Chapter 16
Waves-I
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1 Introduction

1.1 Types

There are two main types of waves.

(i) Mechanical waves
Some physical medium is being disturbed.
The wave is the propagation of a disturbance through a medium.

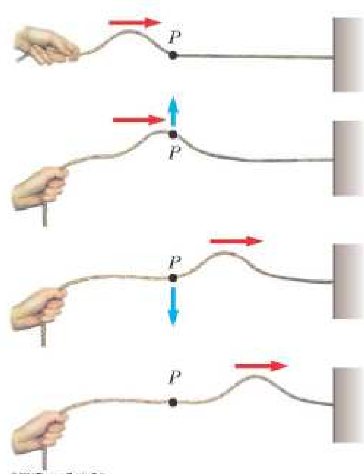
(ii) Electromagnetic waves
No medium is required.
Examples are light, radio waves, x-rays. All electromagnetic waves propagate in vacuum with the same speed $c = 3.0 \times 10^8$ m/s.

(iii) Matter waves (de Broglie wave in quantum mechanics). All microscopic particles such as electrons, protons, neutrons, atoms etc have a wave associated with them governed by Schrödinger's equation.

1.2 General feature of wave

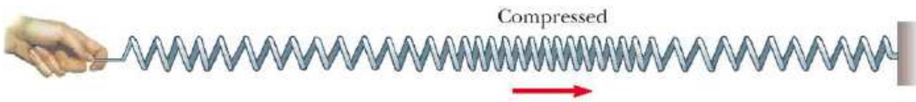
In wave motion, energy is transferred over a distance. Matter is not transferred over a distance. A *disturbance* is transferred through space without an accompanying transfer of *matter*. All waves carry energy. The amount of energy and the mechanism responsible for the transport of the energy differ.

1.3 Transverse wave



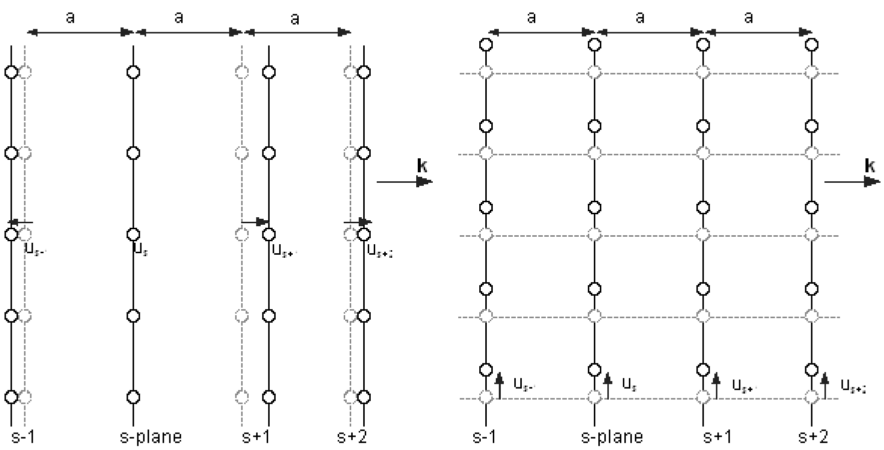
A traveling wave or pulse that causes the elements of the disturbed medium to move *perpendicular* to the direction of propagation is called a transverse wave. The particle motion is shown by the blue arrow, while the direction of the propagation is shown by the red arrow.

1.4 Longitudinal waves



A traveling wave or pulse that causes the elements of the disturbed medium to move parallel to the direction of propagation is called a longitudinal wave. The displacement of the coils is parallel to the propagation. The sound wave is one of examples.

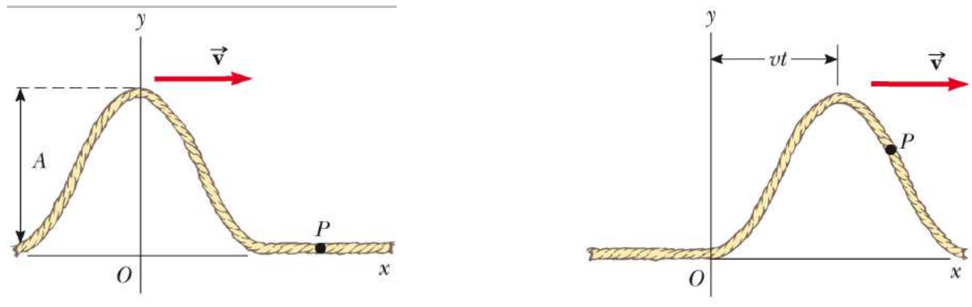
1.5 Phonon in the solid (quantum mechanics of lattice vibration)



In solids, there are both longitudinal waves and transverse waves.

2 Traveling pulse

The shape of the pulse at $t = 0$ is shown. The shape can be represented by $y = f(x)$. This describes the transverse position y of the element of the string located at each value of x at $t = 0$. The speed of the pulse is v. At some time, t, the pulse has traveled a distance vt. The shape of the pulse does not change. The shape of the pulse at t is given by $y = f(x-vt)$.



(a) Pulse at $t = 0$ (b) Pulse at time t

For a pulse traveling to the right,

$$y(x,t) = f(x - vt).$$

For a pulse traveling to the left,

$$y(x,t) = f(x + vt).$$

The function is also called the wave function. The wave function represented the y coordinate of any element located at position x at any time t. The y coordinate is the transverse position. If t is fixed, then the wave function is called the waveform. It defines a curve representing the actual geometric shape of the pulse at that time

((**Example-1**))
We consider the traveling of the Gaussian wave packet.

$$y(x,t) = \frac{1}{\sqrt{2\pi}\sigma} \exp[-\frac{(x - vt)^2}{2\sigma^2}]$$

The Gaussian wave packet propagates along the x axis with the velocity v. The shape of the wave packet remains unchanged.

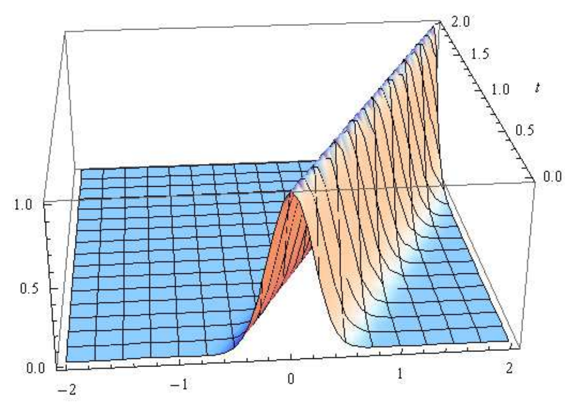


Fig. Plot3D of Gaussian wave packet given by

$$\psi(x,t) = \exp[-\frac{(x-vt)}{2\sigma^2}] \text{ with } v = 1 \text{ and } \sigma = 0.2$$

((**Example-2**)) Propagation of wave packet along the *x* axis.

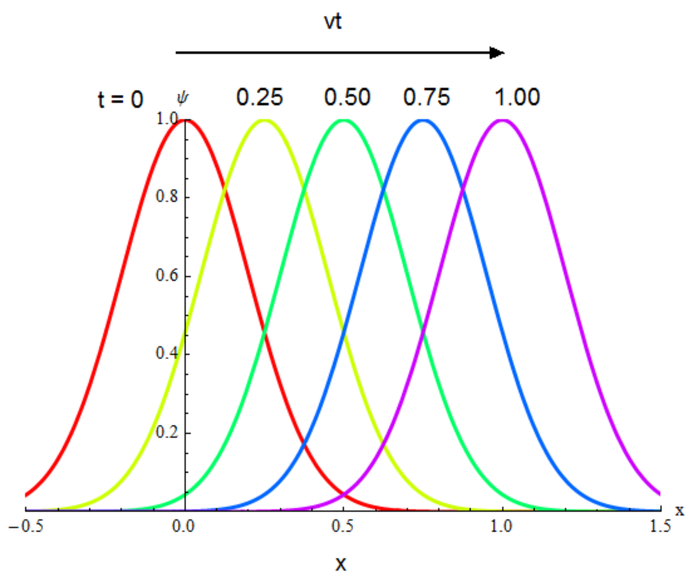


Fig. Plot of Gaussian wave packet

$$\psi(x,t) = \exp[-\frac{(x-vt)}{2\sigma^2}] \text{ with } v = 1 \text{ and } \sigma = 0.2$$

as a function of *x*, where *t* is changed as a parameter, $t = 0 - 1$ with $\Delta t = 0.25$.

((**Note**))

Propagation of the wave packet of the electron (quantum mechanics).

