_r\hat{r}$$

The equation of motion for the particle on the tunnel along the *x*-axis.

$$m\ddot{x} = F_r\cos\theta = -\frac{GmM_E}{{R_E}^3}r\cos\theta = -\frac{GmM_E}{{R_E}^3}x$$

or

$$\ddot{x} = -\omega^2 x \qquad \text{(Simple harmonics)}$$

where

$$\omega = \sqrt{\frac{GM_E}{{R_E}^3}} = \sqrt{\frac{g}{R_E}}$$

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{R_E}{g}} = 5061.43\ s$$

or

$$\frac{T}{2} = 2530.7s = 42.2\min$$

((Note)) Period of satellite

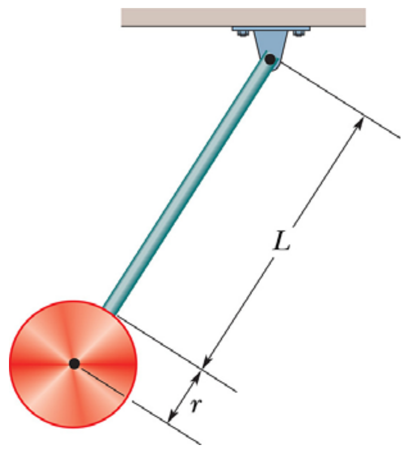
$$v = \sqrt{\frac{M_E G}{R_E}} = 7.910km/s = 4.9miles/s = 17640miles/h$$

$$T = \frac{2\pi r}{v} = 2\pi\sqrt{\frac{R_E}{g}} = 5061s = 1hour24initues21\sec$$

10.6 Example-5

Problem 15-41 (SP-15) (10-th edition)

In Fig., the pendulum consists of a uniform disk with radius r = 10.0 cm and mass 500 g attached to a uniform rod with length L = 500 mm and mass 270 g. (a) Calculate the rotational inertia of the pendulum about the pivot point. (b) What is the distance between the pivot and the center of mass of the pendulum? (c) Calculate the period of oscillation.



$$I = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 + \frac{1}{2}Mr^2 + M(L+r)^2$$
$$= \frac{1}{3}mL^2 + \frac{1}{2}Mr^2 + M(L+r)^2 = 0.205kgm^2$$

where M is the mass of disk and m is the mass of the rod

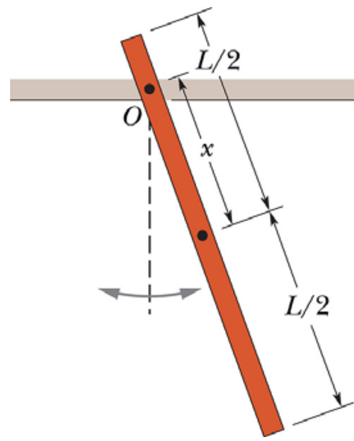
The equation of motion

$$I\ddot{\theta} = -[M(L+r) + m\frac{L}{2}]g\sin\theta$$
$$\approx -[M(L+r) + m\frac{L}{2}]g\theta$$

10.7 Example-6

Problem 15-51 (SP-15) (10-th edition)**

In Fig. a stick of length L = 1.85 m oscillates as a physical pendulum. (a) What value of distance x between the stick's center of mass and its pivot point O gives the least period? (b) What is the least period?



$$\tau = I_0 \ddot{\theta} = -Mgx \sin\theta$$

$$I_0 = \frac{1}{12} ML^2 + Mx^2$$

or

$$\ddot{\theta} = -\frac{mgx}{I}\theta = -\omega^2 \theta \qquad \text{(simple harmonics)}$$

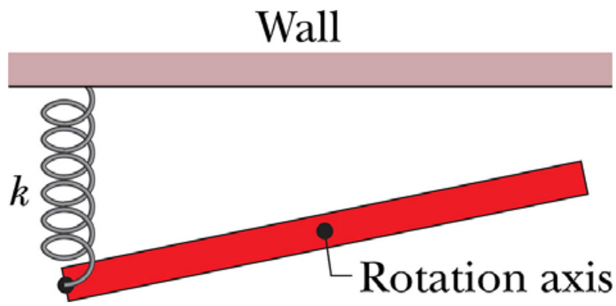
where θ is the angle between the vertical axis and the thin rod, and ω is defined by

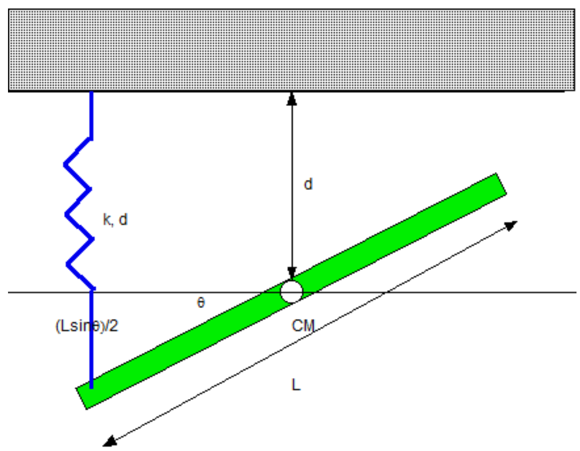
$$\omega = \sqrt{\frac{mgx}{\frac{1}{12}mL^2 + mx^2}} = \sqrt{\frac{gx}{\frac{1}{12}L^2 + x^2}} .$$

10.8 Example-7

Problem 15-53 (HW-15) (10-th edition)**

In the overhead view of Fig., a long uniform rod of mass 0.600 kg is free to rotate in a horizontal plane about a vertical axis through its center. A spring with force constant k = 1850 N/m is connected horizontally between one end of the rod and a fixed wall. When the rod is in equilibrium, it is parallel to the wall. What is the period of the small oscillations that result when the rod is rotated slightly and released?

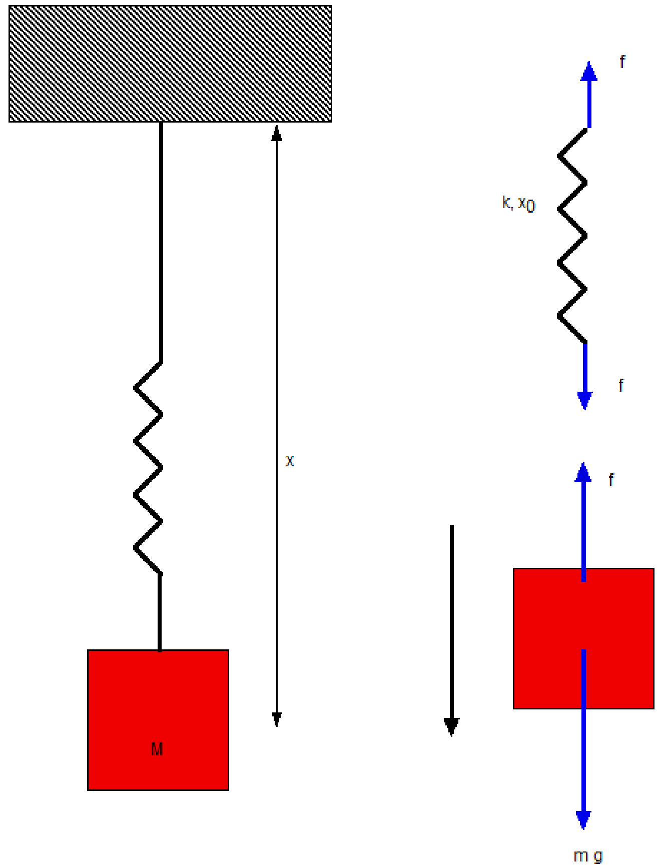




$$\tau = I\alpha = I\ddot{\theta} = -\frac{k}{2}L\sin\theta\frac{L}{2}\cos\theta = -\frac{k}{4}L^2\frac{1}{2}\sin(2\theta)$$

$$I\ddot{\theta} = -\frac{kL^2}{8}\sin(2\theta) \approx -\frac{kL^2}{4}\theta$$

10.9 Example-8



Equation of motion

$$m\ddot{x} = -f + mg$$
$$f = k(x - x_0)$$

or

$$m\ddot{x} = -k(x - x_1)$$

where

$$x_1 = x_0 + \frac{mg}{k}$$

The solution of the differential equation is given by

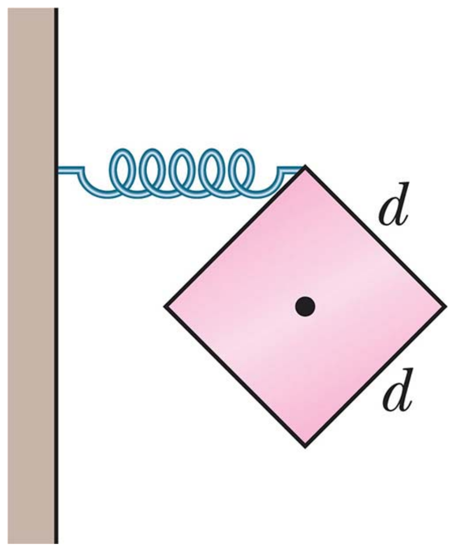
$$\ddot{x} = -\omega^2 (x - x_1)$$
$$x - x_1 = A\cos(\omega t + \phi)$$

with

$$\omega = \sqrt{\frac{k}{m}}$$

10.10
Problem 15-52 (SP-15) (10-th edition)**

The 3.00 kg cube in Fig. has edge lengths d = 6.00 cm and is mounted on an axel through its center. A spring (k = 1200 N/m) connects the cube's upper corner to a rigid wall. Initially the spring is at its rest length. If the cube is rotated 3° and released, what is the period of the resulting simple harmonic oscillation?



((**Solution**))
k = 1200 N/m
M = 3 kg

The moment of inertia is

$$I = \frac{1}{12}M(d^2 + d^2) = \frac{1}{6}Md^2$$

around the axis through its center for the cube with the side d.

The equation of motion:

$$I\ddot{\theta} = -kx(\frac{\sqrt{2}}{2}d) = -k(\frac{\sqrt{2}}{2}d)\theta(\frac{\sqrt{2}}{2}d) = -k\frac{1}{2}d^2\theta$$

or

$$\frac{1}{6}Md^2\ddot{\theta} = -k\frac{1}{2}d^2\theta$$

or

$$\ddot{\theta} = -\frac{3k}{M}\theta = -\omega^2\theta \qquad \text{(simple harmonics)}$$

where

$$\omega = \sqrt{\frac{3k}{M}} = 34.641 \text{ rad/s}$$

The period T is given by

$$T = 2\pi\sqrt{\frac{M}{3k}} = 0.181s$$

The initial condition: $\theta(0) = 3°$ and $\dot{\theta}(0) = 0$

Using this condition, we get the time dependence of θ as

$$\theta(t) = A\cos(\omega t) + B\sin(\omega t)$$
$$\dot{\theta}(t) = -A\omega\sin(\omega t) + B\omega\cos(\omega t)$$
$$\theta(0) = A$$
$$\dot{\theta}(0) = B\omega = 0$$

or

$$\theta(t) = \theta(0)\cos(\omega t)$$

12. Advanced problems

12.1 Serway 15-73 ((numerical calculation))

Consider a bob on a light stiff rod, forming a simple pendulum of length L = 1.20 m. It is displaced from the vertical by an angle θ_{max} and then released. Predict the subsequent angular positions if θ_{max} is small or if it is large. Proceed as follows: Set up and carry out a numerical method to integrate the equation of motion for the simple pendulum;

$$\frac{d^2\theta(t)}{dt^2} = -\frac{g}{L}\sin\theta(t)$$
.

