Chapter 17
Waves-II: Sound

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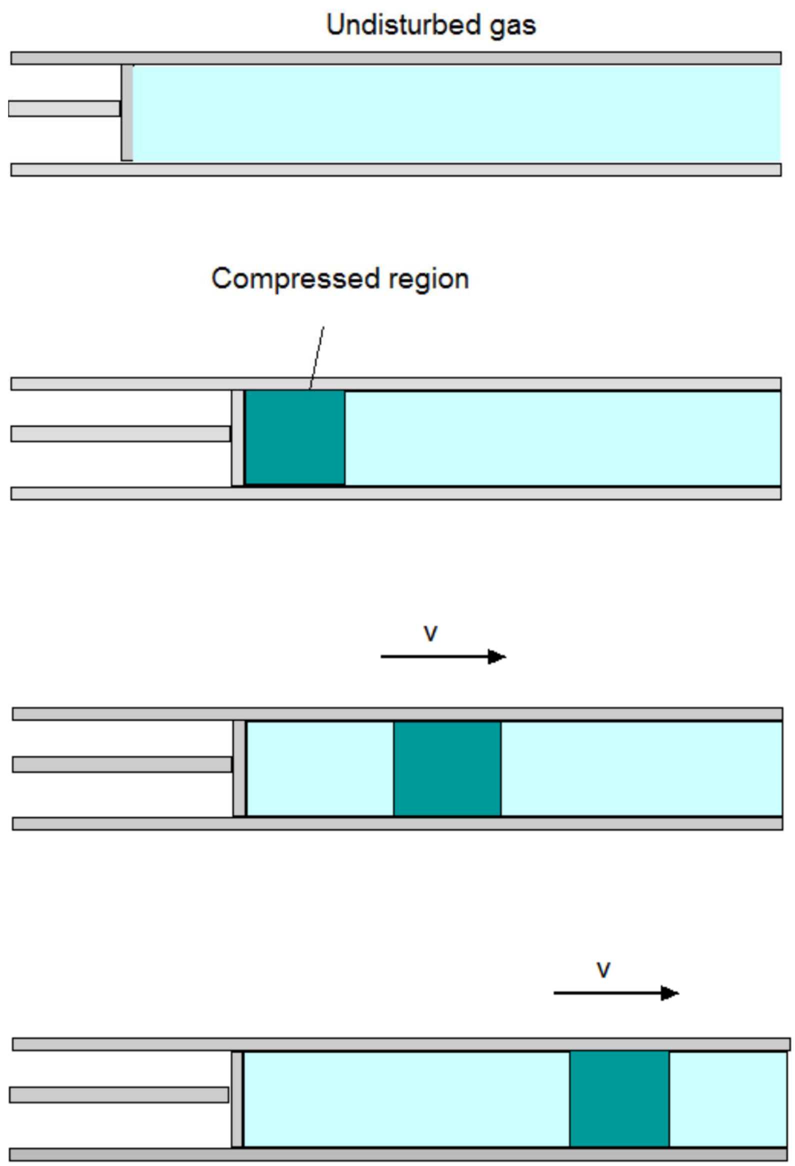
1. Sound pulse

Sound waves are longitudinal waves. They travel through any material medium. The speed of the wave depends on the properties of the medium. We use a compressible gas as an example with a setup as shown below. Before the piston is moved, the gas has uniform density. When the piston is suddenly moved to the right, the gas just in front of it is compressed. Darker region in the diagram



When the piston comes to rest, the compression region of the gas continues to move. This corresponds to a longitudinal pulse traveling through the tube with speed v. The speed of the piston is not the same as the speed of the wave. The light areas are rarefactions (low-pressure region).

2. The speed of sound

2.1 Summary

The speed of sound wave is given by

$$v = \sqrt{\frac{B}{\rho}},$$

where B is the bulk modulus and ρ is the mass density of the medium in which the sound is traveling. As for the sound waves,

$$v = f\lambda.$$

The speed of sound waves in air depends only on the temperature of the air.

$$v = 331 \text{ m/s} + (0.6 \text{ m s}^{-1} \text{ °C}^{-1})\, T_C$$

where T_C is the temperature in Celsius. The speed of sound is $v = 343$ m/s in air at 20 °C and 1493 m/s in water at 25 °C.

2.2 The speed of sound

The formula of the speed of sound can be derived as follows.

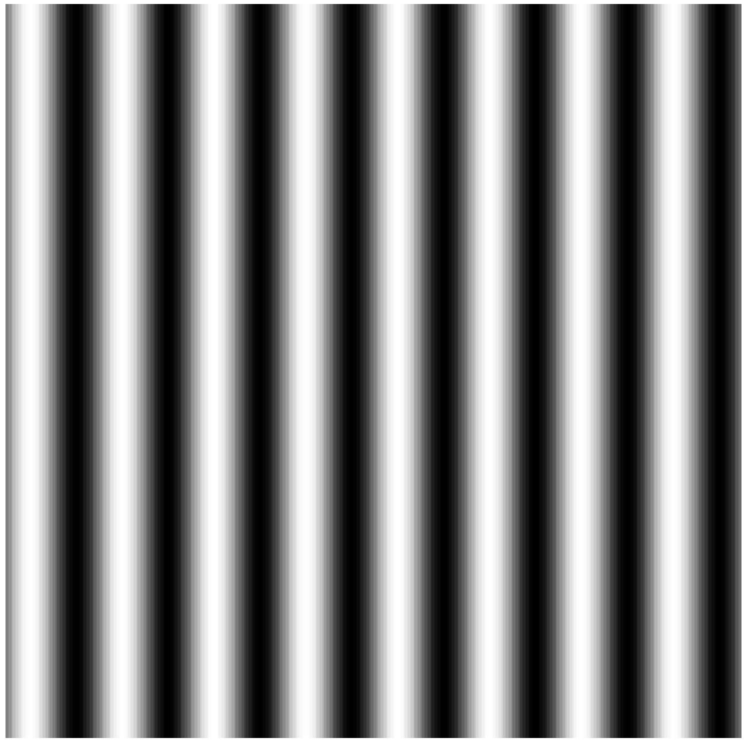
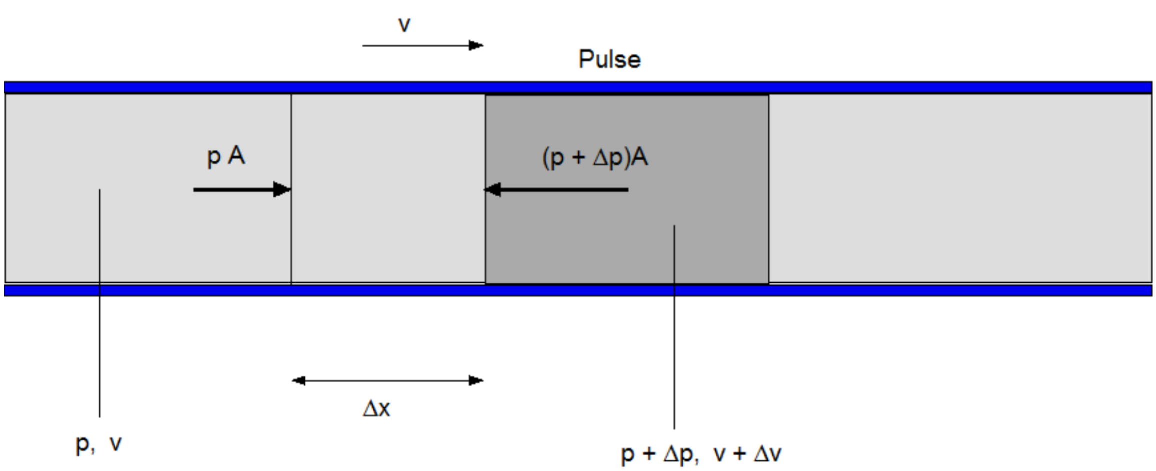


Fig. Pattern of sound wave; bright part (high pressure; compression) and dark part (low pressure; expansion; rarefaction). We use the Mathematica program (Graphics)