The dispersion relation of the electron is rather different from that of light and sound. The energy is expressed by

$$\hbar\omega = \frac{\hbar^2 k^2}{2m},$$

where $\hbar$ (= $h/2\pi$) is the Dirac's constant and h is the Planck's constant. In this case, the probability of finding the wave packet at (x, t) is described by

$$\left|\psi(x,t)\right|^2 = \frac{1}{\sqrt{\pi}\Delta k} \frac{\exp[\dfrac{-(\Delta k)^2 (x - x_0 - \dfrac{k_0 t\hbar}{m})^2}{1 + \dfrac{t^2 (\Delta k)^4 \hbar^2}{m^2}}]}{\sqrt{\dfrac{1}{(\Delta k)^4} + \dfrac{t^2\hbar^2}{m^2}}}$$

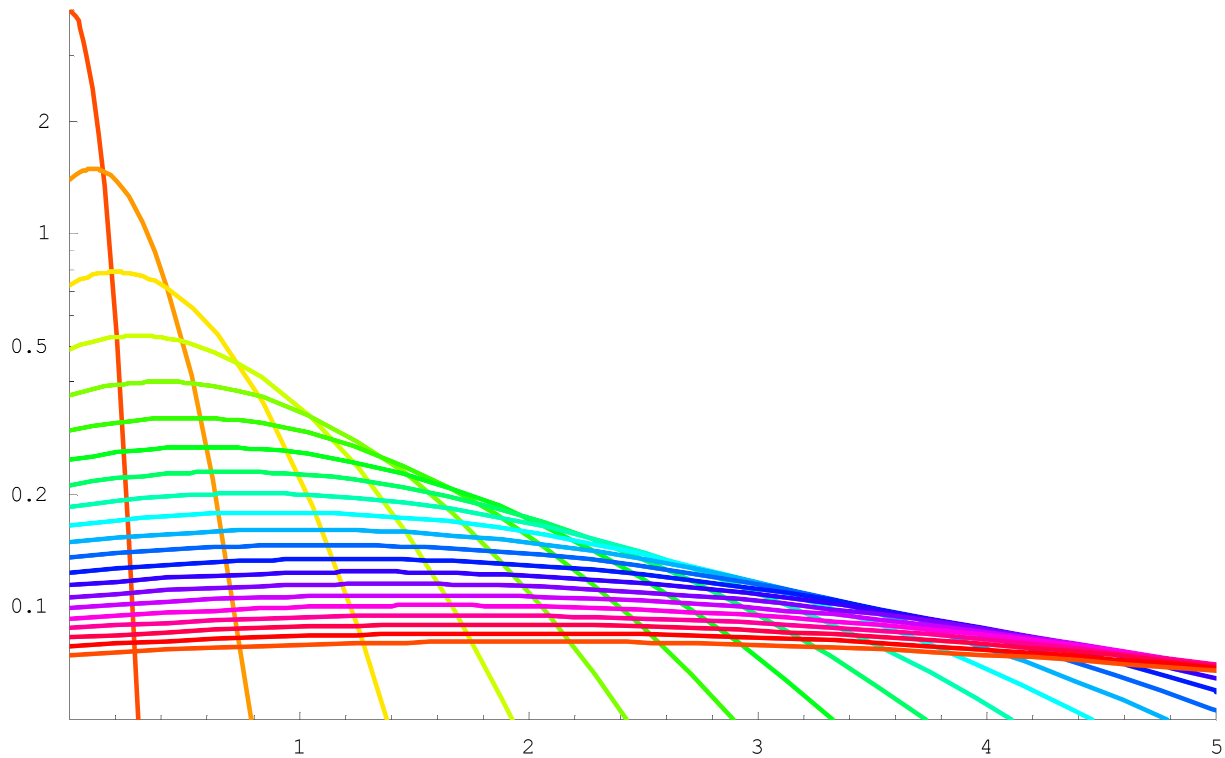


Fig. The plot of $|\psi(x,t)|^2$ as a function of x, where the time t is changed as a parameter.

The position of center:

$$\langle x \rangle = x_0 + \frac{k_0 t\hbar}{m}.$$

The velocity of center

$$\frac{d\langle x \rangle}{dt} = \frac{\hbar k_0}{m} = v_0.$$

The width of the wave packet increases with time t.

$$\Delta x = \frac{1}{\sqrt{2}\Delta k}\sqrt{1 + \frac{t^2\hbar^2}{m^2}(\Delta k)^4},$$

where $\sigma = 1/\Delta k$. The amplitude of the wave packet decreases with time t,

$$A = \frac{\Delta k}{\sqrt{\pi}} \frac{1}{\sqrt{1 + \frac{t^2\hbar^2}{m^2}(\Delta k)^4}}.$$

The evolution of the wave packet is not confined to a simple displacement at a velocity v_0. The wave packet also undergoes a deformation.

The Heisenberg's principle of uncertainty:

$$(\Delta x)(\Delta k) = \frac{1}{\sqrt{2}}\sqrt{1 + \frac{t^2\hbar^2}{m^2}(\Delta k)^4} > \frac{1}{\sqrt{2}}.$$

3 Wave function of the traveling waves

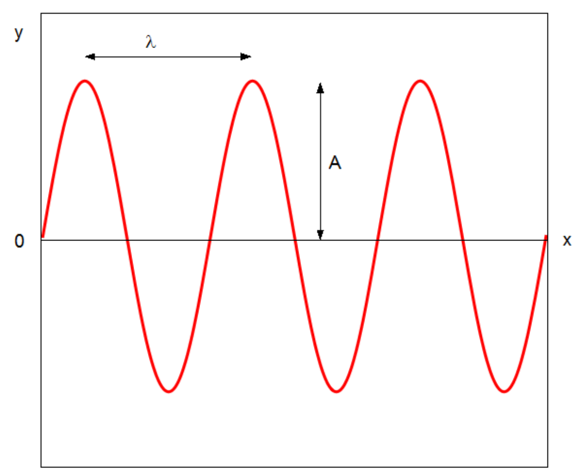
A continuous wave can be created by shaking the end of the string in simple harmonic motion. The shape of the wave is called sinusoidal since the waveform is that of a sine curve. The shape remains the same but moves. We consider the wave function

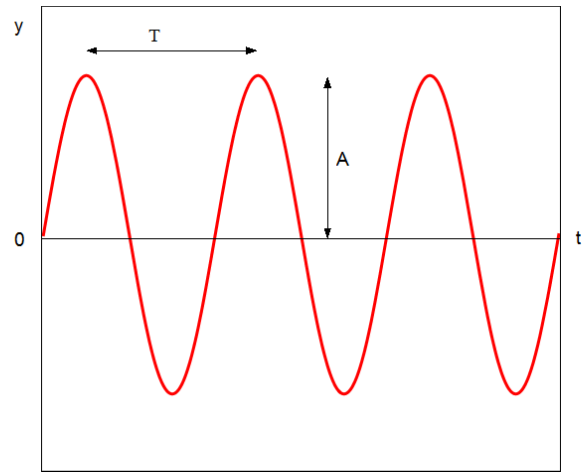
$$y(x,t) = A\sin[k(x - vt)] = A\sin(kx - \omega t),$$

for the travelling wave along the $+x$ direction, where

$$k = 2\pi/\lambda, \qquad \omega = 2\pi f = vk = 2\pi\frac{v}{\lambda},$$

$$v = f\lambda, \qquad T = \frac{1}{f}.$$





Note that $y(x, t)$ is the displacement, A is the amplitude, and $(kx - \omega t)$ is the phase.

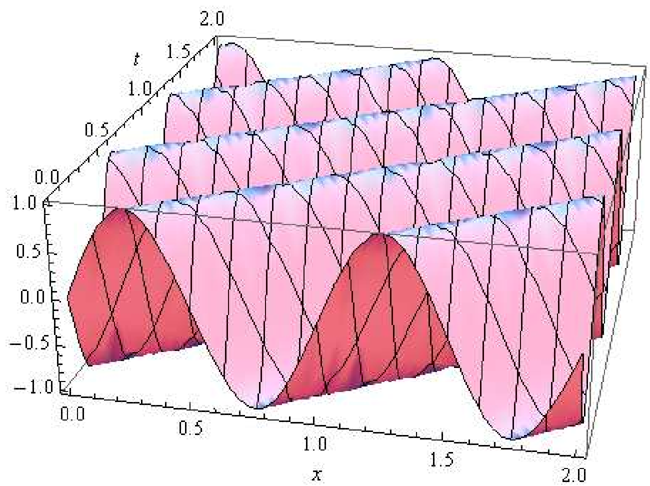


Fig. Plot3D of A sin(kx-ωt) with $A = 1$, $\lambda = 1$, $v = 1.4$.

We also have a wave function given by

$$y(x,t) = A\sin[k(x+vt)]$$

for the travelling wave along the (-x) direction.

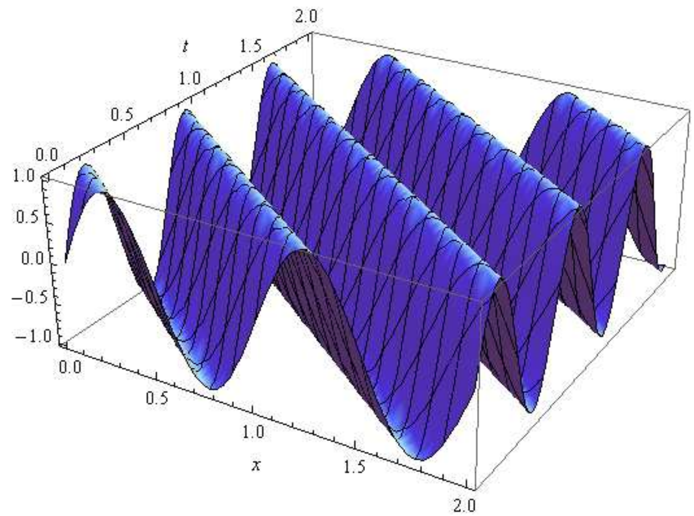


Fig. Plot3D of A sin(kx+ωt) with A = 1, λ = 1, v = 1.4.