Take the initial conditions to be $\theta = \theta_{max}$ and $d\theta/dt = 0$ at $t = 0$. On one trial, choose $\theta_{max} = 5.00°$, and on another trial take $\theta_{max} = 100°$. In each case, find the position $\theta(t)$ as a function of time t. Using the same values of θ_{max}, compare your results for θ with those obtained from $\theta(t) = \theta_{max}\cos(\omega t)$. How does the period for large values of θ_{max} compares with that for the small value of θ_{max}?

((Solution)) We use the Mathematica for the solution.

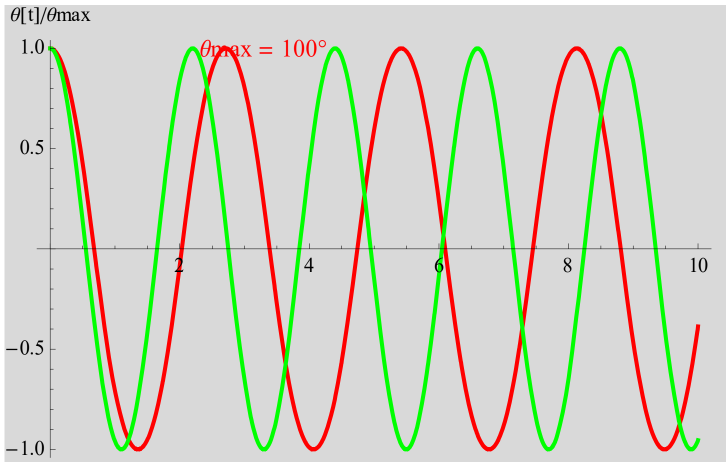


Fig.1 Red: $\theta(t)/\theta_{max}$ for $\theta_{max} = 100°$: Green: $\cos(\omega_0 t)$, where $\omega_0 = \sqrt{\frac{g}{L}} = 2.8577$ rad/s.

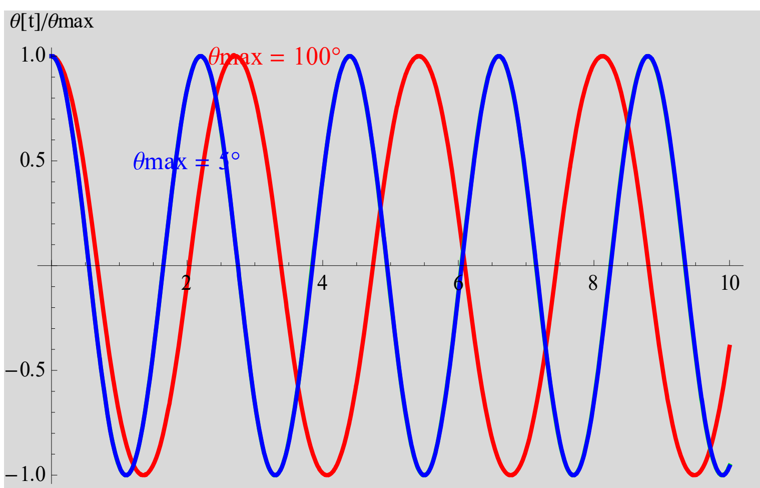
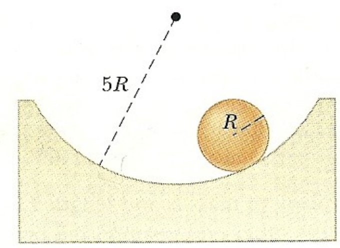


Fig.2 Red: $\theta(t)/\theta_{max}$ for $\theta_{max} = 100°$. Blue: $\theta(t)/\theta_{max}$ for $\theta_{max} = 5.00°$:, where $\omega_0 = \sqrt{\frac{g}{L}} = 2.8577$ rad/s.

12.2 Serway 15-56

A solid sphere (radius = R) rolls without slipping in a cylindrical trough (radius = 5 R) as shown in Fig. Show that, for a small displacements from equilibrium perpendicular to the length of the trough, the sphere executes simple harmonics with a period.

$$T = 2\pi\sqrt{\frac{28R}{5g}}$$



((**Solution**))

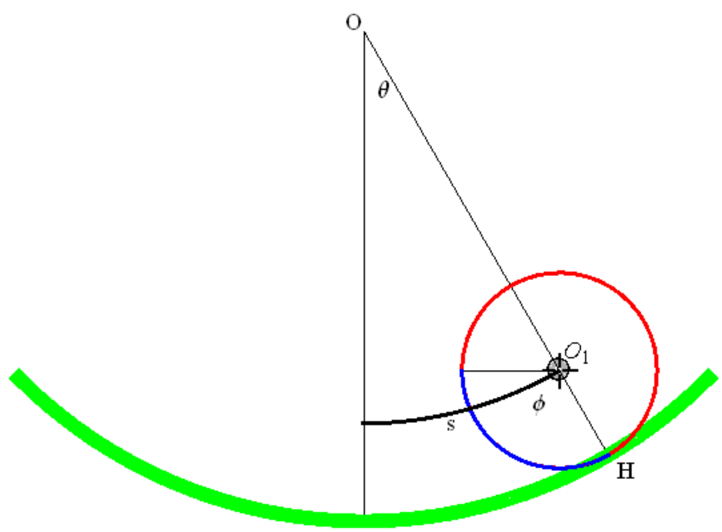


Fig. $OO_1 = 4R$, $O_1H = R$. $s = R\phi = 4R\,\theta$.

Kinetic energy K;

$$K = \frac{1}{2}m{v_c}^2 + \frac{1}{2}I\Omega^2.$$

I is the moment of inertia for the sphere with radius R, $I = \frac{2}{5}mR^2$. Ω is the angular velocity, $\Omega = d\phi/dt$. From the rollinc of the ball without slipping, we have

$$s = 4R\theta = R\phi,$$

or

$$v_c = \frac{ds}{dt} = 4R\dot{\theta} = R\Omega .$$

Then K can be rewritten as

$$K = \frac{1}{2}m(R\Omega)^2 + \frac{1}{2}\frac{2}{5}mR^2\Omega^2 = \frac{7}{10}mR^2\Omega^2 = \frac{7}{10}mR^2(4\dot{\theta})^2 = \frac{56}{5}mR^2\dot{\theta}^2 .$$

The potential energy U is given by

$$U = mg(5R - 4R\cos\theta) ,$$

where the reference point is at the bottom of the cylindrical trough. Since the total energy E is independent of t, we have

$$E = K + U = \frac{56}{5}mR^2\dot{\theta}^2 + mg(5R - 4R\cos\theta) = \text{const}$$

or

$$[\frac{112}{5}mR^2\ddot{\theta} + mg(4R\sin\theta)]\dot{\theta} = 0$$

This is a differential equation for the simple harmonics, when $\sin\theta \approx \theta$.

$$\ddot{\theta} + \omega^2\theta = 0$$

with

$$\omega = \sqrt{\frac{5g}{28R}}$$

The period T is

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{28R}{5g}}$$

13. Exact measurement of the period for the simple pendulum with small maximum angle

"The history of the physics of the pendulum stretches back to the early moments of modern science itself. We might begin with the story, perhaps apocryphal, of Galileo's observation of the swinging chandeliers in the cathedral at Pisa. By using his own heart

rate as a clock, Galileo presumably made the quantitative observation that, for a given pendulum, the time or period of a swing was independent of the amplitude of the pendulum's displacement. Like many other seminal observations in science, this one was only an approximation of reality. Yet it had the main ingredients of the scientific enterprise; observation, analysis, and conclusion. Galileo was one of the first of the modern scientists, and the pendulum was among the first objects of scientific enquiry."
(G.L. Baker and J.A. Blackburn, Oxford University Press, 2005).

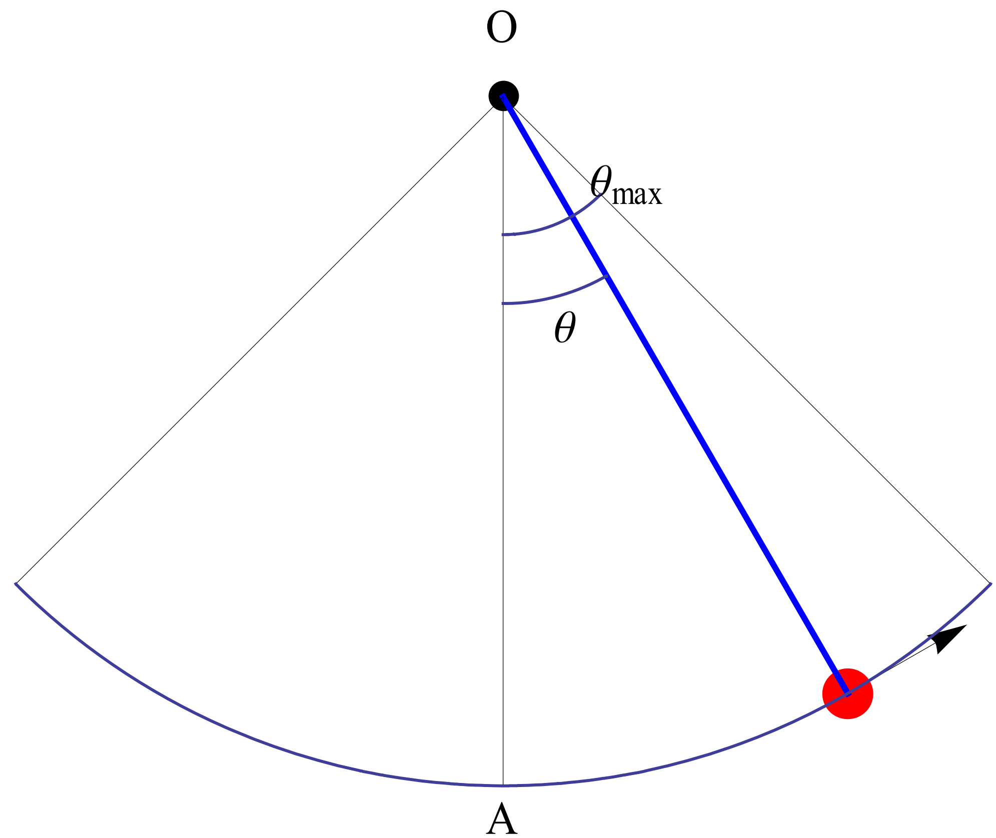


Fig. Simple pendulum with a point mass m. θ_{max} is the maximum of the angle θ.