Let the pressure of the undisturbed air be p and the pressure inside the pulse be $p + \Delta p$, where Δp is positive due to the compression. Consider an element of air of thickness Δx and face area A, moving toward the pulse at speed v. As this element enters the pulse, the leading face of the element encounters a region of higher pressure, which slows the element to speed $v + \Delta v$ in which Δv is negative. This slowing is complete when the rear face of the element reaches the pulse, which requires time interval

$$\Delta t = \frac{\Delta x}{v}$$

Let us apply Newton's second law to the element. During Δt, the average force on the element's trailing face is pA toward the right, and the average force on the leading face is $(p + \Delta p)A$ toward the left (Fig. *b*). Therefore, the average net force on the element during Δt is

$$F = pA - (p + \Delta p)A = -A\Delta p$$

The minus sign indicates that the net force on the air element is directed to the left in Fig. b. The volume of the element is $A\Delta x$, we can write its mass as

$$\Delta m = \rho\Delta V = \rho A\Delta x = \rho A v\Delta t$$

The average acceleration of the element during Δt is

$$a = \frac{\Delta v}{\Delta t}.$$

Thus, from Newton's second law ($F = ma$), we have

$$F = -A\Delta p = (\rho A v\Delta t)\frac{\Delta v}{\Delta t} = (\rho A v)\Delta v,$$

which we can write as

$$\rho v^2 = -\frac{\Delta p}{\frac{\Delta v}{v}}.$$

The air that occupies a volume V ($= Av\Delta t$) outside the pulse is compressed by an amount ΔV ($= A\ \Delta v\Delta t$) as it enters the pulse. Thus,

$$\frac{\Delta V}{V} = \frac{A\Delta v\Delta t}{Av\Delta t} = \frac{\Delta v}{v},$$

leading to

$$\rho v^2 = -\frac{\Delta p}{\frac{\Delta v}{v}} = -\frac{\Delta p}{\frac{\Delta V}{V}} = B,$$

or

$$v = \sqrt{\frac{B}{\rho}}.$$

2.3 Newton's evaluation

We will find dp/dV, the rate of change of pressure with volume. Here Newton used Boyle's law; which says that at constant T,

$$pV = p_0V_0, \quad \text{or} \quad p = \frac{p_0V_0}{V},$$

where p_0 is the equilibrium pressure. Differentiating gives

$$\frac{dp}{dV} = -\frac{p_0V_0}{V^2},$$

i.e., at equilibrium, with $V = V_0$, we have

$$V_0\left(\frac{dp}{dV}\right)_0 = -p_0,$$

or more simply, we have

$$\ln p + \ln V = \ln(p_0V_0).$$

Taking a derivative on both sides,

$$\frac{\Delta p}{p} + \frac{\Delta V}{V} = 0, \quad \text{or} \quad V_0\left(\frac{dp}{dV}\right)_0 = -p_0.$$

The velocity is obtained as

$$\rho v^2 = B = -V_0\left(\frac{dp}{dV}\right)_0 = p_0$$

or

$$v = \sqrt{\frac{p_0}{\rho}}.$$

For air at STP (standard temperature and pressure), we have

p_0 = 1 atm = 1.013 x 10^5 N/m^2 = 1.013 x 10^5 Pa

$\rho = 1.2041$ kg/m^3 at 20°C

$$v_{Newton} = \sqrt{\frac{p_0}{\rho}} = \sqrt{\frac{1.013\times 10^5 N/m^2}{1.2041 kg/m^3}} = 290.06 m/s$$

The experimental velocity is for air at STP is, $v = 343$ m/s.

((**Note**))

At 20 °C and 101.325 kPa, dry air has a density of 1.2041 kg/m^3.

2.4 Correcting Newton's mistake

Instead of Boyle's law, we use the adiabatic gas law, which gives the relation between p and V when no heat is allowed to flow.

$$pV^\gamma = p_0 V_0^\gamma, \qquad \text{or} \qquad p = p_0 V_0^\gamma V^{-\gamma}$$

where $\gamma = C_P/C_V$, $\gamma = 7/5 = 1.40$ for air at STP. From the relation of p vs V, we have

$$\frac{dp}{dV} = -\gamma p_0 V_0^\gamma V^{-\gamma-1}$$

$$\left(\frac{dp}{dV}\right)_0 = -\gamma p_0 V_0^\gamma V_0^{-\gamma-1} = -\gamma p_0 V_0^{-1}$$

$$V_0 \left(\frac{dp}{dV}\right)_0 = -\gamma p_0$$

or more simply, we have

$$\ln p + \gamma \ln V = \ln(p_0 V_0^\gamma)$$

Taking a derivative on both sides,

$$\frac{\Delta p}{p} + \frac{\gamma \Delta V}{V} = 0, \qquad \text{or} \qquad V_0 \left(\frac{dp}{dV}\right)_0 = -\gamma p_0$$

Then we have

$$v = \sqrt{-\frac{V_0}{\rho}\left(\frac{dp}{dV}\right)_0} = \sqrt{\frac{\gamma p_0}{\rho}} = \sqrt{\gamma}\, v_{Newton} = 343.21 m/s$$

2.5 Temperature dependence of the sound velocity

The density of air is calculated as follows.

$$pV = \frac{m}{M} RT$$

$$\rho = \frac{m}{V} = \frac{pM}{RT}$$

The molar mass of oxygen is 15.9994 g. The molar mass of nitrogen is 14.007 g. Then the molar mass of air is

$$M = (\frac{1}{5} \times 2 \times 15.9994 + \frac{4}{5} \times 2 \times 14.007) \times 10^{-3} kg = 28.811 \times 10^{-3} kg$$

Using the gas constant R = 8.314472 J/mol K and p = 1.01325 x 10^5 Pa, we have

$$\rho = \frac{pM}{RT} = \frac{351.107}{T} \text{ (kg/m}^3\text{)}$$

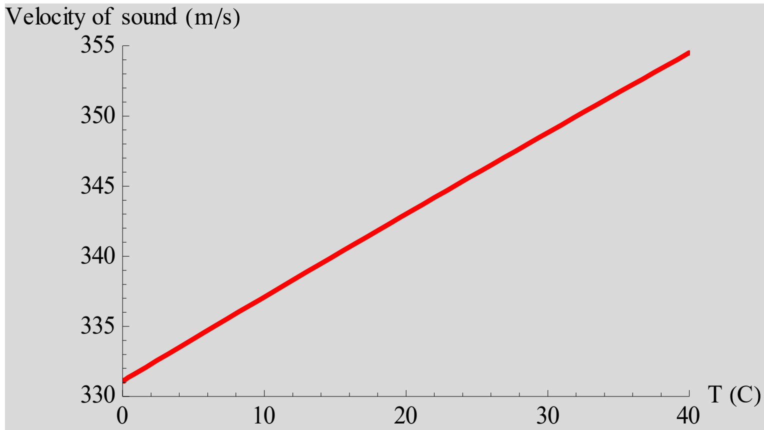
where ρ = 1.285 kg/m^3 for T = 273.15 K (0° C) and ρ = 1.198 kg/m^3 for T = 293.15 K (20°C). Then the velocity of sound is given by

$$\begin{aligned} v &= \sqrt{\frac{\gamma p_0}{\rho}} \\ &= \sqrt{\frac{\gamma p_0 T}{351.107}} \\ &= \sqrt{\frac{\gamma p_0 (273.15 + T_C)}{351.107}} \\ &= 16.988 \sqrt{\gamma} \sqrt{273.15 + T_C} \\ &= 20.10 \sqrt{273.15 + T_C} \\ &= 332.2 \sqrt{1 + \frac{T_C}{273.15}} \end{aligned}$$

((**Note**)) from the Wikipedia.

The sound of velocity of dry (0% humidity) is given by

$$v = 331.3 \sqrt{1 + \frac{T_C}{273.15}} .$$



The value of 331.3 m/s, which represents the 0 °C speed, is based on theoretical (and some measured) values of the heat capacity ratio, γ, as well as on the fact that at 1 atm real air is very well described by the ideal gas approximation. Commonly found values for the speed of sound at 0 °C may vary from 331.2 to 331.6 due to the assumptions made when it is calculated. If ideal gas γ is assumed to be 7/5 = 1.4 exactly, the 0 °C speed is calculated (see section below) to be 331.3 m/s, the coefficient used above.