4 The speed of a traveling wave

The displacement $y(x, t)$ must remain constant when the phase factor is constant.

$$\varphi = kx - \omega t = \text{constant}$$

We take the derivative of this equation, getting

$$0 = k\frac{dx}{dt} - \omega$$
$$\frac{dx}{dt} = \frac{\omega}{k} = v$$

or

$$\omega = vk\ . \qquad \text{(so- called dispersion relation)}$$

5 Wave equation

First we calculate the following from

$$y(x,t) = A\sin[k(x - vt)] = A\sin(kx - \omega t)\,,$$

$$\frac{\partial y}{\partial x} = Ak\cos(kx - \omega t) \qquad \frac{\partial y}{\partial t} = -A\omega\cos(kx - \omega t)$$
$$\frac{\partial^2 y}{\partial x^2} = -Ak^2\sin(kx - \omega t), \qquad \frac{\partial^2 y}{\partial t^2} = -A\omega^2\sin(kx - \omega t)$$

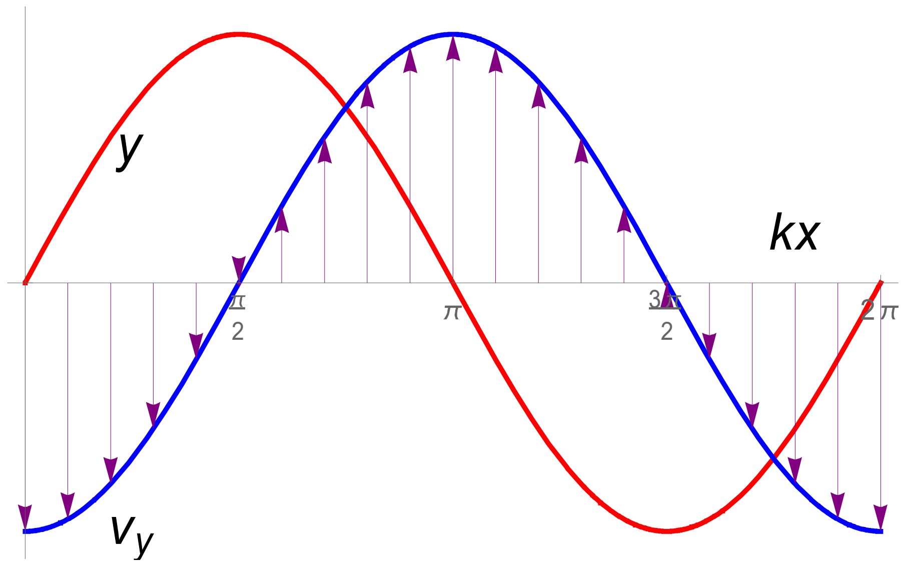


Fig. Plot of y and v_y at $t = 0$, as a function of kx.

From these equations, we get

$$\frac{\partial^2 y}{\partial x^2} = \frac{k^2}{\omega^2}\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}.$$

In general, the wave function satisfies the wave equation given by

$$\boxed{\frac{\partial^2 \psi(x,t)}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 \psi(x,t)}{\partial t^2}}.$$

This applies in general to various types of traveling waves. ψ represents various positions. For a string, it is the vertical displacement of the elements of the string. For a sound wave, it is the longitudinal position of the elements from the equilibrium position. For electromagnetic waves, it is the electric or magnetic field components.

The solution of this wave equation is as follows.

$$\frac{\partial^2}{\partial t^2}\psi = v^2\frac{\partial^2}{\partial x^2}\psi$$

$$\left(\frac{\partial}{\partial t} - v\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} + v\frac{\partial}{\partial x}\right)f = 0$$

We introduce new variables

$$\xi = t - \frac{x}{v}$$

$$\eta = t + \frac{x}{v}$$

So that the equation for ψ becomes

$$\frac{\partial^2 \psi}{\partial \xi \partial \eta} = 0 .$$

The solution obviously has the form

$$\psi = f_1(\xi) + f_2(\eta)$$

where f_1 and f_2 are arbitrary function.

or

$$\psi = f_1(t - \frac{x}{v}) + f_2(t + \frac{x}{v})$$

The function f_1 represents a plane wave moving in the positive direction along the x axis. The function f_2 represents a plane wave moving in the negative direction along the x axis.

((**Another method**)) using the Fourier transform

We use the Fourier transformation technique.

$$\Psi(x,\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega t} \psi(x,t) dt$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} \Psi(x,\omega) d\omega$$

$$\frac{\partial^2}{\partial x^2}\psi - \frac{1}{v^2}\psi = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} [\frac{\partial^2}{\partial x^2}\Psi(x,\omega) + \frac{\omega^2}{v^2}\Psi(x,\omega)] d\omega = 0$$

Then we have

$$\frac{\partial^2}{\partial x^2}\Psi(x,\omega) + k^2\Psi(x,\omega) = 0$$

where $k = \dfrac{\omega}{v}$ (the dispersion relation)

The solution of this equation is

$$\Psi(x,\omega) = g(\omega)\frac{e^{\pm ikx}}{\sqrt{2\pi}} = g(\omega)\frac{e^{\pm i\frac{\omega x}{v}}}{\sqrt{2\pi}}$$

where $g(\omega)$ is an arbitrary function of ω. Finally, we get

$$\psi(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} g(\omega)e^{i\omega(t\pm\frac{x}{v})}d\omega$$

This is an arbitrary function of $(x \pm vt)$.

6 Wave traveling in the string (transverse wave)

6.1 Simple model: the speed of waves on strings

We consider a string symmetrical pulse moving from left to right along a string with speed v. We consider a reference frame, in which pulse remains stationary.

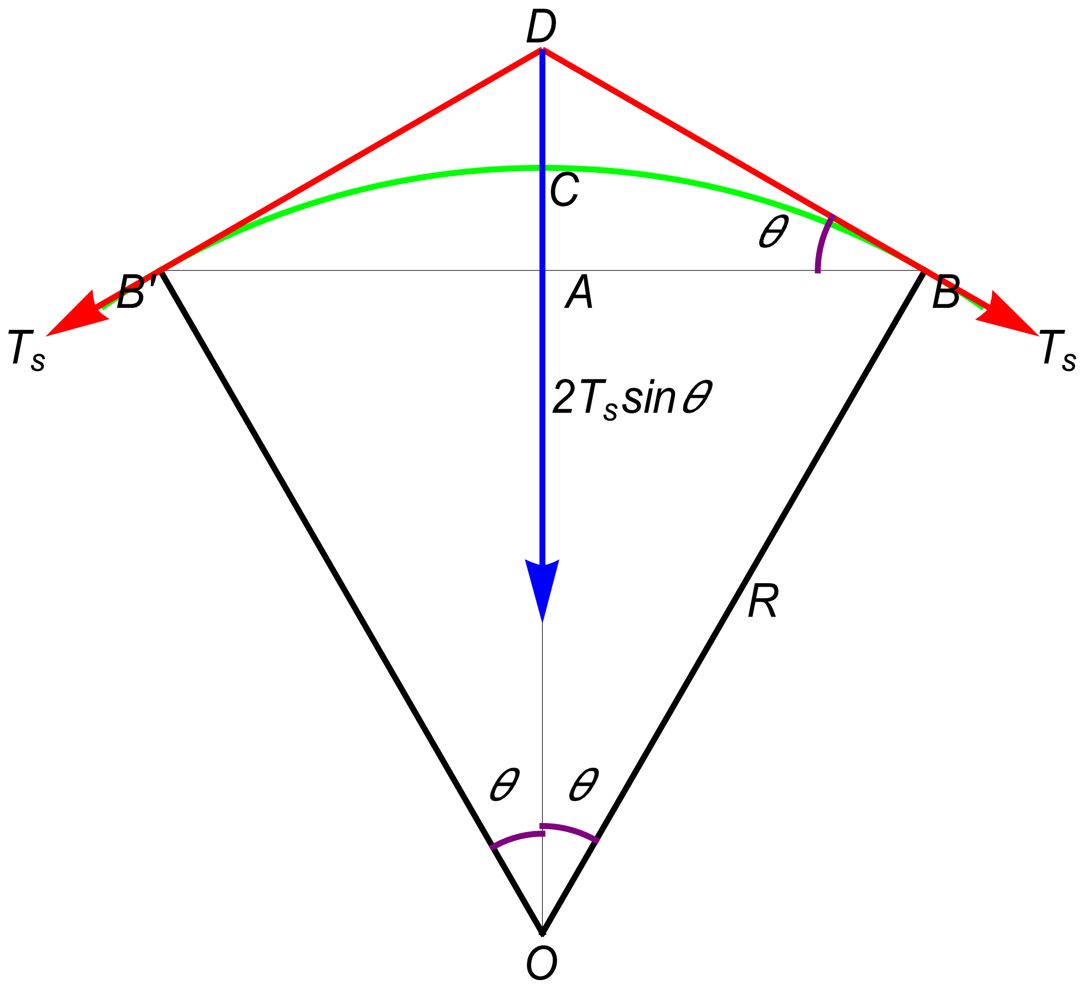
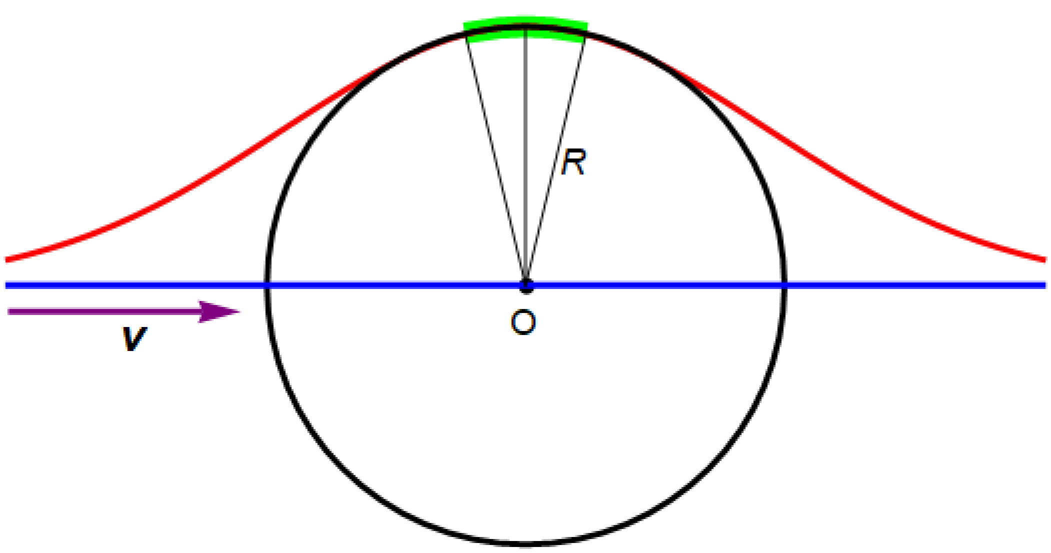