We consider the motion of the simple pendulum. The kinetic energy is given by

$$K = \frac{1}{2} m (l\dot{\theta})^2 ,$$

The potential energy:

$$U = mgl(1 - \cos\theta)$$

The energy conservation

$$E = K + U = \frac{1}{2} m l^2 \dot{\theta}^2 + mgl(1 - \cos\theta)$$

or

$$\frac{1}{2}\omega^2 + \frac{g}{l}(1 - \cos\theta) = const$$

with

$$\omega = \dot{\theta}$$

We now start with

$$\frac{1}{2}\omega^2 + \varepsilon(1 - \cos\theta) = \varepsilon(1 - \cos\theta_{\max})$$

where

$$\varepsilon = \frac{g}{l}, \qquad \omega = \dot{\theta}$$

We note that

(i) $\omega = 0$ for $\theta = \theta_{\max}$
(ii) $\omega = \omega_0$ for $\theta = 0$

So we have

$$\frac{1}{2}\omega^2 + \varepsilon(1 - \cos\theta) = \frac{1}{2}\omega_0{}^2 = \varepsilon(1 - \cos\theta_{\max}) .$$

From this, we get

$$\frac{1}{2}\omega^2 = -\varepsilon(1 - \cos\theta) + \varepsilon(1 - \cos\theta_{\max})$$

or

$$\omega^2 = \dot{\theta}^2 = \varepsilon(\cos\theta - \cos\theta_{\max})$$

Using a formula,

$$\cos\theta = 1 - 2\sin^2\frac{\theta}{2}$$

we have

$$\dot{\theta} = \frac{d\theta}{dt} = 2\sqrt{\varepsilon}\sqrt{\sin^2\frac{\theta_{\max}}{2} - \sin^2\frac{\theta}{2}}$$

Then the period T is given by

$$\frac{T}{4} = \frac{1}{2\sqrt{\varepsilon}}\int_0^{\theta_{\max}} \frac{d\theta}{\sqrt{\sin^2\frac{\theta_{\max}}{2} - \sin^2\frac{\theta}{2}}}.$$

Here we put

$$z = \frac{\sin\frac{\theta}{2}}{\sin\frac{\theta_{\max}}{2}} = \frac{1}{k}\sin\frac{\theta}{2}$$

with

$$k = \sin\frac{\theta_{\max}}{2}.$$

Note that

$$dz = \frac{1}{2k}\cos\frac{\theta}{2}d\theta = \frac{1}{2k}\sqrt{1-\sin^2\frac{\theta}{2}}d\theta = \frac{1}{2k}\sqrt{1-k^2z^2}d\theta$$

Then we have

$$T = \frac{2\pi}{\omega} = \frac{2}{\sqrt{\varepsilon}}\int_0^1 \frac{2kdz}{k\sqrt{1-z^2}\sqrt{1-k^2z^2}} = \frac{4}{\sqrt{\varepsilon}}\int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-k^2z^2}} = \frac{4}{\sqrt{\varepsilon}}K(k^2)$$

or

$$\boxed{T = \frac{4}{\sqrt{\varepsilon}}\text{EllipticK}[k^2]}$$

where

$$\varepsilon = \frac{g}{l}, \qquad k^2 = \sin^2 \frac{\theta_{max}}{2}$$

Note that $K(\xi)$ is the complete elliptic integral of the first kind and is defined by

$$K(\xi) = \int_0^1 \frac{dz}{\sqrt{1-z^2}\sqrt{1-\xi z^2}}$$

In the Mathematica, this function corresponds to EllipticK[ξ]. When $k \to 0$, we have

$$T = \frac{2\pi}{\sqrt{\varepsilon}} = 2\pi\sqrt{\frac{l}{g}}$$

The series expansion of T around $k = 0$ is given by

$$T = \frac{2\pi}{\sqrt{\varepsilon}}(1 + \frac{1}{4}k^2 + \frac{9}{64}k^4 + \frac{25}{256}k^6 + \frac{1225}{16384}k^8 + .\frac{3969}{65536}k^{10} + ...$$

We make a plot of the deviation $(T - T_0)/T_0$ (denoted by %) as a function of θ_{max}, where

$$T_0 = \frac{2\pi}{\sqrt{\varepsilon}} = 2\pi\sqrt{\frac{l}{g}}$$

It is clear that the deviation starts to occur when θ_{max} is larger than $\theta_{max} = 2°$.

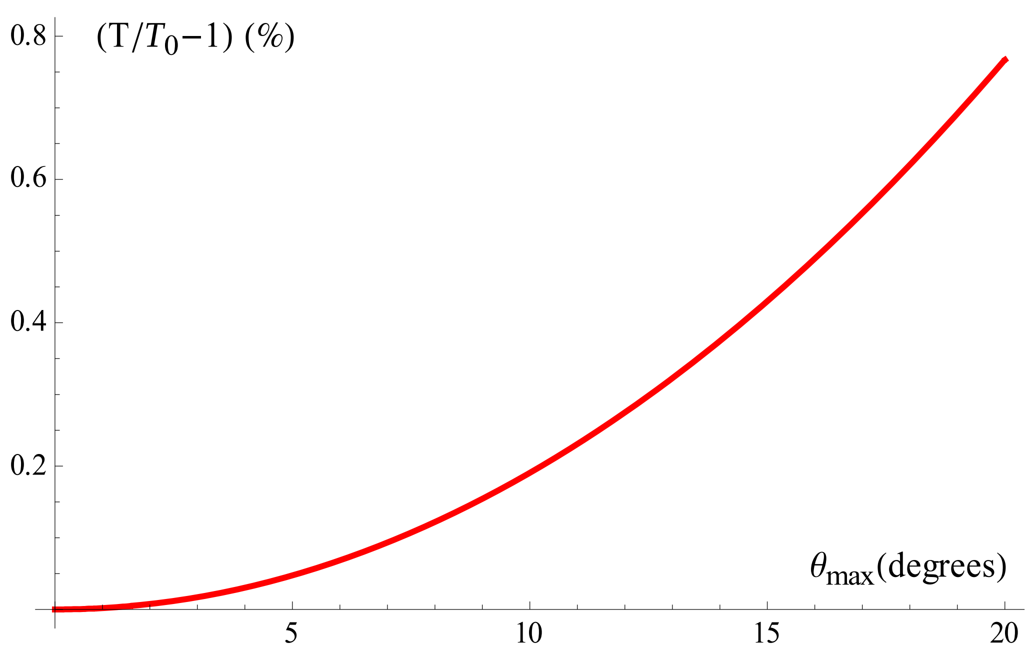


Fig. The deviation $\Delta T/T_0$ x 100 (%) as a function of the maximum angle $\theta_{\max}$. The deviation is defined by that $\Delta T/T_0 = (T - T_0)/T_0$.

We note that the period is independent of $\theta_{\max}$ only for a few degrees. The period becomes dependent of $\theta_{\max}$ as $\theta_{\max}$ increases. The deviation is 0.19 % for $\theta_{\max}$ = 10° and 0.77% for $\theta_{\max}$ = 20°. So if one want to measure the exact period (T_0), one needs to use the small value of $\theta_{\max}$ below a few degrees.

Table The value of the deviation $\Delta T/T_0$ x 100 % for typical values of $\theta_{\max}$ where $\Delta T = T - T_0$

$\theta_{\max}$	$(T/T_0 - 1)$ x 100 %
0	0.
5	0.0476172
10	0.190719
15	0.430058
20	0.766903
25	1.20306
30	1.74088
35	2.38334
40	3.13405
45	3.99733
50	4.9783

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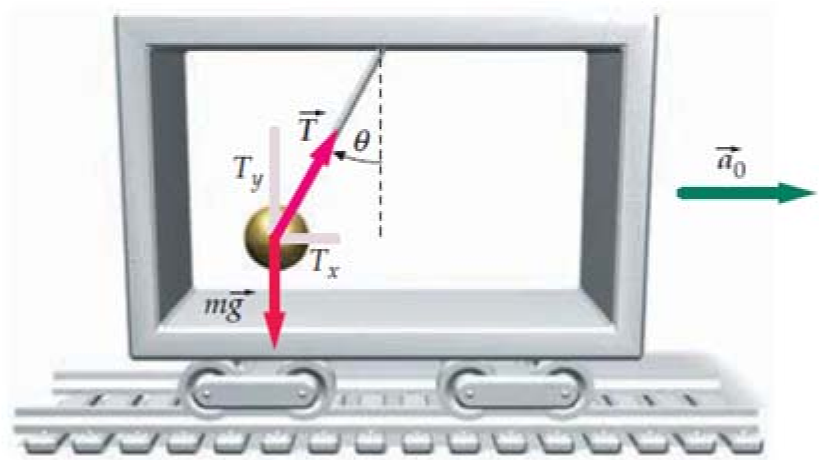
14. Simple pendulum in an accelerated reference frame

We consider a simple pendulum in an accelerated frame using the two examples. These examples come from (P.T. Tipler and G. Mosca, Physics for Scientists and Engineers, 6-th edition W.H. Freeman and Company, 2008) Chapter 14.

((**Example-1**)) **A simple pendulum suspended in the moving cart (with constant acceleration)**

A simple pendulum of length L suspended from the ceiling of a cart (C) that has acceleration $\boldsymbol{a}_{CG}$. Find the period of oscillation for small oscillations of this pendulum.



We start with an equation of motion given by

$$\boldsymbol{T} + m\boldsymbol{g} = m\boldsymbol{a}_{BG}$$

where $\boldsymbol{T}$ is the tension and m is the mass of bob of the simple pendulum. The acceleration of the bob relative to the ramp is equal to the acceleration of the bob(B) relative to the cart plus the acceleration of the cart (C) relative to the ground (G),

$$\boldsymbol{a}_{BG} = \boldsymbol{a}_{BC} + \boldsymbol{a}_{CG}.$$

Then we get

$$\boldsymbol{T} + m\boldsymbol{g} = m(\boldsymbol{a}_{BC} + \boldsymbol{a}_{CG}),$$

or

$$\boldsymbol{T} + m(\boldsymbol{g} - \boldsymbol{a}_{CG}) = m\boldsymbol{a}_{BC}.$$

We define $\boldsymbol{g}_{eff}$ as

$$\boldsymbol{g}_{eff} = \boldsymbol{g} - \boldsymbol{a}_{CG}.$$

Then we have

$$\boldsymbol{T} + m\boldsymbol{g}_{eff} = m\boldsymbol{a}_{BC}$$

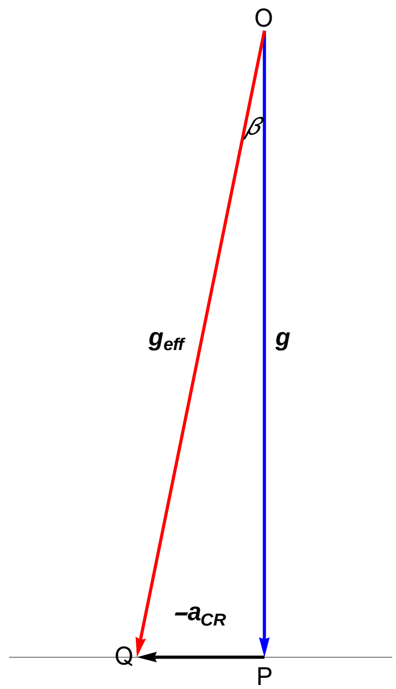


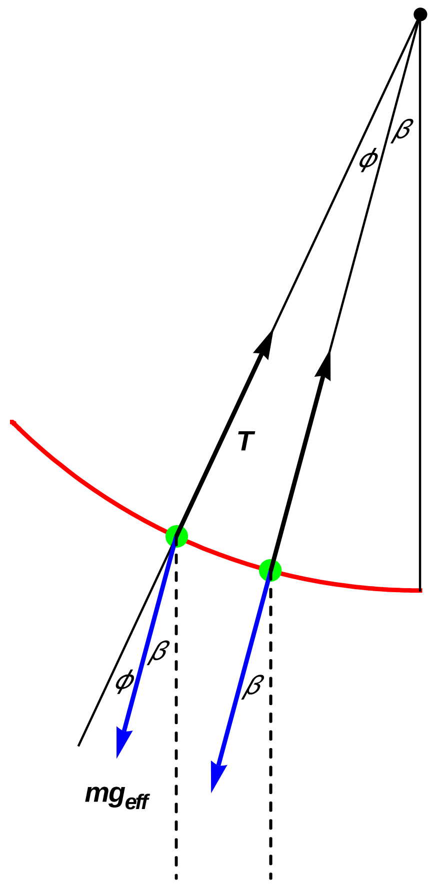
Fig. $\boldsymbol{g}_{eff} = \boldsymbol{g} - \boldsymbol{a}_{CG}$. $g_{eff} = \sqrt{g^2 + a_{CG}{}^2}$. $\angle QOP = \beta$.