This equation is correct to a much wider temperature range, but still depends on the approximation of heat capacity ratio being independent of temperature, and will fail, particularly at higher temperatures. It gives good predictions in relatively dry, cold, low pressure conditions, such as the Earth's stratosphere. A derivation of these equations will be given in a later section.

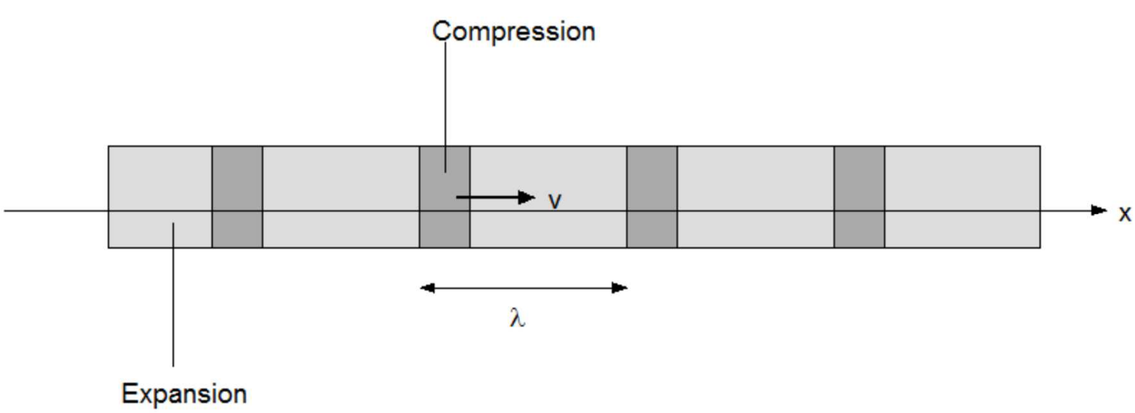
2.6 Speed of transverse wave in a bulk solid ((comparison))

The speed of transverse wave in a bulk solid is given by

$$v_s = \sqrt{\frac{S}{\rho}},$$

where S is the shear modulus of the material and ρ is the density of the material.

3. Wave equation of sound wave



The mathematical description of sinusoidal sound waves is very similar to sinusoidal waves on a string. The distance between two successive compressions (or two successive rarefactions) is the wavelength, λ. As these regions travel along the tube, each element oscillates back and forth in simple harmonic motion. Their oscillation is parallel to the direction of the wave.

The displacement of a small element is

$$s(x,t) = s_{\max} \cos(kx - \omega t)$$

where $s_{\max}$ is the maximum position relative to equilibrium. This is the equation of a displacement wave. k is the wave number. ω is the angular frequency of the piston.

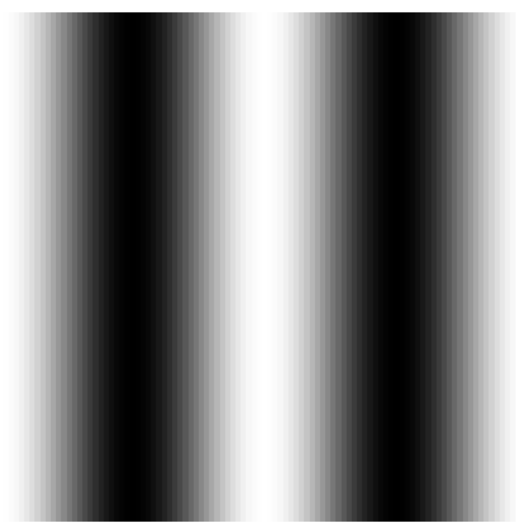
The variation Δp in the pressure of the gas as measured from its equilibrium value is also sinusoidal,

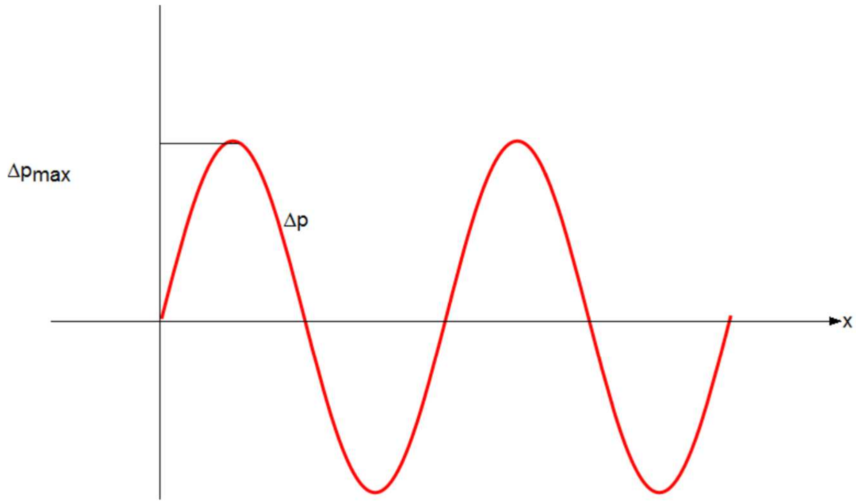
$$\Delta p(x,t) = \Delta p_{\max} \sin(kx - \omega t)$$

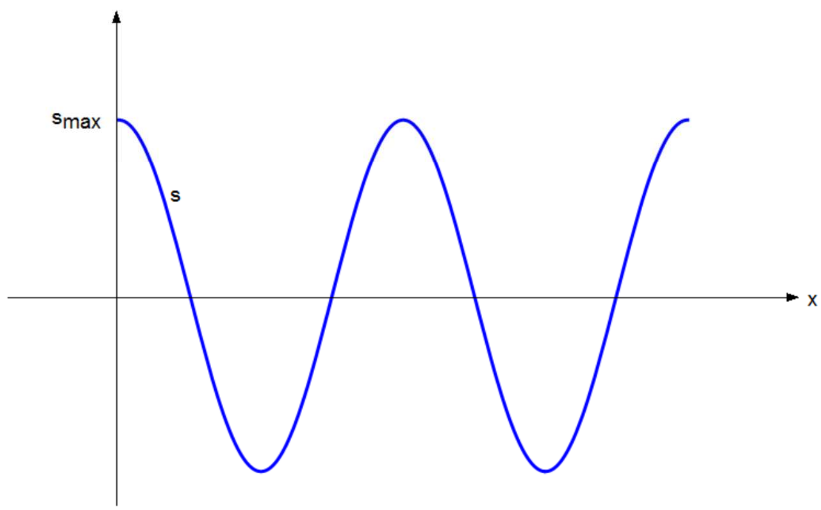
The pressure amplitude, $\Delta p_{\max}$ is the maximum change in pressure from the equilibrium value. The pressure amplitude is proportional to the displacement amplitude,

$$\Delta p_{\max} = \rho v \omega s_{\max}$$

A sound wave may be considered either a displacement wave or a pressure wave. The pressure wave is 90° out of phase with the displacement wave







((Note))

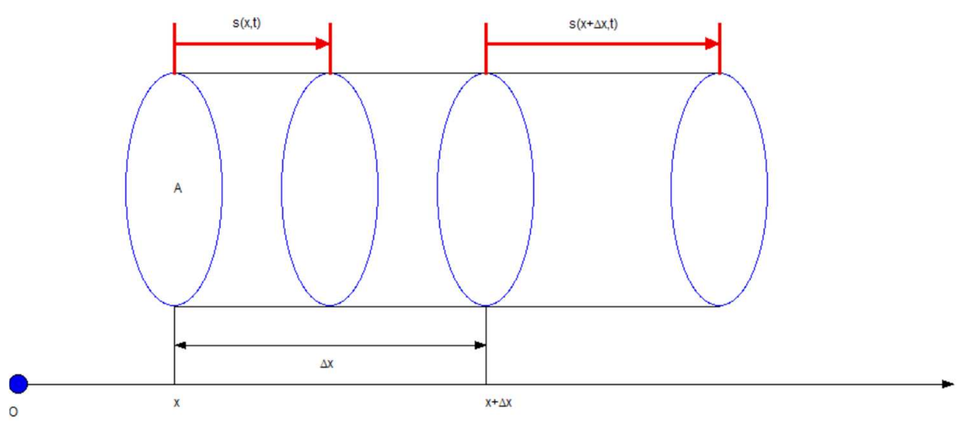


Fig. $s(x,t)$ is the displacement of the element at (x,t) from the position in equilibrium.