Fig. The element of string (Δs) under the tension $T = T_s$. $\overline{OC} = \overline{OB} = R$. The element of string is denoted by thick green line. $\angle \mathrm{AOB} = \theta$. $\angle \mathrm{CBA} = \theta / 2$. $\angle \mathrm{ABD} = \theta$.



We consider one small string element of length Δs. The net force acting in the y direction (vertical line, toward to the origin, centripetal force) is

$$F_y = 2T_s \sin\theta \approx 2T_s\theta .$$

Note that $\mu\Delta s$ is the mass of the element and that Δs is equal to $2R\theta$. At the moment, the string element Δs is moving in an arc of circle. We apply the Newton's second law to this element (centripetal force).

$$\mu\Delta s\frac{v^2}{R} = F_y = 2T_s\theta, \quad \text{or} \qquad \mu 2R\theta\frac{v^2}{R} = F_y = 2T_s\theta$$

The velocity is obtained as

$$\boxed{v = \sqrt{\frac{T_s}{\mu}}} \qquad \text{(m/s)}$$

where tension: T_s (N=kg m/s^2) and μ (kg/m).

6.2 More general case

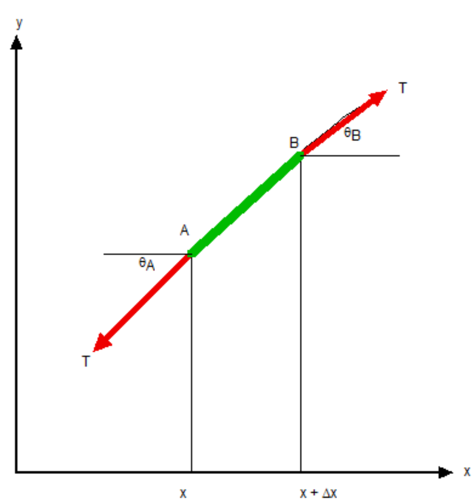


Fig. A snapshot of s travelling wave on a string at time t. Tension T_s on the string (denoted by thick green line).

Suppose that a traveling wave is propagating along a string that is under a tension T_s. Let us consider one small element of length Δx. The ends of the element make small angle θ_A and θ_B with the x axis. The net force acting on the element along the y-axis is

$$\sum F_y = T_s \sin\theta_B - T_s \sin\theta_A$$
$$= T_s(\sin\theta_B - \sin\theta_A) \approx T_s(\tan\theta_B - \tan\theta_A)$$

or

$$F_y = T_s[\left(\frac{\partial y}{\partial x}\right)_B - \left(\frac{\partial y}{\partial x}\right)_A]$$
$$= T_s[\left(\frac{\partial y}{\partial x}\right)_{x+dx} - \left(\frac{\partial y}{\partial x}\right)_x] = \Delta x T_s \frac{\partial}{\partial x}\left(\frac{\partial y}{\partial x}\right) = \Delta x T_s \frac{\partial^2 y}{\partial x^2}$$

where we use the Taylor expansion. We now apply the Newton's second law to the element, with the mass of the element given by $m = \mu\Delta x\sqrt{1+\left(\frac{\partial y}{\partial x}\right)^2} \approx \mu\Delta x$,

$$F_y = ma_y = \mu\Delta x \frac{\partial^2 y}{\partial t^2}$$

Then we have

$$\mu\Delta x \frac{\partial^2 y}{\partial t^2} = T_s \Delta x \frac{\partial^2 y}{\partial x^2}, \qquad \frac{\mu}{T_s}\frac{\partial^2 y}{\partial t^2} = \frac{\partial^2 y}{\partial x^2}$$

which leads to a wave equation given by

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu}{T_s}\frac{\partial^2 y}{\partial t^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

where

$$v = \sqrt{\frac{T_s}{\mu}}$$

6.4 Energy density in wave motion

Although no matter is transported down the string as the wave propagates, the energy is carried along by the wave with velocity *v*. As a piece of the string moves up and down executing simple harmonics, it has kinetic energy as well as potential energy (because the string is stretched like a spring).

The infinitesimal mass of the string is $\Delta m = \mu\Delta x$. The kinetic energy contribution ΔK is given by

$$\Delta K = \frac{1}{2}\Delta m v_y{}^2 = \mu \Delta x \frac{1}{2}\left(\frac{\partial y}{\partial t}\right)^2$$

How about the potential energy? The potential energy ΔU is given by

$$\Delta U = \frac{T_s}{2}\Delta x\left(\frac{\partial y}{\partial x}\right)^2.$$

The derivation of the potential energy will be given in the APPENDIX because of some difficulty. Then the energy density is given by

$$\Delta E = \Delta K + \Delta U = \Delta x\left[\frac{T_s}{2}\left(\frac{\partial y}{\partial x}\right)^2 + \frac{\mu}{2}\left(\frac{\partial y}{\partial t}\right)^2\right].$$

7 Average power for the sinusoidal wave on a string

Waves transport energy when they propagate through a medium. We can model each element of a string as a simple harmonic oscillator. The oscillation will be in the *y* direction. Every element has the same energy.

Each element can be considered to have a mass of Δm (= $\mu\Delta x$). Its kinetic energy is given by

$$dK = \frac{1}{2}(\Delta m)v_y{}^2 = \frac{1}{2}(\mu dx)v_y{}^2 = \frac{1}{2}\mu A^2\omega^2\cos^2(kx-\omega t)dx.$$

Note that $y = A\sin(kx-\omega t)$ and $v_y = \frac{\partial y}{\partial t} = -A\omega\cos(kx-\omega t)$. Integrating over all the elements, the total kinetic energy in one wavelength is

$$\begin{aligned}K_\lambda &= \int_0^\lambda dK = \int_0^\lambda \frac{1}{2}\mu A^2\omega^2\cos^2(kx-\omega t)dx \\ &= \frac{1}{2}\mu A^2\omega^2\frac{1}{2}\int_0^\lambda [1+\cos(2kx-2\omega t)]dx \\ &= \frac{1}{4}\mu\omega^2 A^2\lambda\end{aligned}$$

The potential energy is given by

$$dU = \frac{1}{2}(\mu dx)\omega^2 y^2 = \frac{1}{2}\mu\omega^2 A^2\sin^2(kx-\omega t)dx.$$

Integrating over all the elements, the total potential energy in one wavelength is

$$U_\lambda = \int_0^\lambda dU = \int_0^\lambda \frac{1}{2}\mu\omega^2 A^2 \sin^2(kx-\omega t)dx$$
$$= \frac{1}{2}\mu A^2\omega^2 \frac{1}{2}\int_0^\lambda [1-\cos(2kx-2\omega t)]dx$$
$$= \frac{1}{4}\mu\omega^2 A^2 \lambda$$

Note that U_λ is exactly the same as K_λ (equi-partition law of energy). The total energy in one wavelength of the wave is the sum of the kinetic energy and the potential energy,

$$E_\lambda = K_\lambda + U_\lambda = 2K_\lambda = \frac{1}{2}\mu\omega^2 A^2\lambda \, .$$

The power transmitted by a sinusoidal wave on a stretch string is given by

$$\wp = \frac{E_\lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 \frac{\lambda}{T} = \frac{1}{2}\mu\omega^2 A^2 v = \frac{1}{2}\mu\omega^2 A^2 \sqrt{\frac{T_s}{\mu}} = \frac{1}{2}\sqrt{\mu T_s}\,\omega^2 A^2 .$$

((Example)) **Problem16-26**

A string along which waves can travel is 2.70 m long and has a mass of 260 g. The tension in the string is 36.0 N. What must be the frequency of travelling waves of amplitude 7.70 mm for the average power to be 85.0 W?