The magnitude of g_{eff} is obtained as

$$g_{eff} = \sqrt{g^2 + a_{CG}{}^2}$$

where

$$\tan\beta = \frac{a_{CG}}{g}$$



An equation for the motion of the bob draw the free-body diagram is given by

$$ml\ddot{\phi} = -mg_{eff}\sin\phi$$

where l is the length of the string. This can be written as

$$\ddot{\phi} = -\frac{g_{eff}}{l}\sin\phi \approx -\frac{g_{eff}}{l}\phi = -\omega^2\phi$$

where

$$\omega^2 = \frac{g_{eff}}{l}$$

Then the period T is obtained as

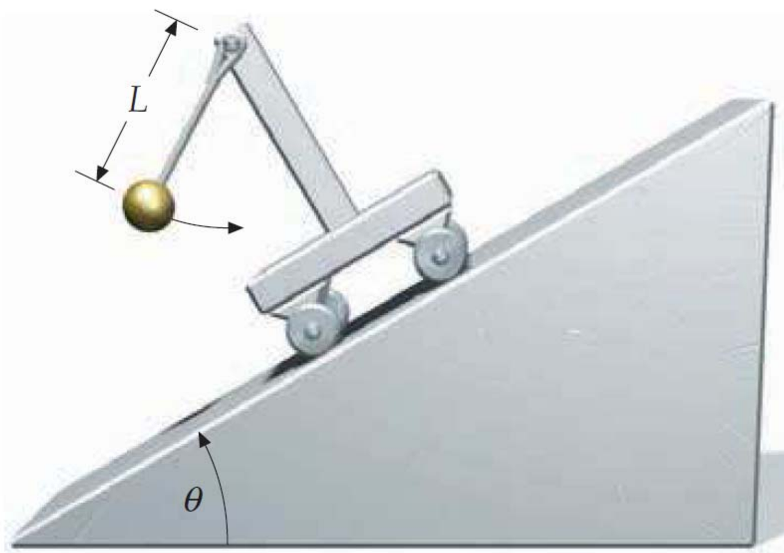
$$T = 2\pi\sqrt{\frac{l}{g_{eff}}}$$

with

$$g_{eff} = \sqrt{g^2 + a_{CG}{}^2}$$

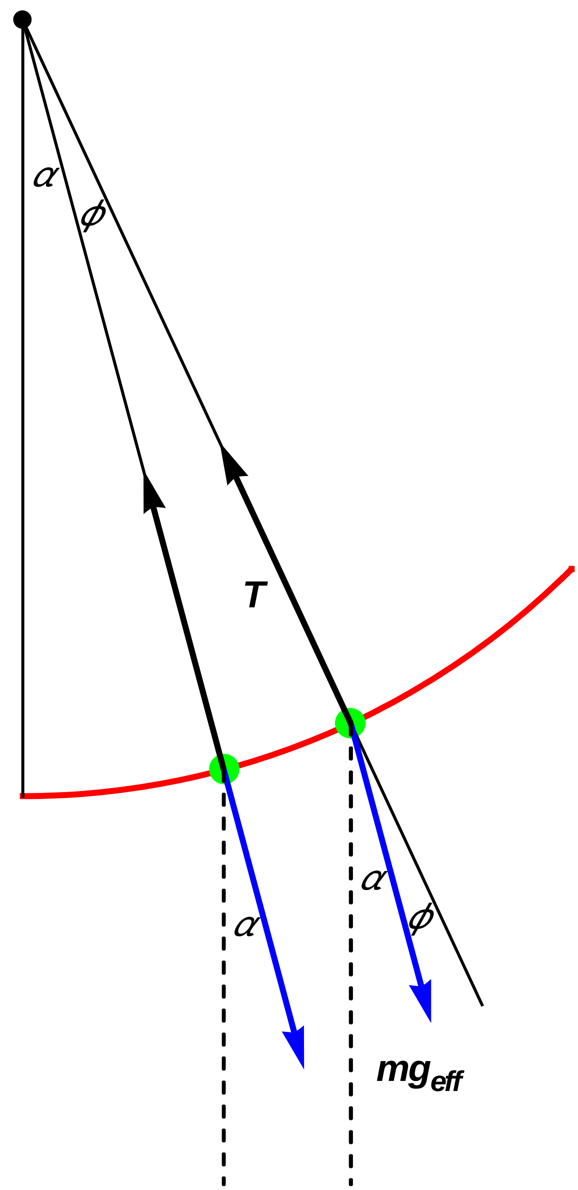
((Example-2))

Simple pendulum on the moving cart (with constant acceleration) on the ramp



A simple pendulum of length *L* is attached to a massive cart that slides without friction down a plane inclined at angle $\theta = \alpha$ with the horizontal, as shown in Figure. Find the period of oscillation for small oscillations of this pendulum
(P.T. Tipler and G. Mosca, Physics for Scientists and Engineers, 6-th edition W.H. Freeman and Company, 2008) Problem 14-65 p.491

The cart accelerates down the ramp with a constant acceleration of $g \sin\alpha$. This happens because the cart is much more massive than the bob, so the motion of the cart is unaffected by the motion of the bob oscillating back and forth. The path of the bob is quite complex in the reference frame of the ramp, but in the reference frame moving with the cart the path of the bob is much simpler—in this frame the bob moves back and forth along a circular arc. To solve this problem we first apply Newton's second law (to the bob) in the inertial reference frame of the ramp. Then we transform to the reference frame moving with the cart in order to exploit the simplicity of the motion in that frame.



Apply Newton's 2nd law to the bob, labeling the acceleration of the bob (B) relative to the ramp (R), $\boldsymbol{a}_{BR}$

$$\boldsymbol{T} + m\boldsymbol{g} = m\boldsymbol{a}_{BR}$$

The acceleration of the bob relative to the ramp is equal to the acceleration of the bob relative to the cart plus the acceleration of the cart relative to the ramp,

$$\boldsymbol{a}_{BR} = \boldsymbol{a}_{BC} + \boldsymbol{a}_{CR}$$

Then we get

$$\boldsymbol{T} + m\boldsymbol{g} = m(\boldsymbol{a}_{BC} + \boldsymbol{a}_{CR})$$

or

$$\boldsymbol{T} + m(\boldsymbol{g} - \boldsymbol{a}_{CR}) = m\boldsymbol{a}_{BC}$$

We define $\boldsymbol{g}_{eff}$ as

$$g_{eff} = \boldsymbol{g} - \boldsymbol{a}_{cR}$$

Then we have

$$\boldsymbol{T} + m\boldsymbol{g}_{eff} = m\boldsymbol{a}_{BC}\,.$$

The magnitude of $\boldsymbol{g}_{\mathrm{eff}}$ is obtained as

$$g_{eff} = g\cos(\alpha),$$

from the geometry of the figure below.

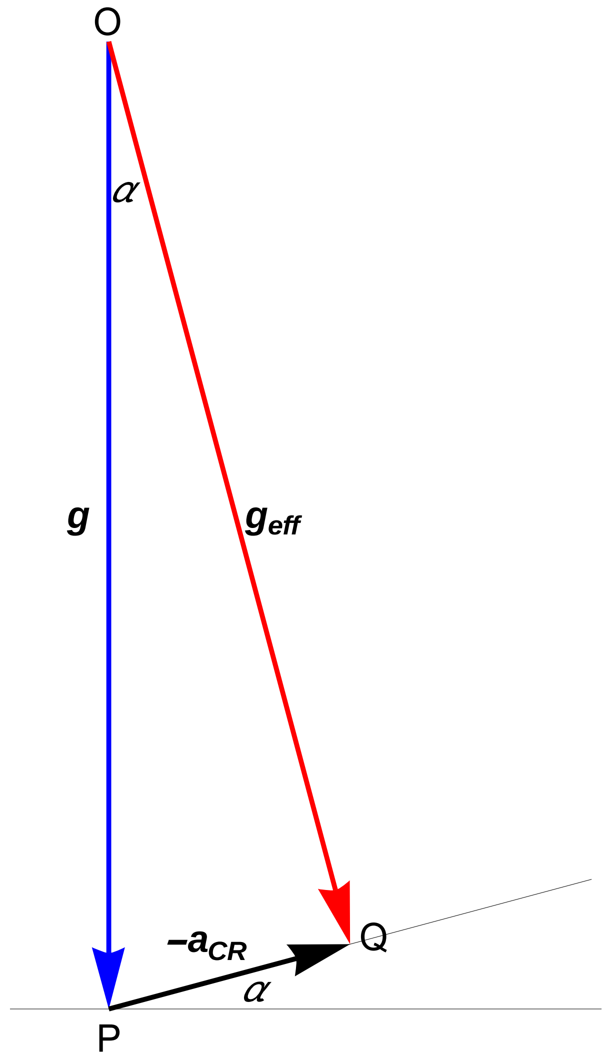


Fig. $\boldsymbol{g}_{eff} = \boldsymbol{g} - \boldsymbol{a}_{CR}$. $\overline{PQ} = |\boldsymbol{a}_{CR}| = g\sin\alpha$. $\overline{OQ} = g_{eff} = g\cos\alpha$. $\angle OQP = 90°$.