$s(x,t)$ is the displacement of the element at (x,t).
In equibrium:

The initial positions: x, and $x+\Delta x$
The initial volume: $A[(x+\Delta x)-x] = A\Delta x$

The deviation from the equilibrium:

The final positions: $x+s(x,t)$, and $x+\Delta x+s(x+\Delta x,t)$
The final volume:

$$A[x+\Delta x+s(x+\Delta x,t)] - A[x+s(x,t)] = A[\Delta x+s(x+\Delta x,t)-s(x,t)]$$

The change in volume ΔV of the cylinder is

$$\Delta V = A[\Delta x+s(x+\Delta x,t)-s(x,t)] - A\Delta x$$

Or

$$\Delta V = A[s(x+\Delta x,t)-s(x,t)]$$

In the limit of $\Delta x \rightarrow 0$, the fractional change in volume dV/V is

$$\frac{dV}{V} = \lim_{\Delta x\rightarrow 0}\frac{A[s(x+\Delta x,t)-s(x,t)}{A\Delta x} = \lim_{\Delta x\rightarrow 0}\frac{[s(x+\Delta x,t)-s(x,t)}{\Delta x} = \frac{\partial s(x,t)}{\partial x}$$

From the definition of *B*, we have

$$\Delta p = -B\frac{\Delta V}{V} = -B\frac{\partial s(x,t)}{\partial x},$$

or

$$\Delta p = -B\frac{\partial s}{\partial x} = Bks_{\max}\sin(kx-\omega t)$$
$$= \Delta p_{\max}\sin(kx-\omega t)$$

where

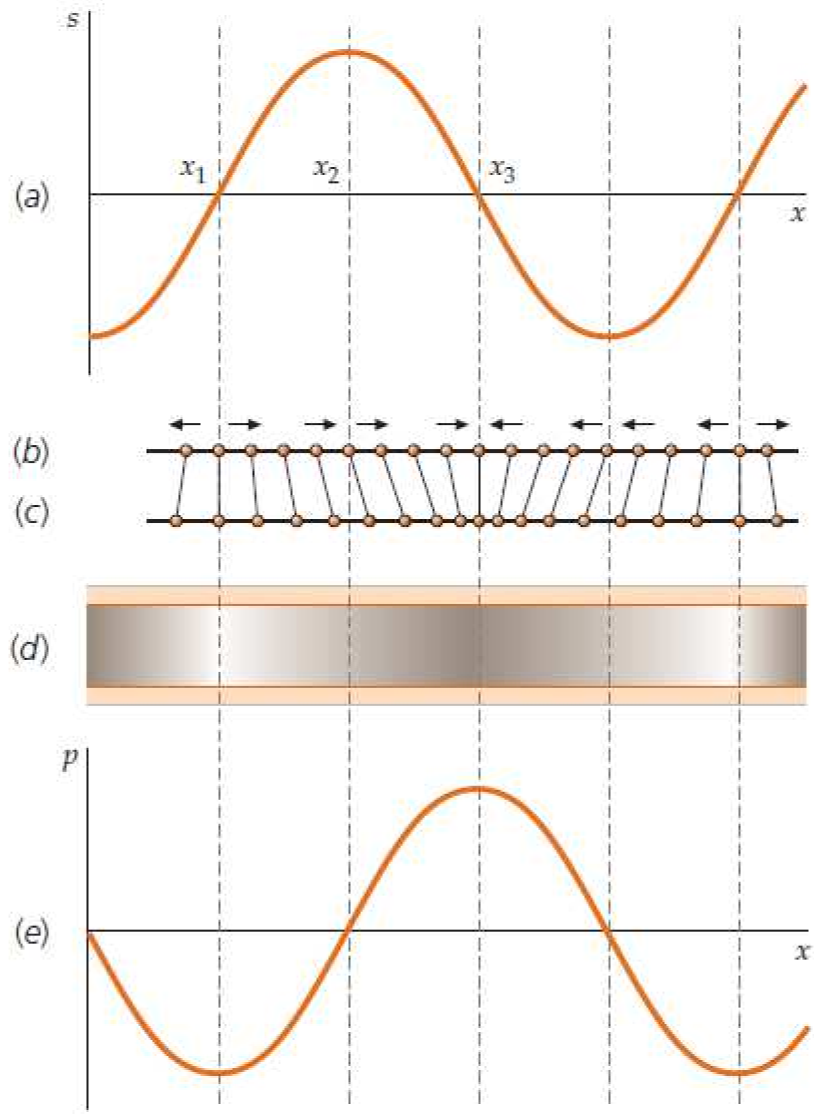
$$s(x,t) = s_{\max}\cos(kx-\omega t)$$

$$\Delta p_{\max} = Bks_{\max} = \rho v^2 ks_{\max} = \rho v\omega s_{\max},$$

and

$B = \rho v^2$, $\omega = vk$

((Note)) Relation between the pressure and displacement
P.A. Tipler and G. Mosca, Physics For Scientists and Engineers, 6-th edition (W.H. Freeman, 2008)



4. Derivation of the intensity of sound

We derive the expression for the intensity given by

$$I = \frac{1}{2}\rho v \omega^2 s_{\max}{}^2.$$

Consider a thin slice of air of thickness *dx*, area *A*, and mass *dm*, oscillating back and forth as the sound wave passes through it. The kinetic energy *dK* of the slice of air is

$$dK = \frac{1}{2} dm v_s{}^2. \qquad (1)$$

Here v_s is not the speed of the wave but the speed of the oscillating element of air,

$$v_s = \frac{\partial s}{\partial t} = -\omega s_{\max} \sin(kx - \omega t).$$

Using this relation and putting $dm = \rho A dx$ allow us to rewrite Eq.(1) as

$$dK = \frac{1}{2}(\rho A dx)(-\omega s_{\max})^2 \sin^2(kx - \omega t) \qquad (2)$$

The total kinetic energy *K* in one wavelength λ is

$$K_\lambda = \int_0^\lambda \frac{1}{2}(\rho A dx)(-\omega s_{\max})^2 \sin^2(kx - \omega t) = \frac{1}{2}\rho A \omega^2 s_{\max}{}^2 \int_0^\lambda \sin^2(kx - \omega t) dx$$
$$= \frac{1}{4}\rho A \omega^2 s_{\max}{}^2 \lambda$$

The *average* rate at which kinetic energy is transported is

$$P_{avg} = \frac{1}{T} K_\lambda = \frac{1}{4}\rho A \omega^2 s_{\max}{}^2 \lambda$$
$$= \frac{1}{4}\rho A \omega^2 s_{\max}{}^2 \frac{\lambda}{T} = \frac{1}{4}\rho A v \omega^2 s_{\max}{}^2$$

We assume that *potential* energy is carried along with the wave at this same average rate. The wave intensity *I*, which is the average rate per unit area at which energy of both kinds is transmitted by the wave, is then,

$$I = \frac{P_{avg}}{A} = \frac{1}{2}\rho v \omega^2 s_{\max}{}^2$$

5. Sound level

The intensity of a sound wave at a surface is the average rate per unit area at which energy is transferred by the wave through or onto the surface,

$$I = \frac{energy / time}{area} = \frac{power}{area} = \frac{P}{A} \text{ (W/m}^2\text{)}$$

where P is the time rate of energy transfer (power) of the sound wave and A is the area of the surface intercepting the sound.