((Solution))

$$\mu = \frac{0.26kg}{2.70m} = 0.0963kg/m$$
$$T_s = 36.0N$$
$$A = 7.70\times 10^{-3}m$$

The velocity is given by

$$v = \sqrt{\frac{T_s}{\mu}} = 19.33m/s$$

The average power is

$$P_{avg} = \frac{1}{2}\mu v\omega^2 A^2 = \frac{1}{2}\sqrt{\mu T_s}\,\omega^2 A^2 = 85W$$

The angular frequency is

$$\omega = 1.24\times 10^3 \text{ rad/s},$$

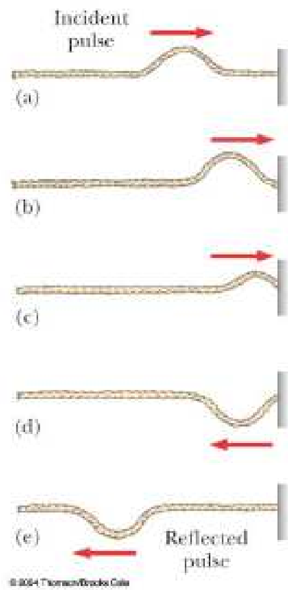
or

$f = 198$ Hz.

8 Reflection and transmission of waves

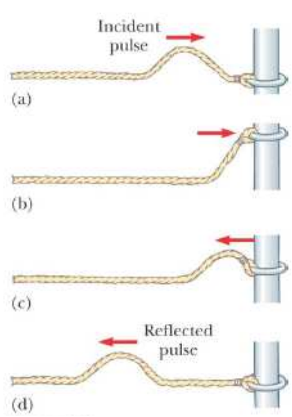
8.1 Reflection of a Wave, Fixed End

When the pulse reaches the support, the pulse moves back along the string in the opposite direction. This is the reflection of the pulse. The pulse is inverted when it is reflected from a fixed boundary.



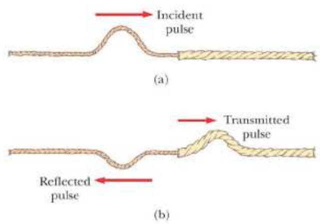
8.2 Reflection of a wave, free end

With a free end, the string is free to move vertically. The pulse is reflected. The pulse is not inverted when reflected from a free end.



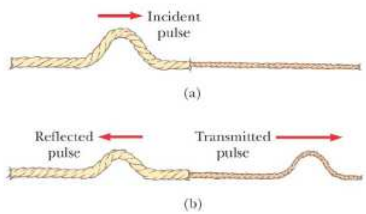
8.3 Transmission of a wave (I)

Assume a light string is attached to a heavier string. The pulse travels through the light string and reaches the boundary. The part of the pulse that is reflected is inverted. The reflected pulse has a smaller amplitude



8.4 Transmission of a wave (II)

Assume a heavier string is attached to a light string. Part of the pulse is reflected and part is transmitted. The reflected part is not inverted

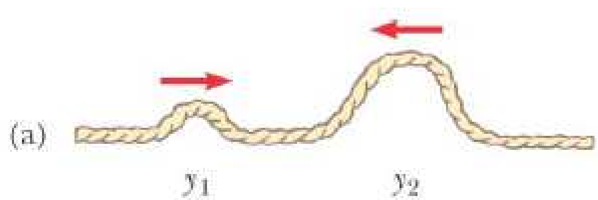


((**Note**))

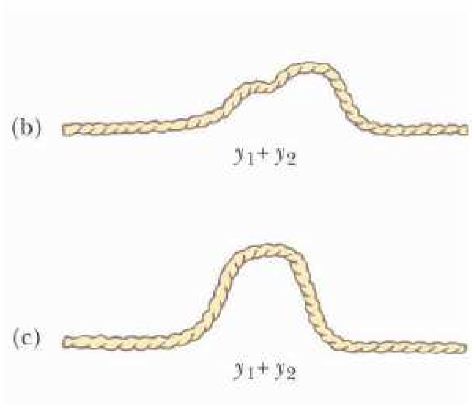
Conservation of energy governs the pulse. When a pulse is broken up into reflected and transmitted parts at a boundary, the sum of the energies of the two pulses must equal the energy of the original pulse

9. Superposition of waves

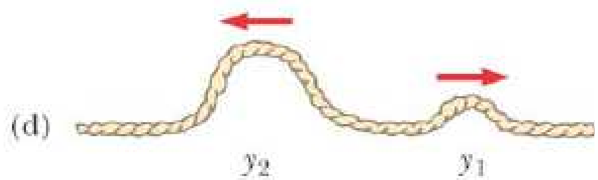
If two or more traveling waves are moving through a medium and combine at a given point, the resultant position of the element of the medium at that point is the sum of the positions due to the individual waves. Waves that obey the superposition principle are linear waves. In general, linear waves have amplitudes much smaller than their wavelengths



Two pulses are traveling in opposite directions. The wave function of the pulse moving to the right is y_1 and for the one moving to the left is y_2. The pulses have the same speed but different shapes. The displacement of the elements is positive for both



When the waves start to overlap (b), the resultant wave function is $y_1 + y_2$. When crest meets crest (c) the resultant wave has a larger amplitude than either of the original waves



The two pulses separate. They continue moving in their original directions. The shapes of the pulses remain unchanged.

((**Mathematica**))

Superposition of two Gaussian wave packets traveling in the +*x* and –*x* directions

((**Example-1**))

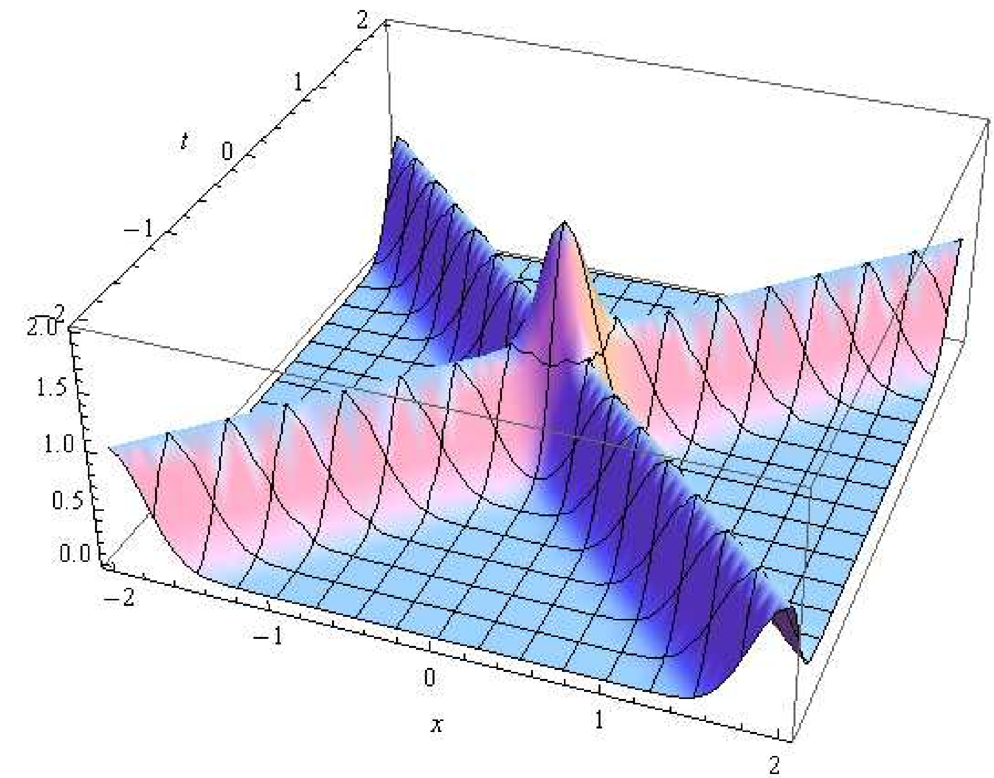


Fig. Plot3D of superposition of two Gaussian wave packets which is expressed by $\psi = \exp[-\frac{(x-vt)^2}{2\sigma^2}] + \exp[-\frac{(x+vt)^2}{2\sigma^2}]$, with σ = 0.2 and v = 1.

((**Example-2**))

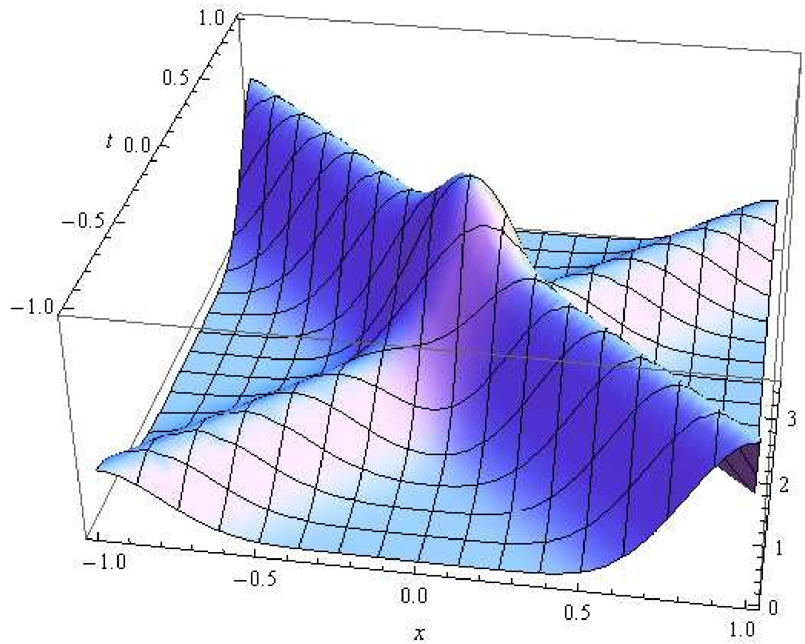


Fig. Plot3D of superposition of two Gaussian wave packets which is expressed by $\psi = \exp[-\frac{(x-vt)^2}{2\sigma^2}] + 2.5\exp[-\frac{(x+vt)^2}{2\sigma^2}]$, with $\sigma = 0.2$ and $v = 1$.