Suppose that $\boldsymbol{a}_{BC} = 0$

$$\boldsymbol{T} = -m\boldsymbol{g}_{eff}$$

An equation for the motion of the bob draw the free-body diagram is given by

$$ml\ddot{\phi} = -mg_{eff}\sin\phi ,$$

where l is the length of the string. This can be written as

$$\ddot{\phi} = -\frac{g_{eff}}{l}\sin\phi \approx -\frac{g\cos\alpha}{l}\phi = -\omega^2\phi ,$$

where

$$\omega^2 = \frac{g\cos\alpha}{l} .$$

Then the period T is obtained as

$$T = 2\pi\sqrt{\frac{l}{g_{eff}}} = 2\pi\sqrt{\frac{l}{g\cos\alpha}} .$$

When $\alpha = 0$, T is equal to the conventional value of T,

$$T = 2\pi\sqrt{\frac{l}{g}} .$$

15 Link

Tacoma Narrows Bridge collapse (forced oscillation)
http://www.youtube.com/watch?v=j-zczJXSxnw

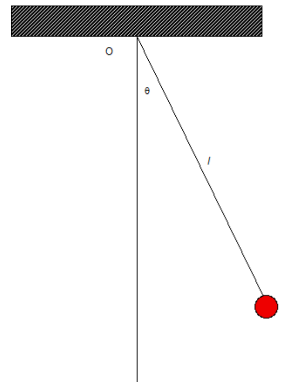
Simple harmonics motion Wikipedia
http://en.wikipedia.org/wiki/Simple_harmonic_motion

Simple pendulum:
Physics of simple pendulum, a case study of nonlinear dynamics
http://physics.binghamton.edu/Sei_Suzuki/suzuki.html

Lecture Note (University of Rochester)
http://teacher.pas.rochester.edu/phy121/LectureNotes/Contents.html

Appendix Nonlinear oscillation (Challenging topics)

A.1 Formulation



We consider the motion of mass *m* hanging from a ceiling with a string. The mass of the string is neglected. We set up an equation of motion,

$$I\ddot{\theta} = -mgl\sin\theta \qquad (1)$$

where *I* is the moment of inertia and is given by

$$I = ml^2$$

Equation (1) can be rewritten as

$$\ddot{\theta} + \varepsilon\sin\theta = 0$$

where

$$\varepsilon = \frac{g}{l} = \omega_0{}^2. \qquad \text{or} \qquad \omega_0 = \sqrt{\frac{g}{l}}$$

For small θ,

$$\ddot{\theta} + \omega_0{}^2(\theta - \frac{\theta^3}{6} + \frac{\theta^3}{120} + ...) = 0$$

The solution of this equation is given by the nonlinear terms

$$\theta = \alpha_1\cos(\omega t + \delta) + \alpha_3\cos(3\omega t + \delta) + \alpha_5\cos(5\omega t + \delta) + ...$$

where $|\alpha_1| > |\alpha_2| > |\alpha_3|$ and δ is a phase factor;

$$\omega^2 = \omega_0{}^2(1-\frac{1}{8}\alpha_1{}^2)$$

$$T = \frac{2\pi}{\omega} = \frac{2\pi}{\omega_0}(1+\frac{\alpha^2}{16}) = T_0(1+\frac{\alpha^2}{16})$$

$$\alpha_3 \approx \frac{1}{-9\omega^2+\omega_0{}^2}\frac{1}{24}\omega_0{}^2\alpha_1{}^3 \approx -\frac{\alpha_1{}^3}{192}$$

with

$$T_0 = \frac{2\pi}{\omega_0} = 2\pi\sqrt{\frac{l}{g}}\ .$$

In the limit of $\theta \to 0$,

$$\ddot{\theta} + \omega_0{}^2\theta = 0$$

corresponding to a simple harmonics,

$$\theta = \alpha_1 \cos(\omega_0 t + \delta)$$

A.2 Energy conservation

The multiplication of $\dot{\theta}$ on both sides and integration with respect to t lead to

$$\ddot{\theta}\dot{\theta} + \varepsilon \sin\theta\dot{\theta} = 0$$

or

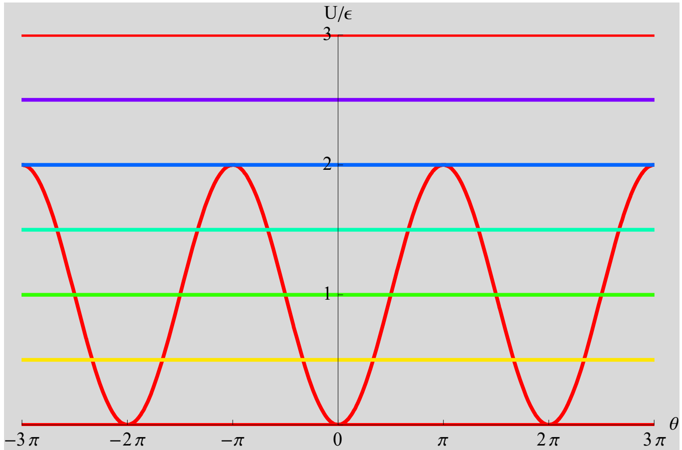
$$\frac{d}{dt}[\frac{1}{2}\dot{\theta}^2 + \varepsilon(1-\cos\theta)] = 0$$

or

$$E = \frac{1}{2}\dot{\theta}^2 + \varepsilon(1-\cos\theta)$$
$$= \frac{1}{2}v^2 + \varepsilon(1-\cos\theta)$$

where the first term is the kinetic energy and the second term is a potential energy defined by

$$U = \varepsilon(1-\cos\theta)$$



For small θ, U can be approximated by

$$U = \varepsilon\left(\frac{\theta^2}{2} - \frac{\theta^4}{24} + \frac{\theta^6}{720} - \frac{\theta^8}{40320} + \ldots\right)$$

Here we use an initial condition

$$\theta(t = 0) = \theta_0$$
$$\dot{\theta}(t = 0) = v_0$$

Then the total energy E is expressed by

$$E = \frac{1}{2}{v_0}^2 + \varepsilon(1 - \cos\theta_0)$$

In other words, E increases as v_0 increases and it also increases as θ_0. increases.

A.3 Phase space of θ vs $v = d\theta/dt$

From

$$v = \dot{\theta} \qquad\qquad \dot{v} = -\varepsilon\sin\theta$$

we have

$$\frac{dv}{d\theta} = \frac{\dot{v}}{\dot{\theta}} = -\frac{\varepsilon}{v}\sin\theta$$

This is a differential equation (separation variable type). So we can solve easily as

$$\int v dv = -\varepsilon \int \sin\theta d\theta$$

$$\frac{1}{2}v^2 - \varepsilon\cos\theta = C$$

From the energy conservation law,

$$E = \frac{1}{2}v^2 + \varepsilon(1-\cos\theta) = C + \varepsilon$$

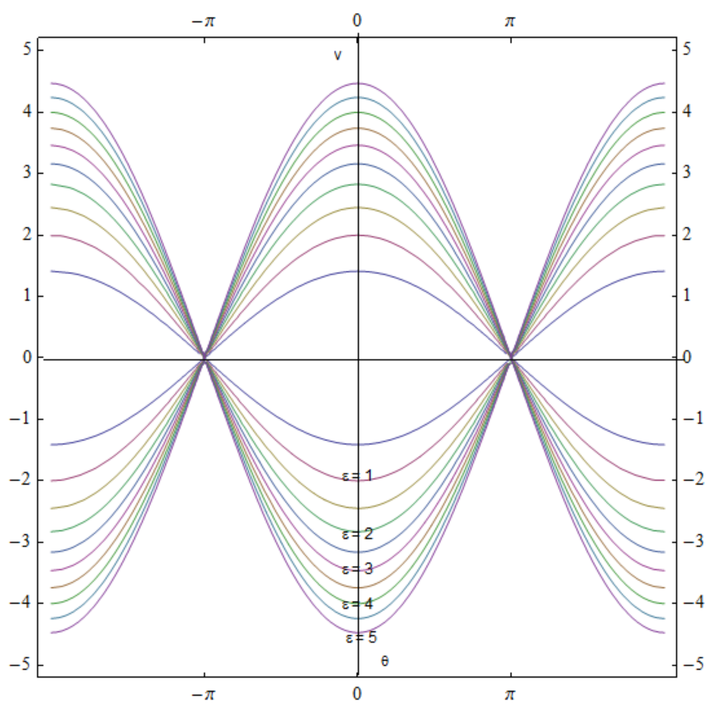
or

$$C = E - \varepsilon$$

(a) ContourPlot of

$$\frac{1}{2}v^2 + \varepsilon(1-\cos\theta) = E$$

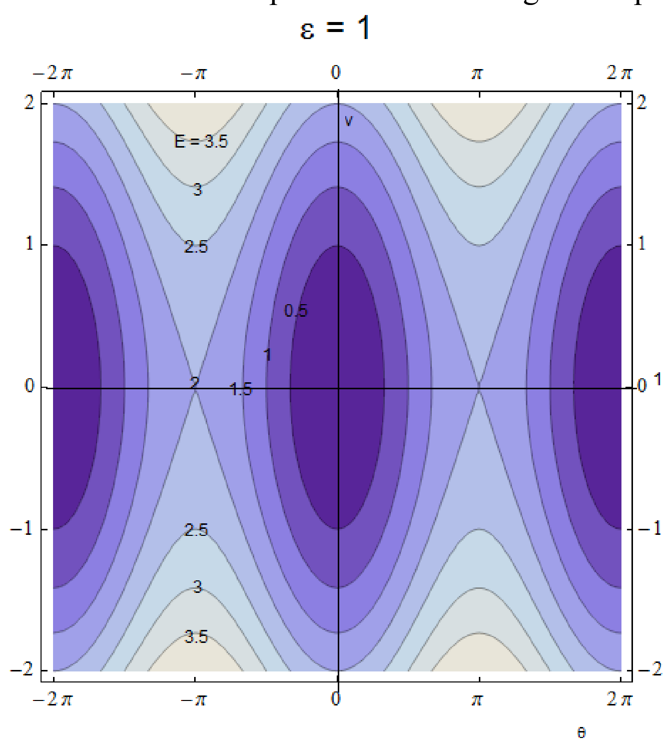
in the θ vs v plane, where $E = 2\varepsilon$ and the parameter ε is varied as a parameter. The character of the orbit inside and outside of the curve are rather different. Such a family of curve is called sepratrices.



(b) ContourPlot of

$$\frac{1}{2}v^2 + \varepsilon(1-\cos\theta) = E$$

in the θ vs v plane, where $\varepsilon = 1$ and the parameter E is changed as a parameter.



Fig, Phase space (v vs θ) for $\varepsilon = 1$. E = 0.5, 1.0, 1.5, 2.0, 2.5, 3.0, and 3.5. $E = 2\varepsilon = 2$ is the highest energy of the potential.

B. Numerical calculation using Mathematica for simple pendulum

B.1 Formulation

For convenience, the differential equation is separated into two parts

$$v = \dot{\theta} \qquad \dot{v} = -\varepsilon \sin\theta$$

1. First we solve this differential equation using numerical method (*Mathematica*, NDSolve)

2. We analyze the Fourier component by using fast Fourier transform (FFT) (*Mathematica* , Fourier).

   a. We choose the N (= 2^n) data, typically the data between 0 and T (= t_{max}).
   b. The minimum time division is $\Delta = T/N$, which corresponds to the maximum ω_{max} =

$$\omega_{max} = \frac{2\pi}{T}N$$

c. The Fourier spectrum is plotted as a function of the channel number scaled by $(2\pi)/T$.

For simplicity, we consider the case of $\theta_0 = 0$. Then we have

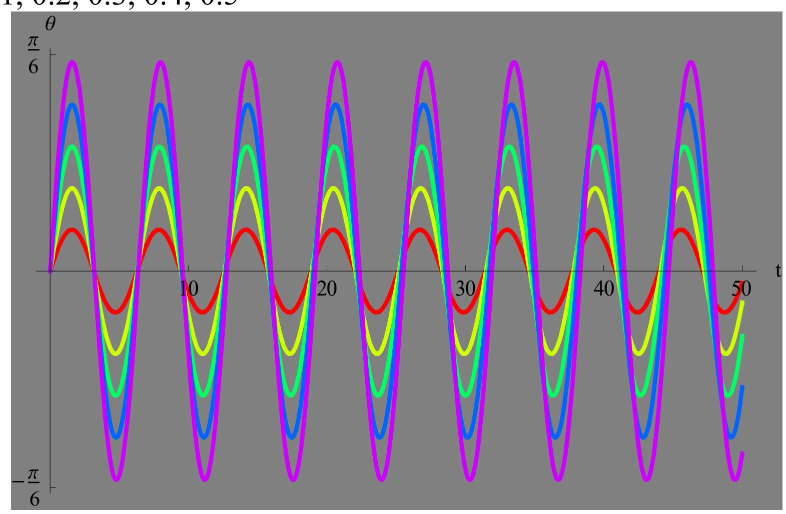
$$E = \frac{1}{2}{v_0}^2$$

When $v_0 < 2\sqrt{\varepsilon}$ (or $E < 2\varepsilon$), the value of θ is limited between $-\pi$ and π. When $v_0 > 2\sqrt{\varepsilon}$ (or $E > 2\varepsilon$), the value of θ is unlimited.