As a sound wave spreads out from its source, its intensity falls off because the area of the wave front grows larger, and therefore the wave energy per unit area grows smaller. The intensity at a distance r from a point source that emits sound waves of power P_s is

$$I = \frac{P_s}{4\pi r^2}.$$

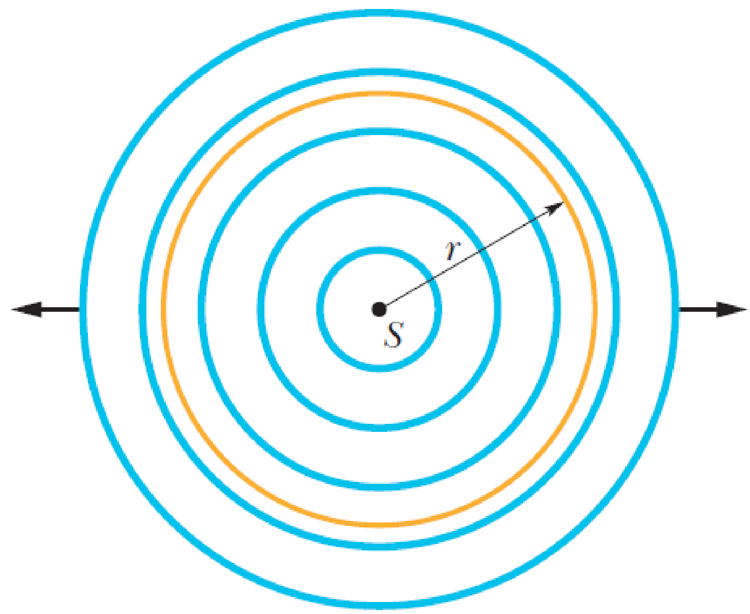


Fig. A point source S emits sound waves uniformly in all directions. The waves pass through an imaginary sphere of radius r that is centered on S.

6. The decibel scale

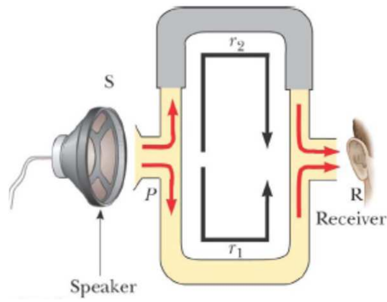
The sound intensity level b of a sound wave is defined by the equation,

$$\beta = 10 dB \log_{10} \frac{I}{I_0}$$

where I is in the units of W/m^2, I_0 is a reference intensity, chosen to be 10^{-12} W/m^2, approximately the threshold of human hearing at 1000 Hz. Sound intensity levels are expressed in decibels, or dB.

7. Interference in Sound Waves

7.1 One source with different paths



Sound from S can reach R by two different paths. The upper path can be varied. A constructive interference occurs when the phase difference is expressed by

$$\Delta\phi = k\Delta r = \frac{2\pi}{\lambda}\Delta r = 2n\pi ,$$

or

$$\Delta r = |\boldsymbol{r}_2 - \boldsymbol{r}_1| = n\lambda .$$

A destructive interference occurs, when the phase difference is expressed by

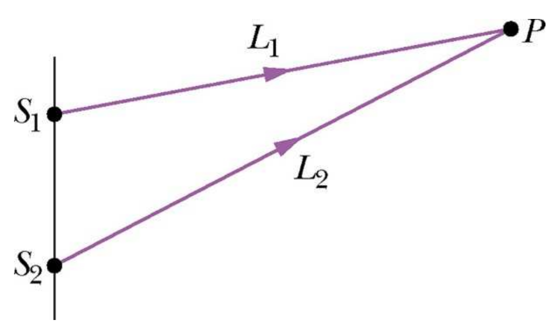
$$\Delta\phi = k\Delta r = \frac{2\pi}{\lambda}\Delta r = (2n+1)\pi ,$$

or

$$\Delta r = |\boldsymbol{r}_2 - \boldsymbol{r}_1| = (n + \frac{1}{2})\lambda$$

A phase difference may arise between two waves generated by the same source when they travel along paths of unequal lengths.

7.2 Two sources with different paths



We consider two point sources of sound waves S_1 and S_2. The two sources are in phase and emit sound waves of the same frequency. Waves from both sources arrive at point P whose distance from S_1 and S_2 is L_1 and L_2, respectively. The two waves interfere at point P. The same interference condition holds valid for the two-point sources located at different places (S_1 and S_2).

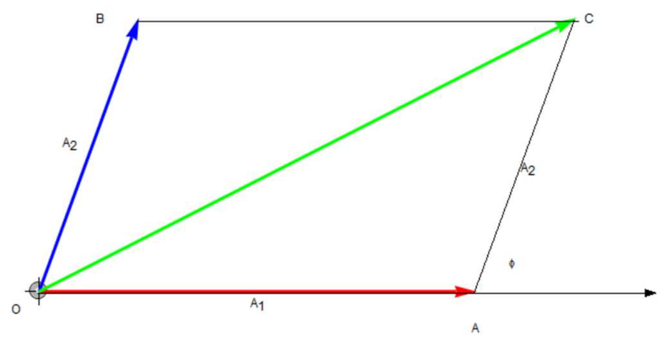
The resultant intensity of two waves can be calculated using the phasor diagram;

Wave-1

$$A_1 \sin(kx - \omega t)$$

Wave-2

$$A_2 \sin(kx - \omega t + \phi)$$



ϕ is the phase difference between the path S_1P and the path S_2P. Then we have the intensity given by

Since the intensity is defined by $I = \frac{1}{2}\rho v \omega^2 s_{\max}{}^2$,

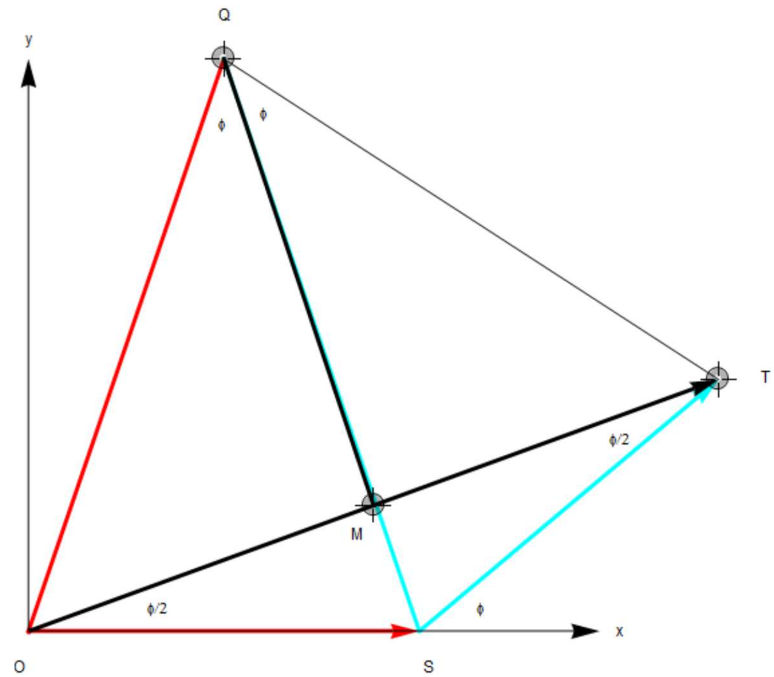
$$I \propto \left|\overrightarrow{OC}\right|^2$$

$$= A_1{}^2 + A_2{}^2 - 2A_1A_2\cos(\pi - \phi)$$

$$= A_1{}^2 + A_2{}^2 + 2A_1A_2\cos\phi$$

7.3 Phasor diagram for $A_1 = A_2$

This phasor diagram is also used for the calculation of the double slits (Young) interference. We consider the sum of the vectors given by $\overrightarrow{OS}$ and $\overrightarrow{ST}$. The magnitudes of these vectors are the same. The angle between $\overrightarrow{OS}$ and $\overrightarrow{ST}$ is ϕ (the phase difference).