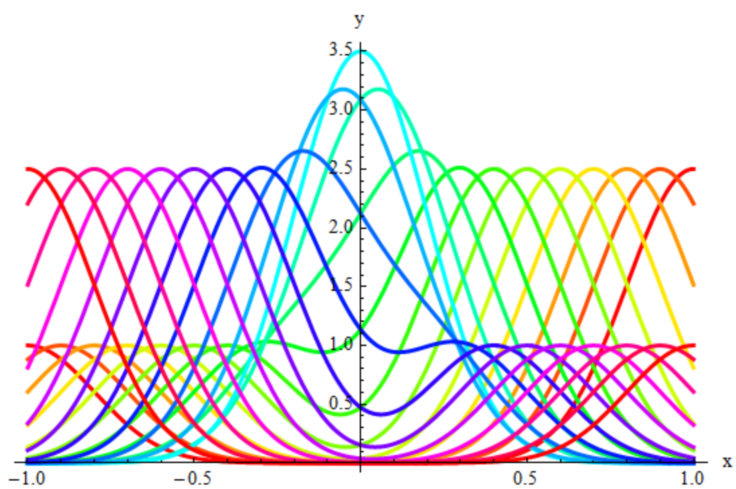


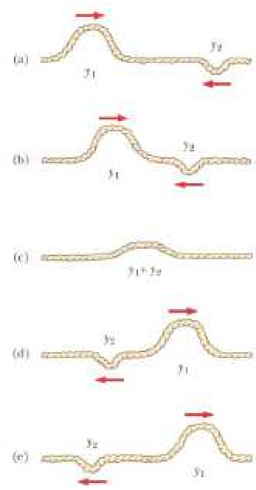
Fig. Plot of ψ as a function of x, where t is changed as a parameter. t = -1 to 1 with Δt = 0.1

10 Interference of pulse waves

Two traveling waves can pass through each other without being destroyed or altered. A consequence of the superposition principle. The combination of separate waves in the same region of space to produce a resultant wave is called interference

Constructive interference occurs when the displacements caused by the two pulses are in the same direction. The amplitude of the resultant pulse is greater than either individual pulse.

Destructive interference occurs when the displacements caused by the two pulses are in opposite directions. The amplitude of the resultant pulse is less than either individual pulse. When they overlap, their displacements partially cancel each other.



((Example))

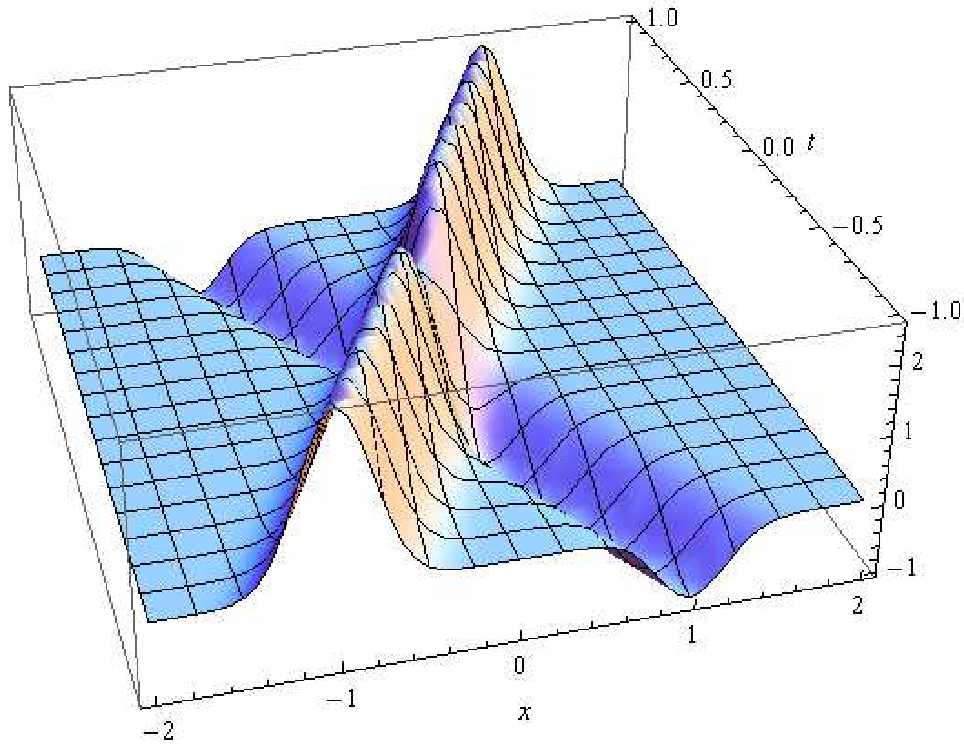


Fig. Plot3D of superposition of two Gaussian wave packets which is expressed by

$$\psi = 2.5\exp[-\frac{(x-vt)^2}{2\sigma^2}] - \exp[-\frac{(x+vt)^2}{2\sigma^2}]$$, with $\sigma = 0.2$ and $v = 1$.

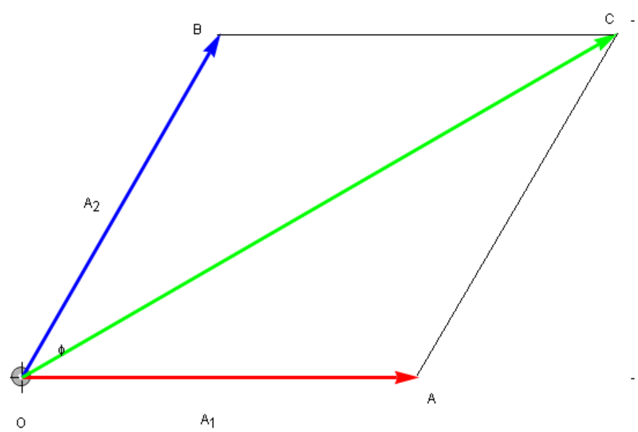
11. Superposition of sinusoidal waves: phasor diagram

Assume two waves are traveling in the same direction, with the same frequency, wavelength and amplitude. The waves differ in phase.

We consider the resultant wave of the two waves given by

$$y_1 = A_1 \sin(kx - \omega t)$$
$$y_2 = A_2 \sin(kx - \omega t + \phi)$$

The phasor diagram is shown in this figure. The wave y_1 corresponds to the vector $\overrightarrow{OA} = (A_1, 0)$ and the wave y_2 corresponds to the vector $\overrightarrow{OA} = (A_1, 0)$ and $\overrightarrow{OB} = (A_2 \cos\phi, A_2 \sin\phi)$.



Note that the validity of the use of phasor diagram for this is discussed in the Appendix.

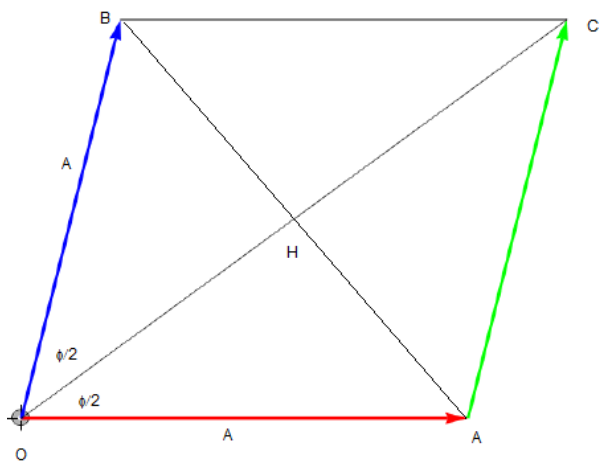
Then the amplitude of the resultant wave is given by $\left|\overrightarrow{OC}\right|$ and is calculated as follows.

$$\overrightarrow{OB} = (A_2 \cos\phi, A_2 \sin\phi)$$
$$\overrightarrow{OA} = (A_1, 0)$$
$$\overrightarrow{OC} = (A_1 + A_2 \cos\phi, A_2 \sin\phi)$$
$$\left|\overrightarrow{OC}\right| = \sqrt{(A_1 + A_2 \cos\phi)^2 + (A_2 \sin\phi)^2}$$
$$= \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \sin\phi}$$

When $\phi = 0$ (in-phase), $\left|\overrightarrow{OC}\right|$ becomes maximum. $\left|\overrightarrow{OC}\right| = A_1 + A_2$.

When $\phi = 0$ (out-of-phase), $\left|\overrightarrow{OC}\right|$ becomes minimum. $\left|\overrightarrow{OC}\right| = \left|A_1 - A_2\right|$

We now consider the special case where $A_1 = A_2 = A$. $\left|\overrightarrow{OC}\right| = 2\left|\overrightarrow{OH}\right| = 2A\cos\dfrac{\phi}{2}$.