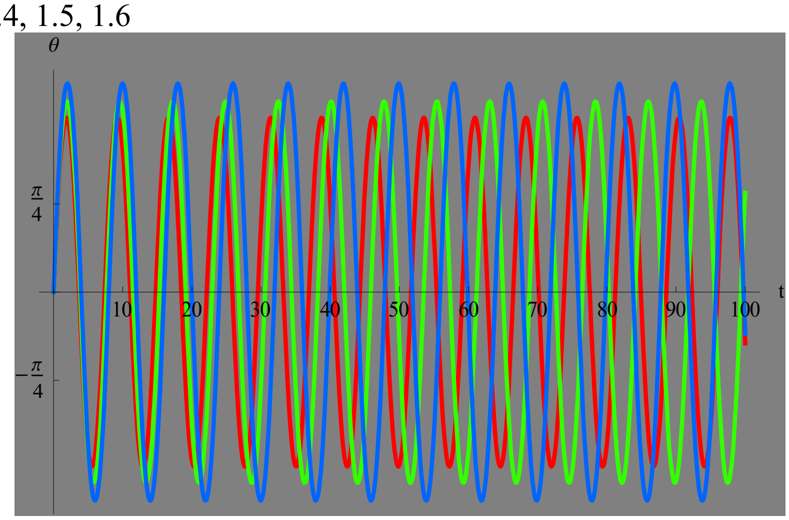
B.2 Time dependence (θ vs t)

We show the time dependence of θ as a function of t with v_0 being changed as a parameter. Here we choose. $\varepsilon = 1$

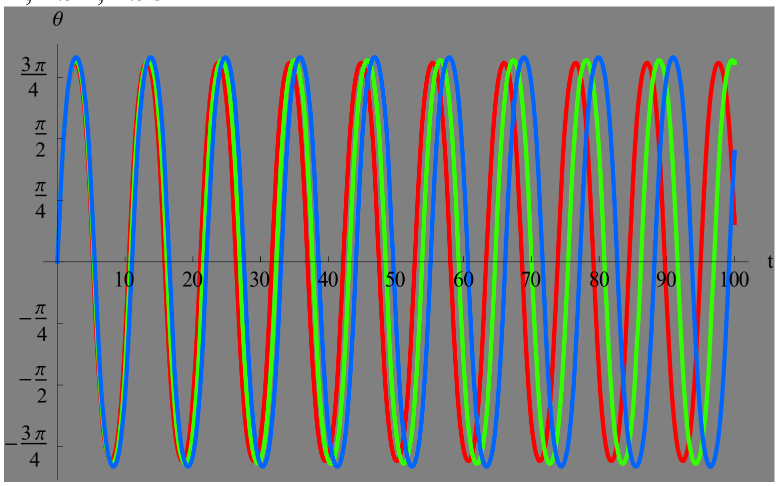
(a) $v_0 = 0.1, 0.2, 0.3, 0.4, 0.5$



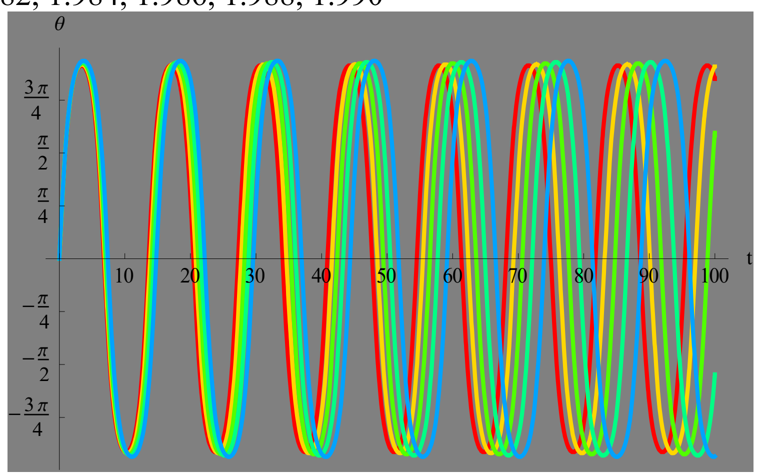
(b) $v_0 = 1.4, 1.5, 1.6$



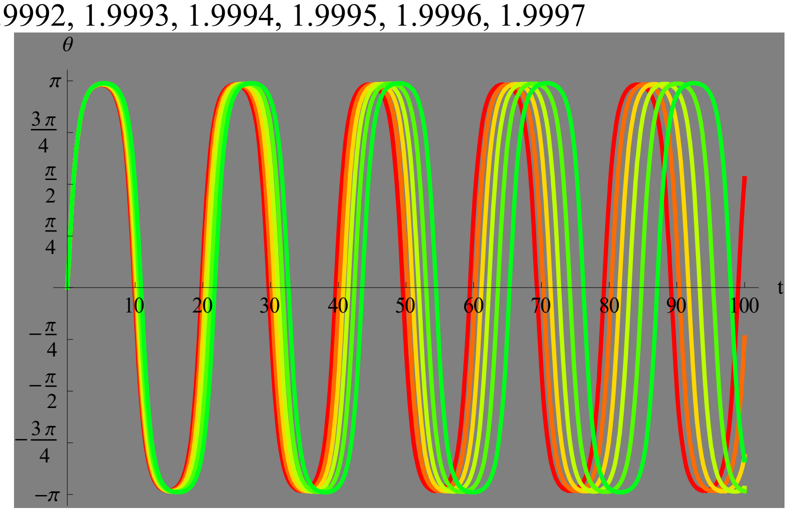
(c) v_0 = 1.91, 1.92, 1.93



(d) v_0 = 1.982, 1.984, 1.986, 1.988, 1.990

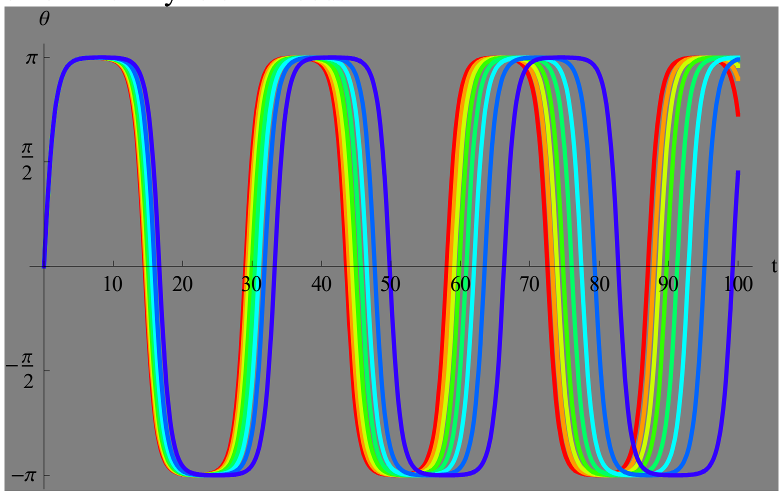


(e) v_0 = 1.9992, 1.9993, 1.9994, 1.9995, 1.9996, 1.9997



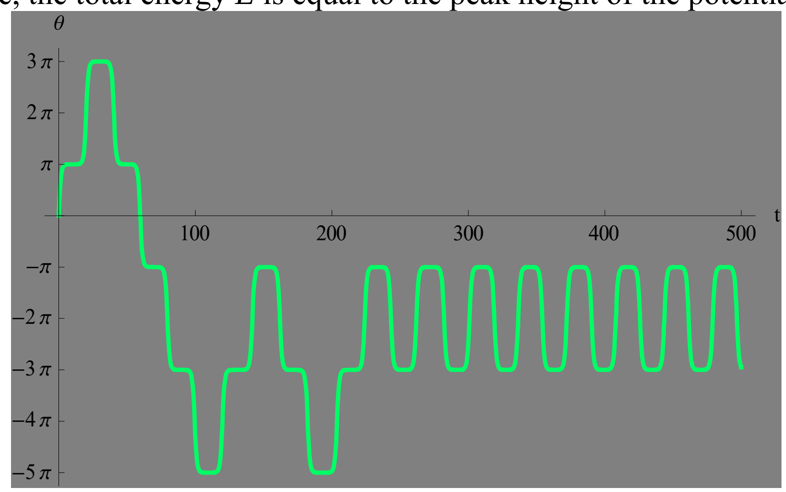
(f) v_0 = 1.999992, 1.999993, 1.999994, 1.999995, 1.999996, 1.999997, 1.999998, 1.999999

As v_0 approaches 2.0, the shape of θ vs t curve becomes square-like. This means that the effect of the nonlinearity is enhanced.



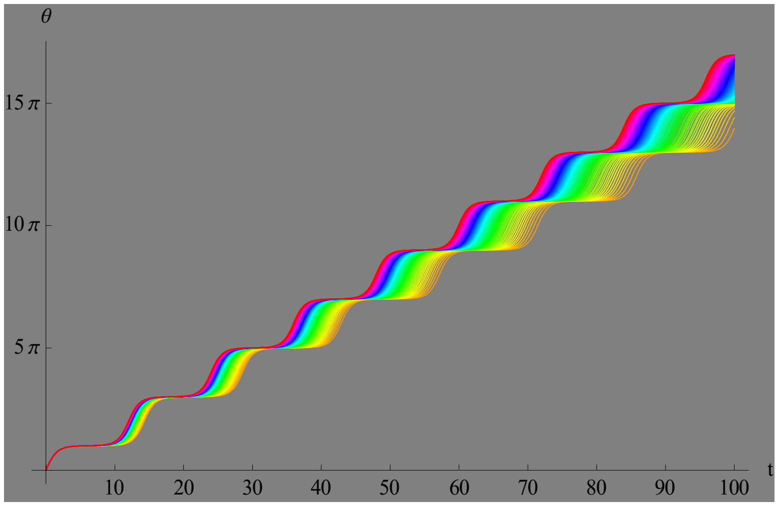
(g) $v_0 = 2$

In this case, the total energy E is equal to the peak height of the potential energy.

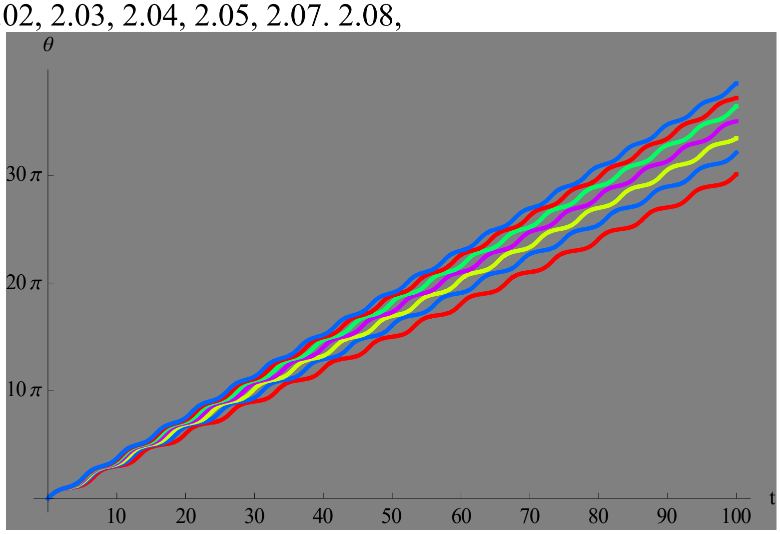


(h) v_0 = 2.00001, 2.00002, 2.00003, 2.00004, 2.0005, 2.0006, 2.0007, 2.00008, 2.00009, 2.0001

For $v_0>2$, the value of θ increases with increasing t. In other words, the pendulum rotates around the rotational axis.