In this figure, $\overline{QO} = \overline{QS} = \overline{QT} = R$. $\angle SOM = \angle STM = \phi / 2$. Then we have

$$\overline{OT} = 2\overline{OM} = 2\overline{OS}\cos\frac{\phi}{2} = 2A\cos\frac{\phi}{2}.$$

The resultant intensity is proportional to $\left(\overline{OT}\right)^2$,

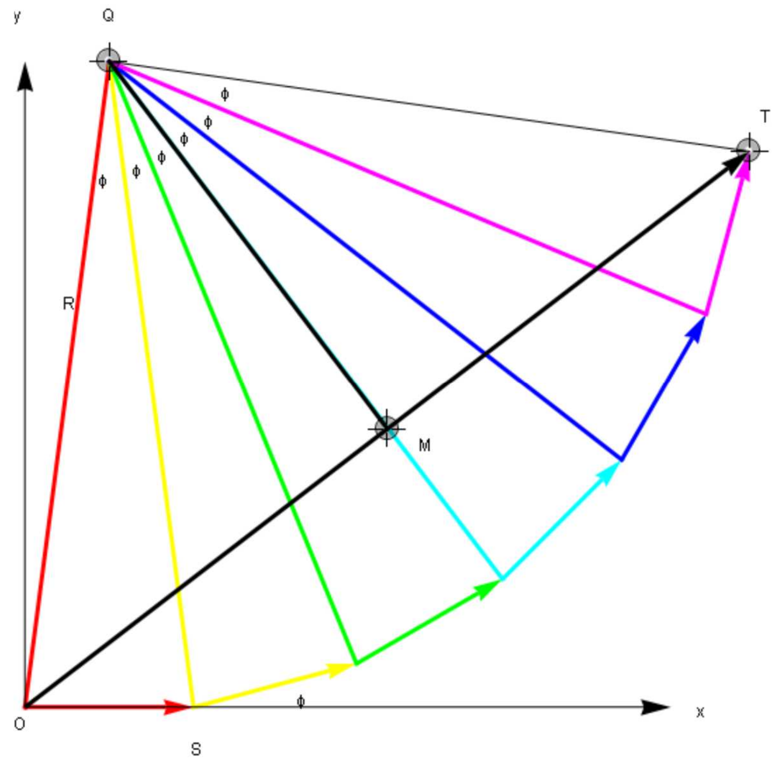
$$I \propto \left(\overline{OT}\right)^2 = 4A^2\cos^2\frac{\phi}{2} = 2A^2(1+\cos\phi).$$

Note that the radius R is related to $\overline{OS}$ (= A) through a relation

$$A = 2R\sin\frac{\phi}{2}.$$

7.4 Interference of many sources with the same amplitude

We consider the case for the interference of N waves.



$$\overline{OQ} = R$$

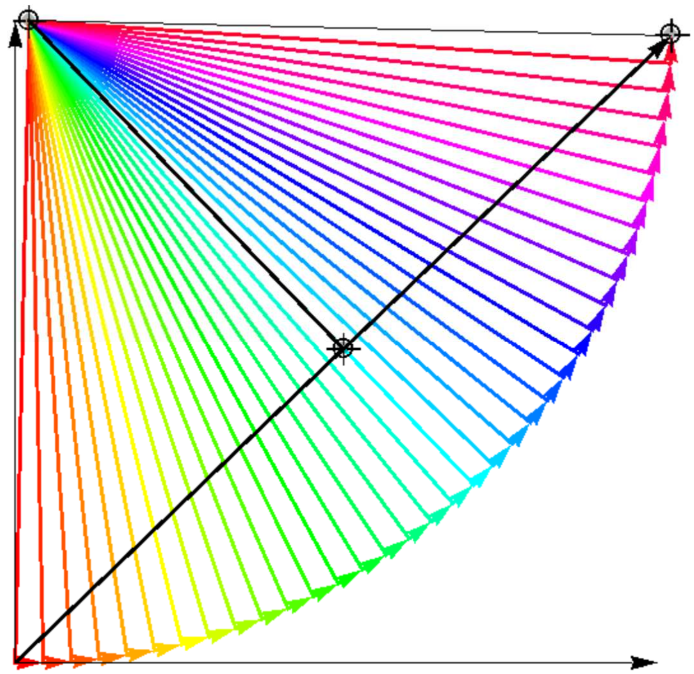
$$\overline{OS} = A = 2R\sin\frac{\phi}{2}$$

$$\overline{OT} = 2\overline{OM} = 2R\sin\frac{N\phi}{2} = A\frac{\sin\frac{N\phi}{2}}{\sin\frac{\phi}{2}}$$

The resultant intensity is proportional to $\left(\overline{OT}\right)^2$,

$$I \propto \left(\overline{OT}\right)^2 = A^2\frac{\sin^2(\frac{N\phi}{2})}{\sin^2(\frac{\phi}{2})}.$$

This intensity is a periodic function of ϕ with the period 2 π.
This is an example for $N = 36$ sources.



8. Standing wave patterns in pipes

8.1 Overview

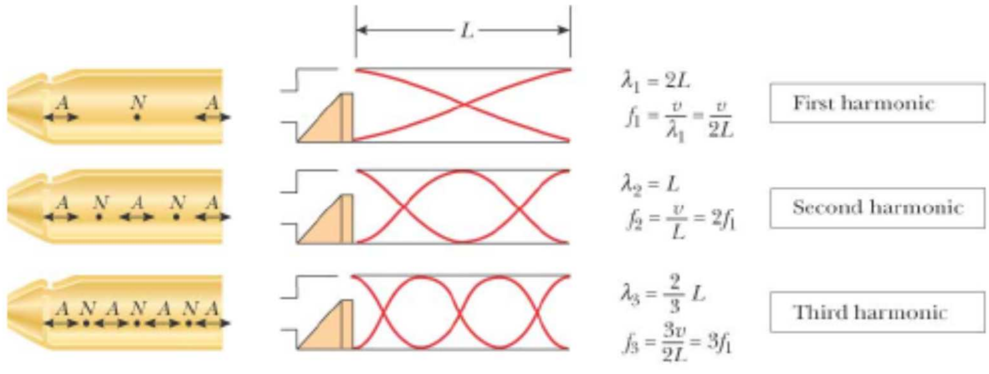
Standing waves can be set up in air columns as the result of interference between longitudinal sound waves traveling in opposite directions. The phase relationship between the incident and reflected waves depends upon whether the end of the pipe is opened or closed.

A closed end of a pipe is a displacement node in the standing wave. The wall at this end will not allow longitudinal motion in the air. The reflected wave is 180° out of phase with the incident wave. The closed end corresponds with a pressure antinode. It is a point of maximum pressure variations

The open end of a pipe is a displacement antinode in the standing wave. As the compression region of the wave exits the open end of the pipe, the constraint of the pipe is removed and the compressed air is free to expand into the atmosphere. The open end corresponds with a pressure node. It is a point of no pressure variation

8.2 Standing waves in an open tube

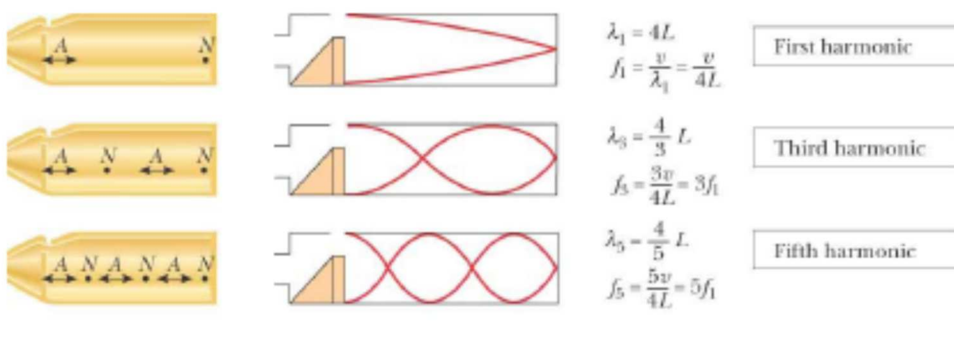
Both ends are displacement antinodes. The fundamental frequency is $v/2L$. This corresponds to the first diagram. The higher harmonics are $f_n = nf_1 = n\,(v/2L)$ where n = 1, 2, 3, …



(a) Open at both ends

8.3 Standing waves in a tube closed at one end

The closed end is a displacement node. The open end is a displacement antinode. The fundamental corresponds to $1/4\lambda$. The frequencies are $f_n = nf = n\,(v/4L)$ where $n = 1, 3, 5, \ldots$

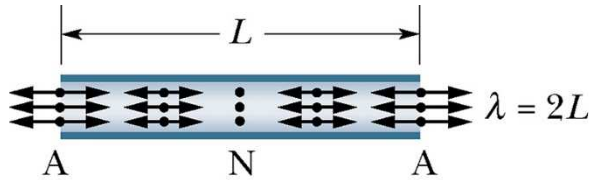


(b) Closed at one end, open at the other

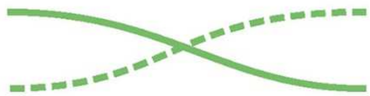
8.4 Conclusion

In conclusion, in a pipe open at both ends, the natural frequencies of oscillation form a harmonic series that includes all integral multiples of the fundamental frequency. In a pipe closed at one end, the natural frequencies of oscillations form a harmonic series that includes only odd integral multiples of the fundamental frequency.