(a) Constructive interference

When $\phi = 0$, then $\cos(\phi/2) = 1$. The amplitude of the resultant wave is $2A$. The crests of one wave coincide with the crests of the other wave. The waves are everywhere in phase. The waves interfere constructively.

(b) Destructive interference

When $\phi = \pi$, then $\cos(\phi/2) = 0$. Also any even multiple of π. The amplitude of the resultant wave is 0. Crests of one wave coincide with troughs of the other wave. The waves interfere destructively

(c) Intermediate state.

When ϕ is other than 0 or an even multiple of π, the amplitude of the resultant is between 0 and $2A$. The wave functions still add

12. Standing waves in a string

We consider the wavefunctions for two sinusoidal waves having the same amplitude, frequency, and wavelength but traveling in opposite directions in the same medium.

$$
\begin{aligned}
y(x,t) &= A\sin[k(x - vt)] + A\sin[k(x + vt)] \\
&= 2A\sin(kx)\cos(\omega t)
\end{aligned}
$$

with $k = \dfrac{2\pi}{\lambda}$, where λ is the wavelength. The equation represents the wave function of a standing wave. A standing wave is an oscillation pattern with a stationary outline.

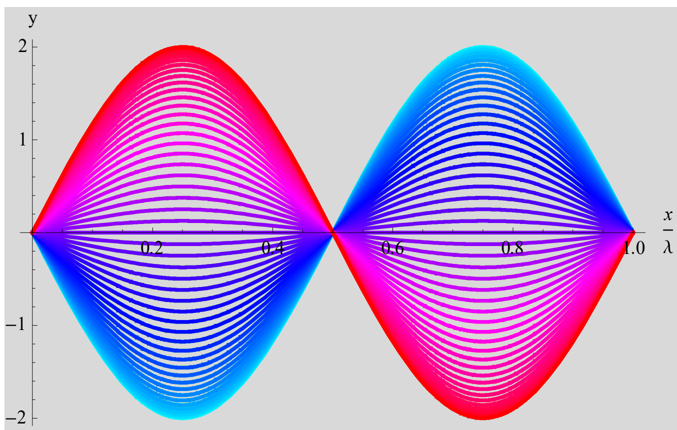
The points of zero amplitude are called nodes;

$x/\lambda = 0,\ 1/2,\ 1,\ ,\ 3/2, \ldots..$

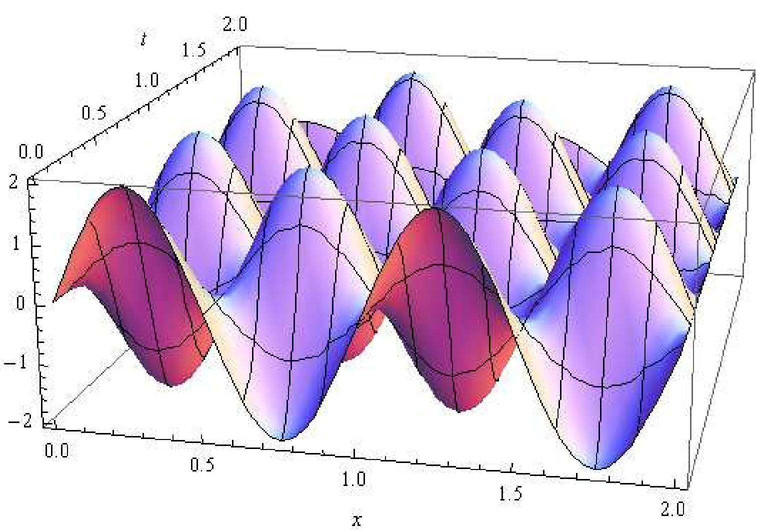
The positions in the medium at which the maximum displacement occurs are called anticode;

$x/\lambda = 1/4,\ 3/4,\ 5/4, \ldots..$

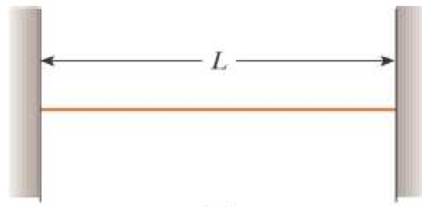
The distance between adjacent antinodes is equal to $\lambda/2$. The distance between adjacent nodes is equal to $\lambda/2$. The distance between a node and an adjacent antinode is $\lambda/4$.



We have a 3D plot of $y(x, t)$ in the x-t plane.



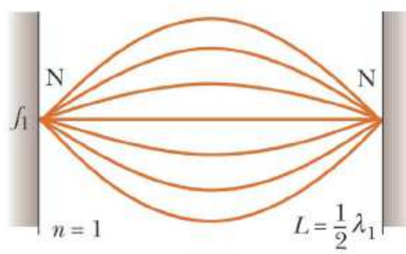
13 Resonance (standing waves in a string)



(a)

We consider a string fixed at both ends. The string has length L. Standing waves are set up by a continuous superposition of waves incident on and reflected from the ends. There is a boundary condition on the waves.

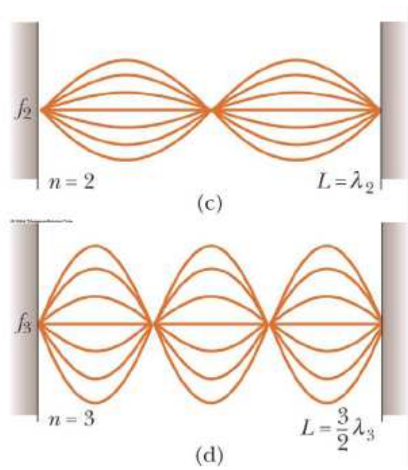
The ends of the strings must necessarily be nodes. They are fixed and therefore must have zero displacement. The boundary condition results in the string having a set of normal modes of vibration. Each mode has a characteristic frequency. The normal modes of oscillation for the string can be described by imposing the requirements that the ends be nodes and that the nodes and antinodes are separated by l/4



(b)

This is the first normal mode that is consistent with the boundary conditions. There are nodes at both ends. There is one antinode in the middle. This is the longest wavelength mode

$$\frac{1}{2}\lambda_1 = L$$
$$\lambda_1 = 2L$$



Consecutive normal modes add an antinode at each step. The second mode (c) corresponds to $\lambda_2 = L$. The third mode (d) corresponds to $\lambda_3 = 2L/3$

The wavelengths of the normal modes for a string of length L fixed at both ends are

$$\lambda_n = 2L/n \text{ with } n = 1, 2, 3, \ldots$$

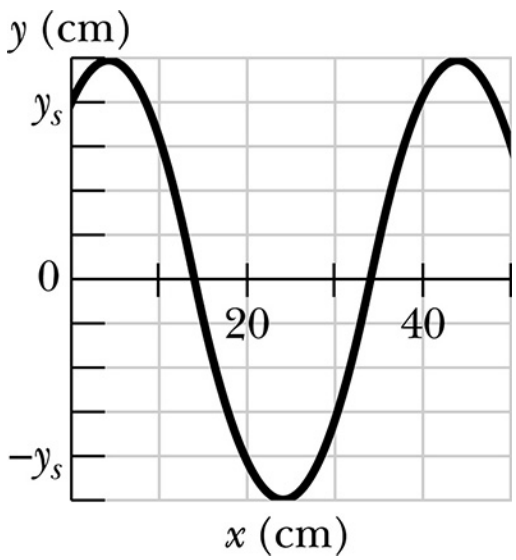
where n is the nth normal mode of oscillation. These are the possible modes for the string The natural frequencies are given by

$$f_n = \frac{v}{\lambda_n} = \frac{nv}{2L} = \frac{n}{2L}\sqrt{\frac{T_s}{\mu}}$$

14. Typical problems

14.1 Problem 16-21 (SP-16)

A sinusoidal transverse wave is travelling along a string in the negative direction of an x axis. Figure shows a plot of the displacement as a function of position at time $t = 0$; the scale of the y axis is set by $y_s = 4.0$ cm. The string tension is 3.6 N, and its linear density is 25g/m. Find the (a) amplitude, (b) wavelength, (c) wave speed, and (d) period of the wave. (e) Find the maximum transverse speed of a particle in the string. If the wave is of the form $y(x,t) = y_m \sin(kx \pm \omega t + \phi)$, what are (f) k, (g) ω, (h) ϕ, and (i) the correct choice of sign in front of ω?



((**Solution**))
Since the wave travels along the negative x axis,

$$y = y_m \sin(kx + \omega t + \phi)$$
$$\frac{dy}{dt} = y_m \omega \cos(kx + \omega t + \phi)$$

$$\mu = 2.50 \times 10^{-3} kg/m$$
$$T_s = 3.6N$$

(a) The amplitude: $y_m = 5.00 \times 10^{-2} m$

(b) The wavelength is $\lambda = 0.40$ m. The wavenumber is

$$k = \frac{2\pi}{\lambda} = 15.71/m$$

(c) The velocity is given by

$$v = \sqrt{\frac{T_s}{\mu}} = 12.0 m/s$$

(d) The angular frequency ω is

$$\omega = kv = 188.5 \text{ rad/s}$$

The period T is

$$T = \frac{2\pi}{\omega} = 0.033s$$

(e) The maximum transverse velocity is

$$y_m \omega = 9.43 m/s$$

(f) $k = \frac{2\pi}{\lambda} = 15.71/m$

(g) $\omega = kv = 188.5 \text{ rad/s}$

(h) $y = y_m \sin(kx + \phi)$

At $x = 0$ m, $y = 4.00$ cm

$$4.00cm = 5.00cm \sin(\phi)$$
$$\sin\phi = 0.8$$
$$\phi = 0.927$$

(i) plus sign.