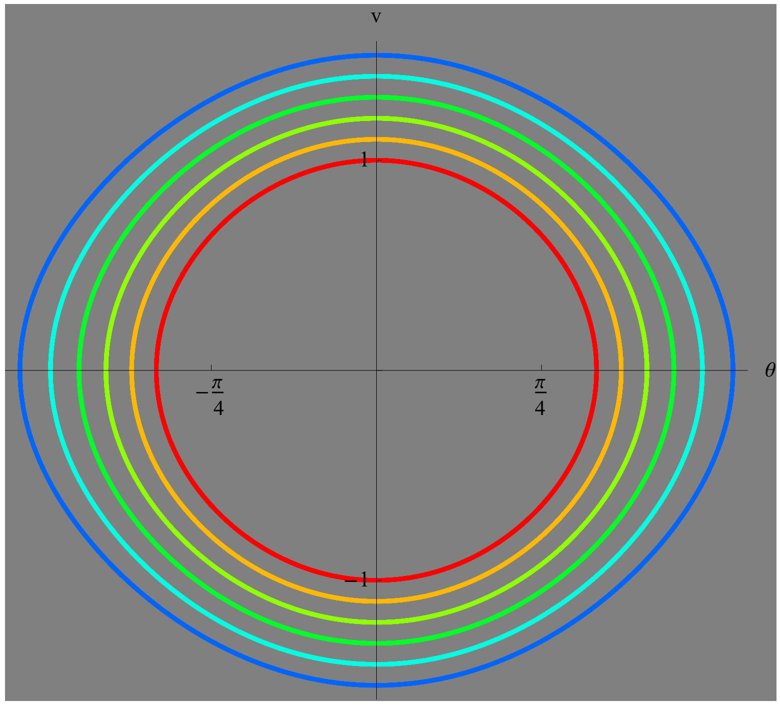
(i) v_0 = 2.02, 2.03, 2.04, 2.05, 2.07. 2.08,



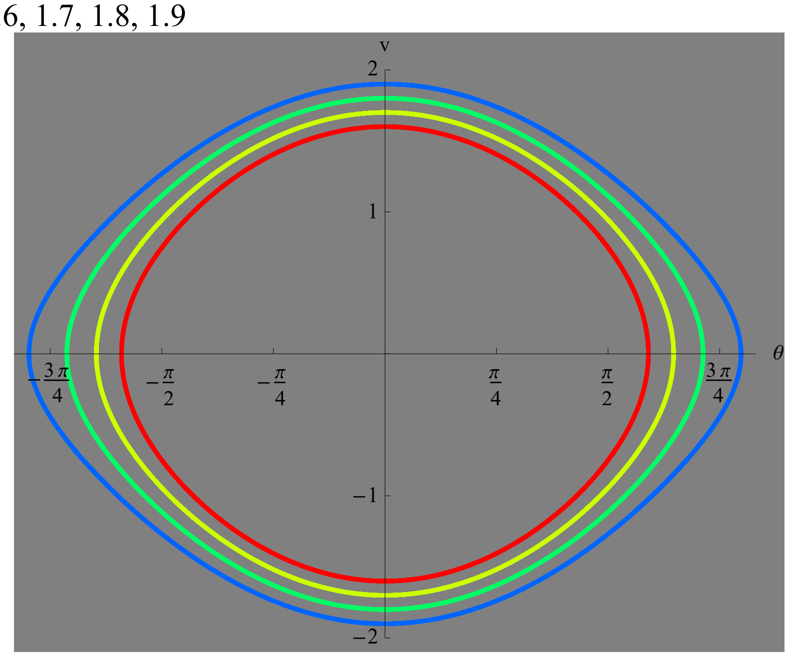
B.3 Phase space (v vs θ)

We make a plot of the phase space for $\varepsilon = 1$, when v_0 is changed as a parameter.

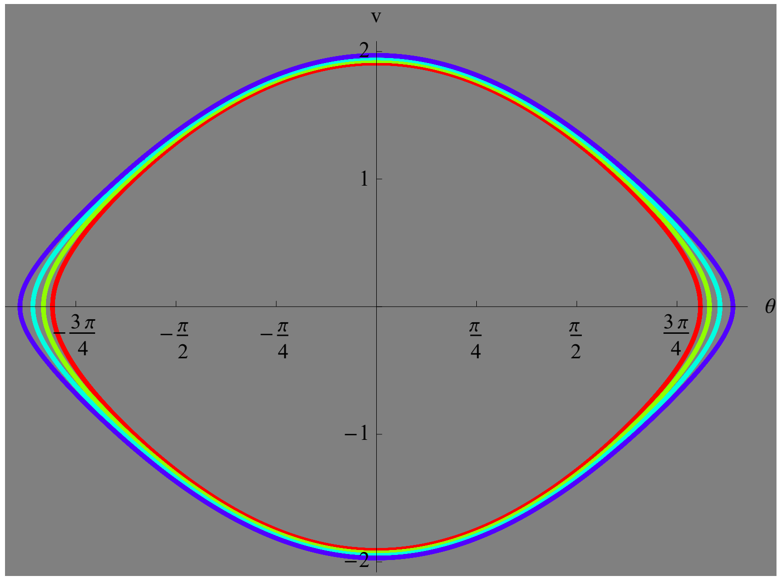
(a) v_0 = 1.0, 1.1, 1.2, 1.3, 1.4, 1.5



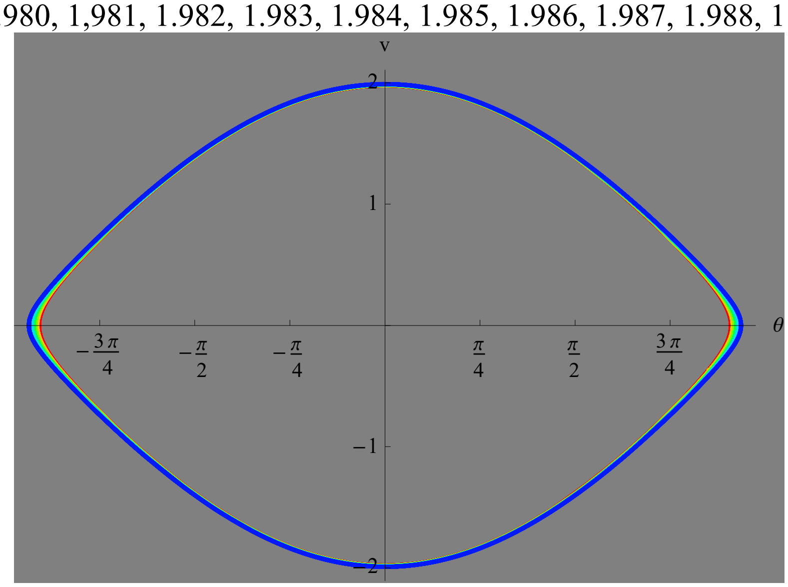
(b) v_0 = 1.6, 1.7, 1.8, 1.9



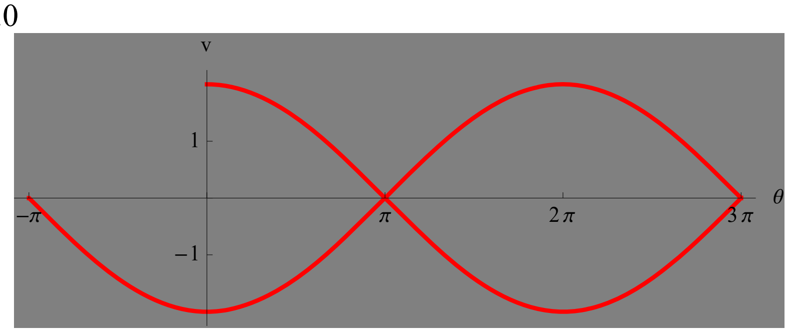
(c) v_0 = 1.91, 1.92, 1.93, 1.94, 1.95, 1.96, 1.97



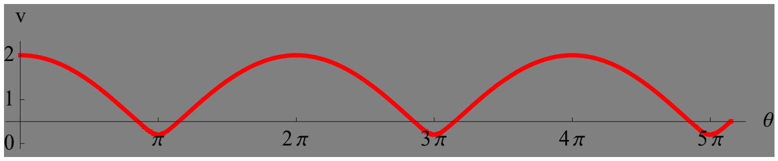
(d) v_0 = 1.980, 1,981, 1.982, 1.983, 1.984, 1.985, 1.986, 1.987, 1.988, 1.989, 1.990.



(e) v_0 = 2.0



(f) v_0 = 2.01



B.4 Fourier analysis using Fast Fourier transform (FFT)

The time dependence of q can be analyzed using the FFT analysis. The Fourier spectrum is the square of th absolute amplitude as a function of angular frequency ω. In the small limit of $v_o = 0$,

$$\omega_0 = \sqrt{\varepsilon} \ .$$

In the present case, $\omega_0 = 1$ since $\varepsilon = 1$.

As v_o approaches $2\sqrt{\varepsilon}$, the time dependence of θ changes from sinusoidal wave to a square wave, indicating the enhancement of nonlinearity in the system. There appear 3ω, 5ω, 7ω, .. components. The frequency of the fundamental mode rapidly decreases.

$\varepsilon = 1$.

(a) $v_0 = 0.5$

ω ($\approx$ 1) component is mainly observed.

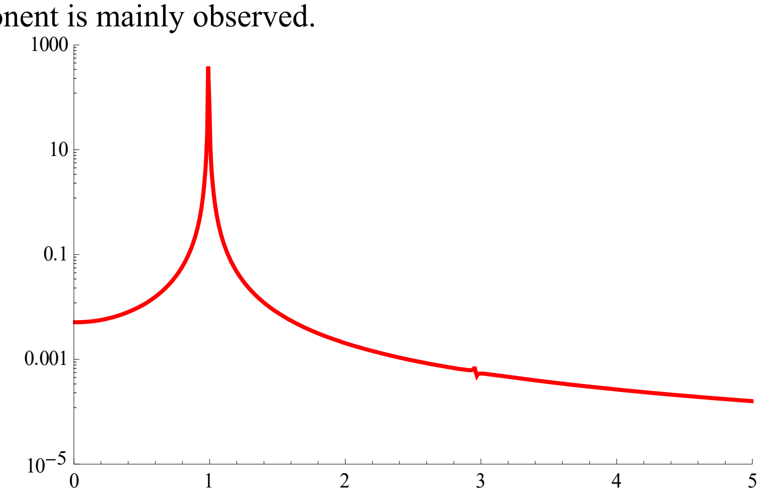
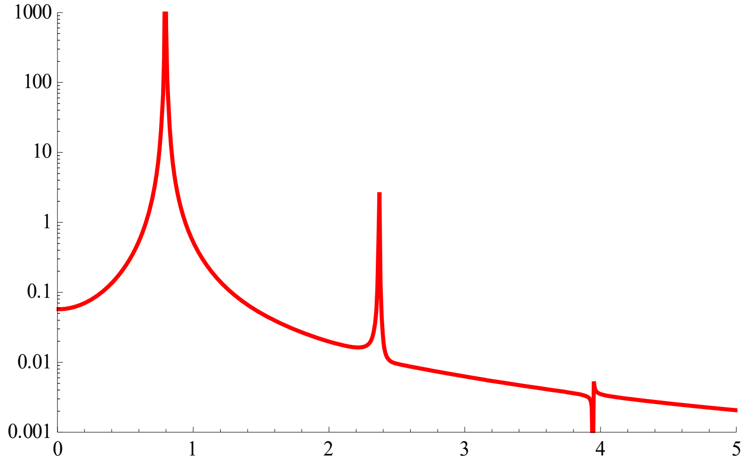


Fig. Fourier spectrum obtained from the FFT analysis of the θ vs t. The x axis is the angular frequency w and the y axis is the squares of the absolute value of complex amplitude. The fundamental harmonic is observed at $\omega = 1$.