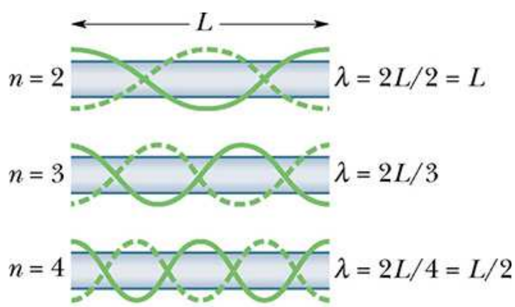
((Note))



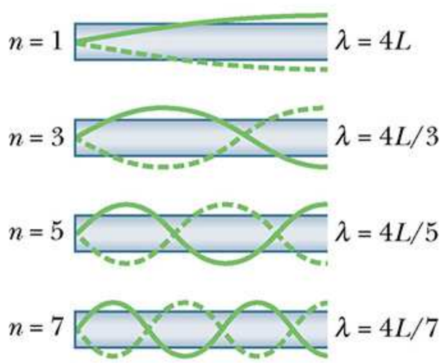
(a)



(b)



(a)



(b)

9 Beat

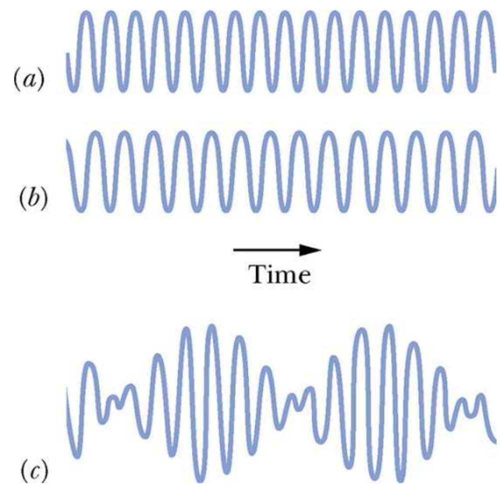
Temporal interference will occur when the interfering waves have slightly different frequencies. Beating is the periodic variation in amplitude at a given point due to the superposition of two waves having slightly different frequencies

Beats arises when two waves having slightly different frequencies, f_1 and f_2, are detected together. The beat frequency is given by

$$f_{net} = f_1 - f_2.$$

or

$$T_{net} = \frac{2\pi}{f_{net}} = \frac{2\pi}{f_1(1-\frac{f_2}{f_1})} = T_1\frac{1}{1-\frac{f_2}{f_1}}$$



The number of amplitude maxima one hears per second is the beat frequency. It equals the difference between the frequencies of the two sources. The human ear can detect a beat frequency up to about 20 beats/sec

We now consider two sound waves given by

$$s_1 = s_0\cos(\omega_1 t)$$
$$s_2 = s_0\cos(\omega_2 t)$$

The resultant displacement is

$$\begin{aligned} s &= s_1 + s_2 \\ &= s_0[\cos(\omega_1 t) + \cos(\omega_2 t)] \\ &= 2s_0\cos[(\frac{\omega_1+\omega_2}{2})t]\cos[(\frac{\omega_1-\omega_2}{2})t] \end{aligned}$$

The first cosine represents the rapid oscillations of the wave at the angular frequency $(\frac{\omega_1+\omega_2}{2})$. The second cosine represents the slow variation in the amplitude of the wave, producing the beat.

Suppose that ω_1 is very close to ω_2: $\omega_1 \approx \omega_2 \approx \omega$ and $\omega_{beat} = \omega_1 - \omega_2$

$$s \approx 2s_0\cos(\omega t)\cos[(\frac{\omega_{beat}}{2})t]$$

T is the repeat time of the slowly varying envelope,

$$T = \frac{T_0}{2} = \frac{1}{2}\frac{2\pi}{\frac{\omega_{beat}}{2}} = \frac{2\pi}{\omega_{beat}}$$

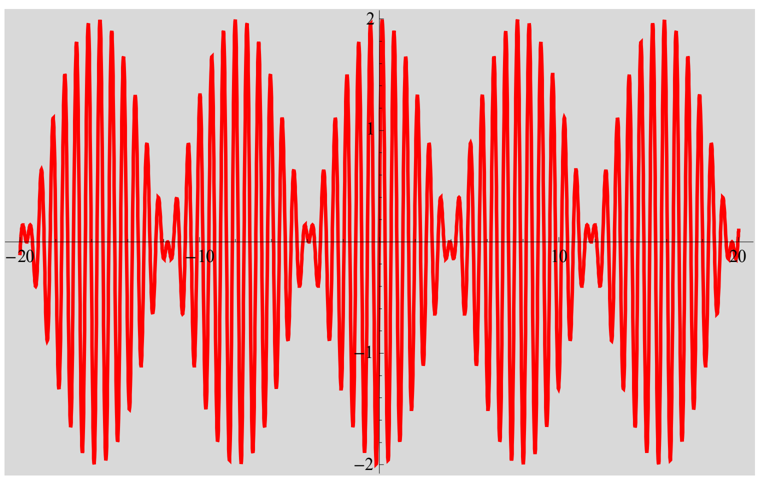
So the beat frequency is

$$f_{\text{beat}} = f_1 - f_2.$$

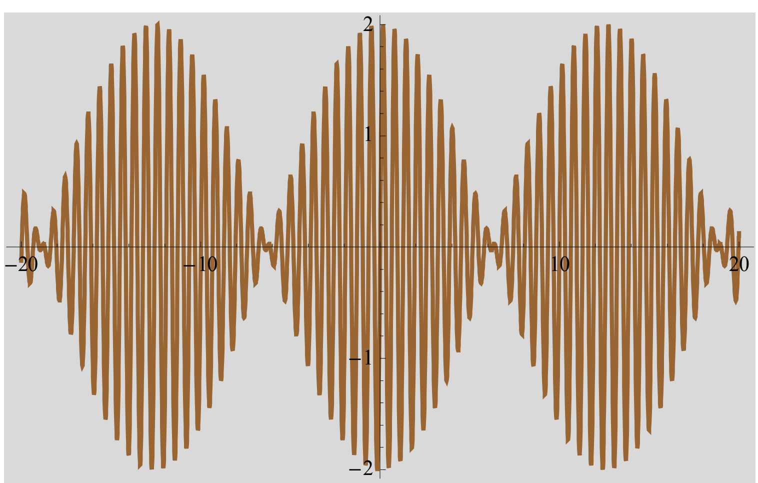
((**Mathematica**))

$$g_1(t) = \sin(10t)$$
$$g_2(t) = \sin(at)$$

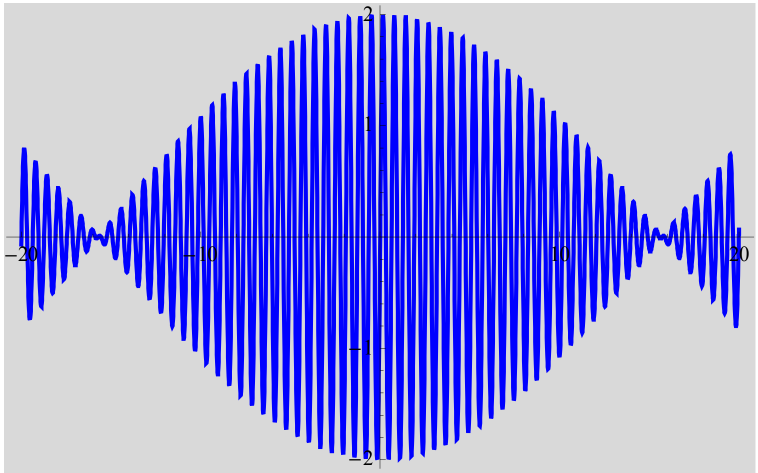
(a) $a = 9.2$; $T_{\text{net}}/T_0 = 12.5$



(b) $a = 9.5$, $T_{\text{net}}/T_0 = 20$.