14.2 Problem 16-38 (SP-16): Phasor diagram

Four waves are to be sent along the same string, in the same direction:

$$y_1(x,t) = (4.00mm)\sin(2\pi x - 400\pi t)$$
$$y_2(x,t) = (4.00mm)\sin(2\pi x - 400\pi t + 0.7\pi)$$
$$y_3(x,t) = (4.00mm)\sin(2\pi x - 400\pi t + \pi)$$
$$y_4(x,t) = (4.00mm)\sin(2\pi x - 400\pi t + 1.7\pi)$$

What is the amplitude of the resultant wave?

((**Solution**))

$$\boldsymbol{A} = (4,0)$$
$$\boldsymbol{B} = (4\cos(0.7\pi), 4\sin(0.7\pi))$$
$$\boldsymbol{C} = (4\cos(\pi), 4\sin(\pi)) = (-4,0)$$
$$\boldsymbol{D} = (4\cos(1.7\pi), 4\sin(1.7\pi)) = (-4\cos(0.7\pi), -4\sin(0.7\pi))$$

Then we have

$$\boldsymbol{A} + \boldsymbol{B} + \boldsymbol{C} + \boldsymbol{D} = 0$$

The amplitude of the resultant wave is zero.

14.3 Problem 16-57 (SP-16): Standing wave

The following two waves are sent in opposite directions on a horizontal string so as to create a standing wave in a vertical plane:

$$y_1(x,t) = (6.00cm)\sin(4.00\pi x - 400\pi t)$$
$$y_2(x,t) = (6.00cm)\sin(4.00\pi x + 400\pi t)$$

with *x* in meters and *t* in seconds. An antinode is located at point A. In the time interval that point takes to move from maximum upward displacement to maximum downward displacement, how far does each wave move along the string?

((**Solution**))

y_m = 6.00 mm = 6.00 x 10^{-3} m

k = 4.00 π (1/m)

ω = 400 π (rad/s)

$$T = \frac{2\pi}{\omega} = 5\times10^{-3}s$$

$$v = \frac{\omega}{k} = \frac{400\pi}{4\pi} = 100m/s$$

$$y_1(x,t) = y_m\sin(kx-\omega t)$$
$$y_2(x,t) = y_m\sin(kx+\omega t)$$

$$\begin{aligned} y &= y_m\sin(kx-\omega t) + y_m\sin(kx+\omega t) \\ &= 2y_m\sin(kx)\cos(\omega t) \\ &= 12.00 \text{ x } 10^{-3}\sin(4\pi x)\cos(400\pi t) \end{aligned}$$

When cos(ωt) changes from 1 to -1, it takes half the period (= T/2).

We define the two phases for the two waves,

$$\phi_a = kx - \omega t$$
$$\phi_b = kx + \omega t$$

From the condition that both phases do not change

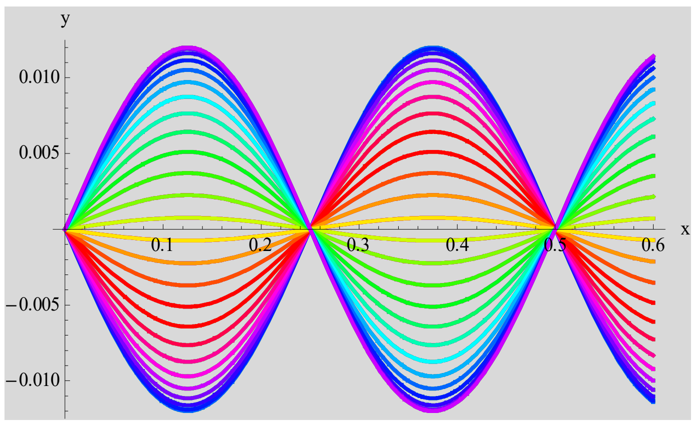
$$\Delta\phi_a = k\Delta x - \omega\Delta t = 0$$

$$\Delta x = \frac{\omega}{k}\Delta t = v\Delta t = v\frac{T}{2} = 100\times\frac{5}{2}\times10^{-3}s = 0.25m$$

$$\Delta\phi_b = k\Delta x + \omega\Delta t = 0$$

$$\Delta x = -\frac{\omega}{k}\Delta t = -v\Delta t = -v\frac{T}{2} = -100\times\frac{5}{2}\times10^{-3}s = -0.25m$$

One wave moves along the + x direction by 0.25 m, while the other wave moves along the (-x) direction by 0.25 m.



APPENDIX-1

A. Derivation of the potential energy

Vibration and Sound

P.M. Morse second edition (McGraw-Hill, 1948, New York)

The total force along the y direction for the element ($x - x+\Delta x$) is given by

$$F_y = \Delta x T_s \frac{\partial^2 y}{\partial x^2}.$$

Then we can imagine changing the string from the equilibrium form (y = 0) to the final form by making its intermediate form be ky, where k changes from zero to unity. The force on any element of string of the form ky is

$$\Delta x T_s \frac{\partial^2 (ky)}{\partial x^2}.$$

As we displace the string from equilibrium by changing k, the element of displacement is ydk. The work required to bring this element of string into place is

$$\int_0^1 \Delta x T_s \frac{\partial^2 (ky)}{\partial x^2} kydk = \Delta x T_s \frac{\partial^2 y}{\partial x^2} y \int_0^1 kdk = \frac{1}{2} \Delta x T_s \frac{\partial^2 y}{\partial x^2} y \,.$$

Then the total work is given by

$$W = \int_0^L \frac{1}{2} T_s y \frac{\partial^2 y}{\partial x^2} dx = \frac{1}{2} T_s \int_0^L y \frac{\partial^2 y}{\partial x^2} dx = \frac{1}{2} [T_s y \left(\frac{\partial y}{\partial x} \right)]_0^L - \frac{1}{2} T_s \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx$$

$$= -\frac{1}{2} T_s \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx$$

by integrating by part. The first term is equal to zero since $y = 0$ at $x = 0$ and L. The potential energy U is related to the work W by

$$U = -W = \frac{1}{2} T_s \int_0^L \left(\frac{\partial y}{\partial x} \right)^2 dx \,.$$

Then the potential energy of the element ($x - x+\Delta x$) is

$$\Delta U = \frac{1}{2} T_s \left(\frac{\partial y}{\partial x} \right)^2 \Delta x \,.$$

B Superposition of sinusoidal waves: (using the complex plane)

Assume two waves are traveling in the same direction, with the same frequency, wavelength and amplitude. The waves differ in phase.

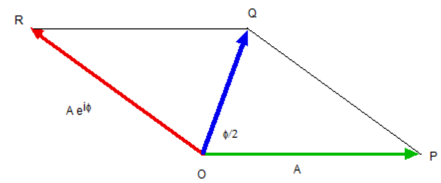
$$y_1 = A\sin(kx - \omega t) = A\cos(kx - \omega t - \frac{\pi}{2}) = \mathrm{Re}[Ae^{i(kx-\omega t+\frac{\pi}{2})}] = \mathrm{Re}[Ae^{i(kx-\omega t\frac{\pi}{2})}]$$

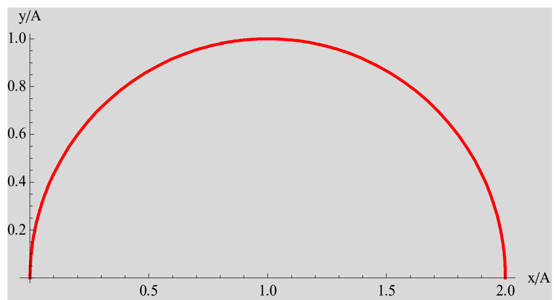
$$y_2 = A\sin(kx - \omega t + \phi) = \mathrm{Re}[Ae^{i\phi} e^{i(kx-\omega t+\frac{\pi}{2})}]$$

$$y = y_1 + y_2 = 2A\cos(\frac{\phi}{2})\sin(kx - \omega t + \frac{\phi}{2})$$
$$= \mathrm{Re}[(A + Ae^{i\phi})e^{i(kx-\omega t+\frac{\phi}{2})}]$$

The resultant wave function, y, is also sinusoidal. It has the same frequency and wavelength as the original waves. The amplitude of the resultant wave is $2A\cos(\phi/2)$, while the phase of the resultant wave is $\phi/2$.

We use the phasor diagram in the complex plane. As ϕ approaches π, the magnitude of the vector OQ becomes zero and the angle becomes $\pi/2$.





APPENDIX-II

W. Lawrence Bragg, Lecture on Waves and Vibrations
https://www.youtube.com/watch?v=pc93R2u3pjE

The reflection of waves is described and their expansion and compression is then illustrated experimentally. Sir Lawrence demonstrated the effect of waves crossing each other and explains this effect with the aid of models and animated diagrams. The Doppler Effect is described and illustrated dramatically by means of ASDIC recordings. Finally, Sir Lawrence considers and demonstrates the effect when a body is travelling through a medium faster than the waves travel in that medium.

Waves and Vibrations - with Sir Lawrence Bragg