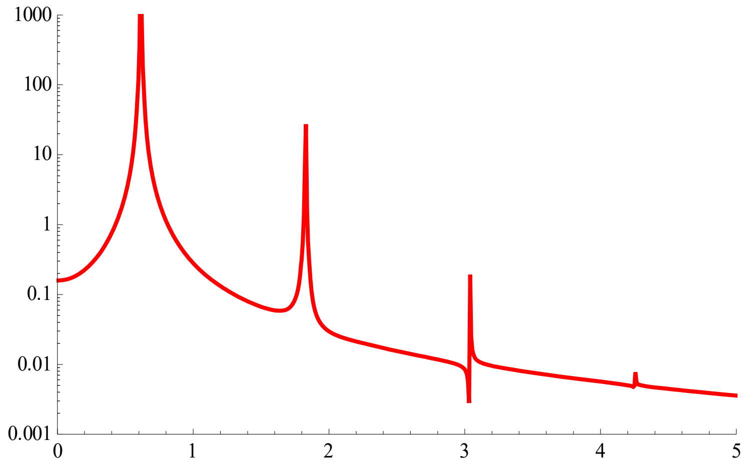
(b) v_0 =1.6

ω and 3ω component are mainly observed.



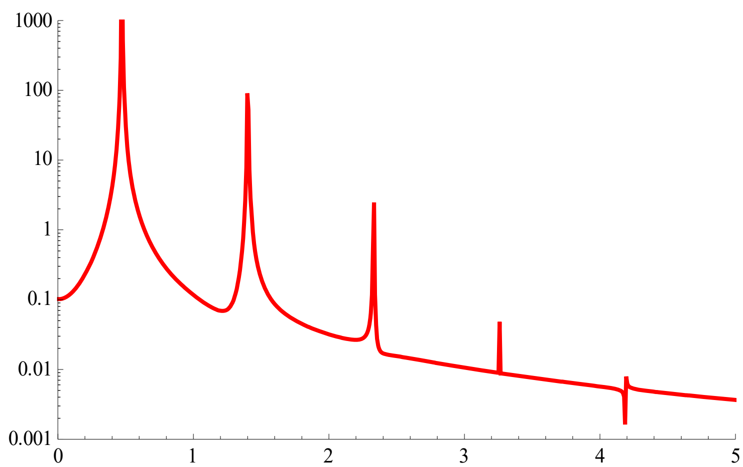
(c) $v_0 = 1.9$

ω, 3ω, and 5ω component are mainly observed.



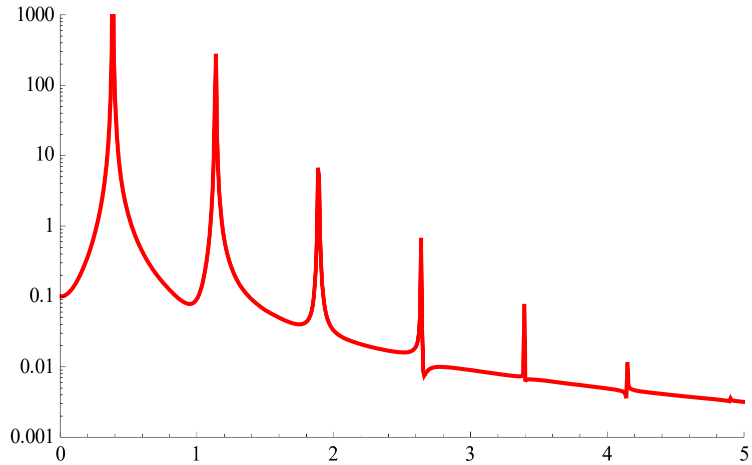
(d) $v_0 = 1.981$

ω, 3ω, 5ω, and 7ω component are mainly observed.



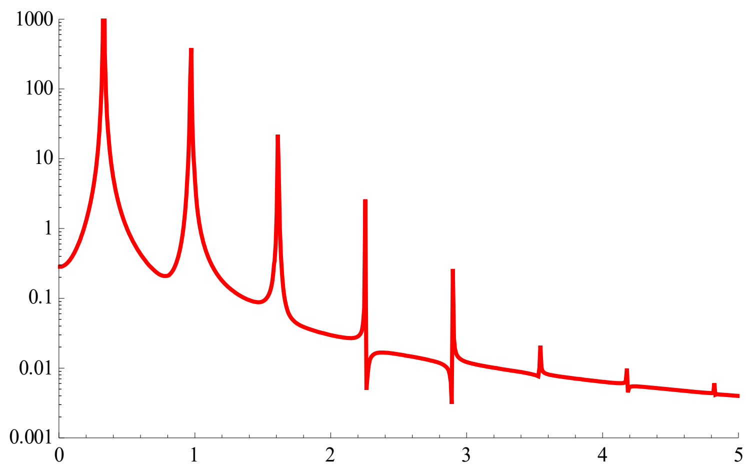
(e) $v_0 = 1.9962$

ω, 3ω, 5ω, 7ω, and 9ω component are mainly observed.



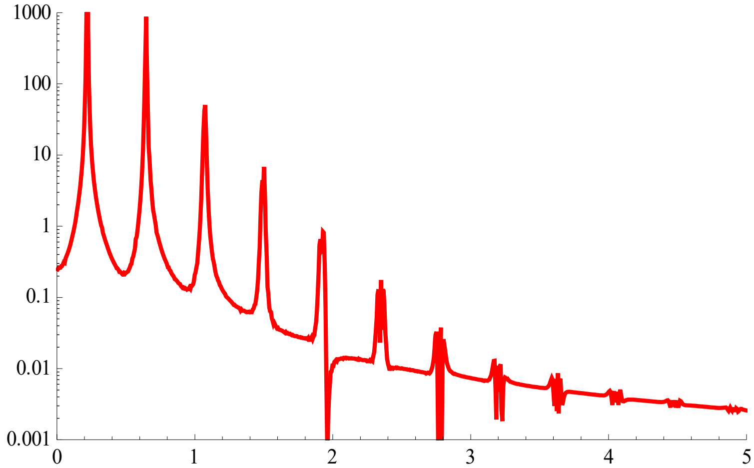
(f) $v_0 = 1.9991$

ω, 3ω, 5ω, 7ω, 9ω, 11ω, and 13ω component are mainly observed.

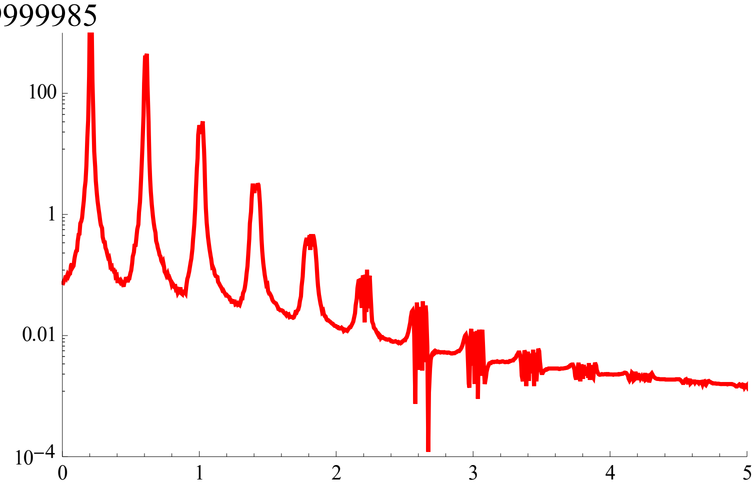


(g) $v_0 = 1.999995$

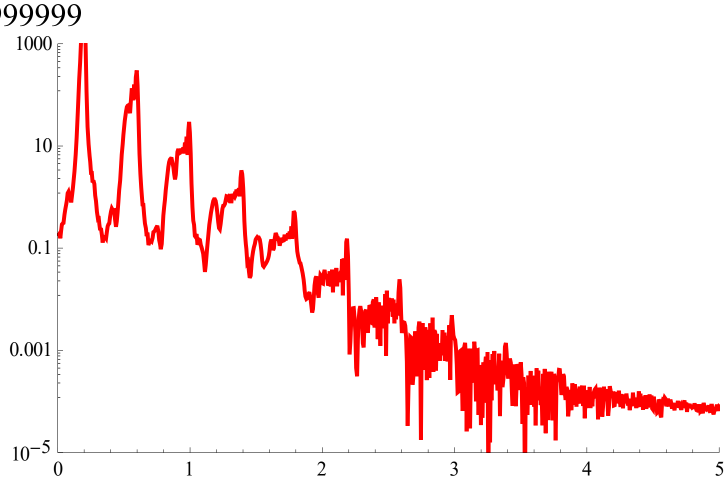
ω, 3ω, 5ω, 7ω, 9ω, 11ω, 13ω, 15ω, and 17ω component are mainly observed.



(h) $v_0 = 1.9999985$

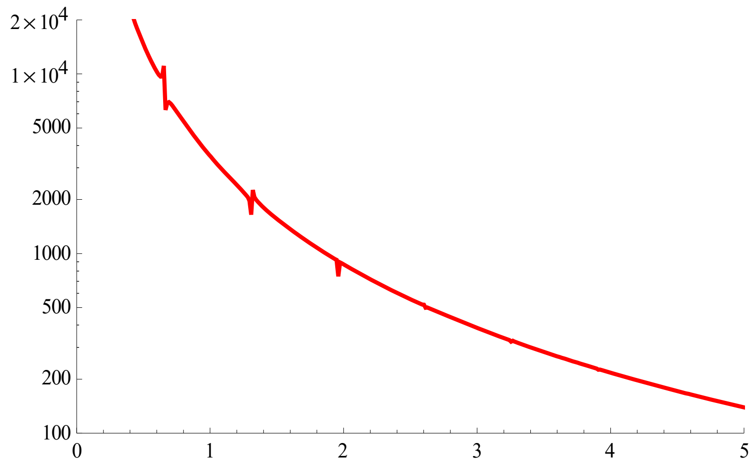


(i) $v_0 = 1.9999999$



The Fourier spectrum drastically changes at $v_0 = 2$. The transition of the behavior occurs from oscillatory to non-oscillatory.

(j) $v_0 = 2.001$



2 × 10^4
1 × 10^4
5000
2000
1000
500
200
100
0
1
2
3
4
5