(c) $a = 9.8$, $T_{\text{net}}/T_0 = 50$.



((**Note**)) **Comment on the beat frequency**

We consider the two waves whose frequencies are very close to each other.

$$\sin[2\pi(f_0 + f_{\text{mod}})t] + \sin[2\pi(f_0 - f_{\text{mod}})t] = 2\sin(2\pi f_0 t)\cos(2\pi f_{\text{mod}} t)$$

The beat frequency is

$$f_{beat} = (f + f_{\text{mod}}) - (f - f_{\text{mod}}) = 2 f_{\text{mod}}$$

The period of the wave $\sin(2\pi f_0 t)$ is $T_0 = \dfrac{1}{f_0}$, while the period of the wave $\cos(2\pi f_{\text{mod}} t)$ is

$$\Delta t = \frac{1}{f_{\text{mod}}}.$$

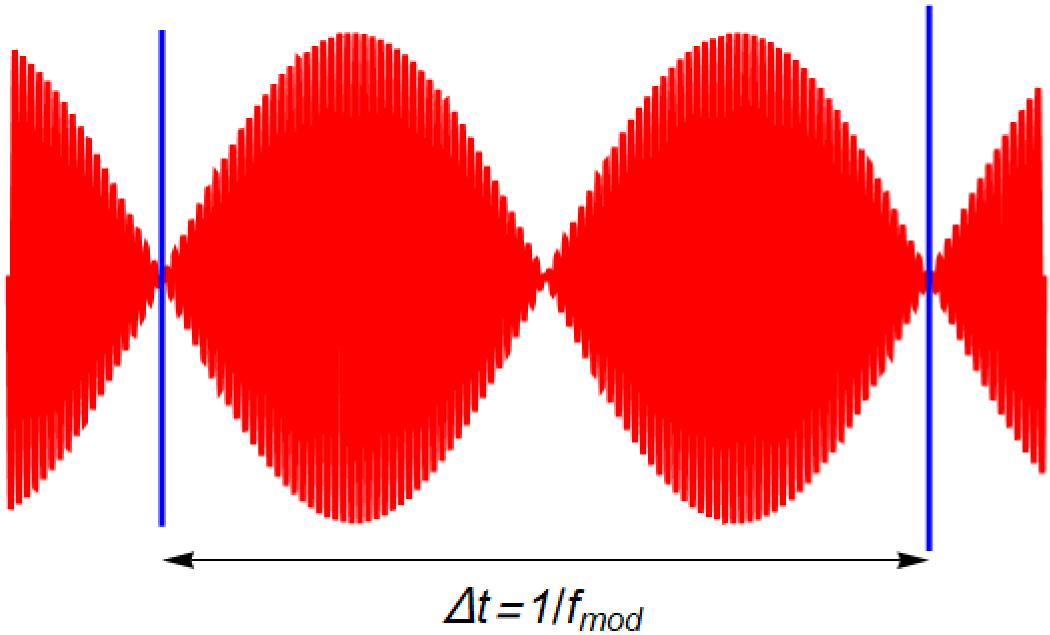


Fig. Beat pattern with two frequencies with $(f + f_{\text{mod}})$ and $(f - f_{\text{mod}})$ with the average frequency *f*. The oscillation of envelope with the frequency f_{mod} is superimposed with the fast oscillation with frequency *f*. $\Delta t = \dfrac{1}{f_{\text{mod}}}$.

$f_{beat} = 2 f_{\text{mod}} = \dfrac{2}{\Delta t}$.

10. Wave analysis: Fast Fourier transform ((Mathematica))

Method of Fast Fourier Transform
1. Give the number of divisons, N.
2. Give the upper limit of the time domain, T.
3. Calculate the distance of each division, Δ=T/N
4. Make the list of the value of f(t) at t = kΔ, where k = 1, 2, 3, ... N.
5. Use the program "Fourier, for the FFT .
6. Claculate the division of the Frequency domain, $\omega 0=2\pi/T$
7. Clacculate the upper limit of rge frequency domain, $\omega max=2\pi/\Delta$
8. One get the Fourier spectrum as a function of k (0, 1, ..., N) in the units of $\omega 0$.

((Example-1))FFT analysis of superposition of waves with several different frequencies

```
N1 = 2048; T = 10 π; Δ = T/N1
```

$\frac{5\pi}{1024}$

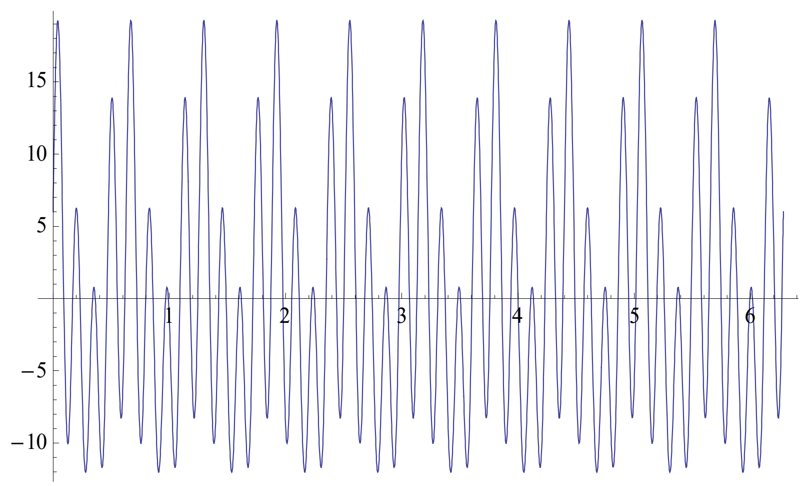
```
ω0 = 2 π/T
```

$\frac{1}{5}$

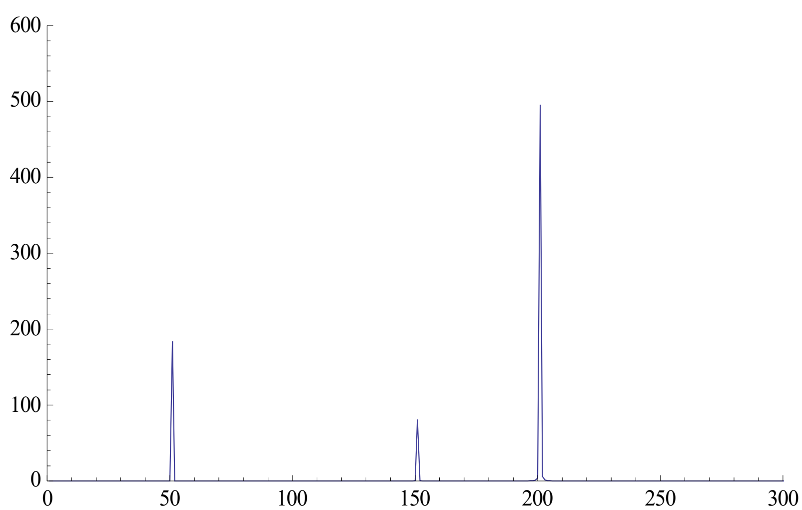
```
f1[t_] = 6 Cos[10 t] + 4 Sin[30 t]  + 10 Sin[40 t]
```

6 Cos[10 t] + 4 Sin[30 t] + 10 Sin[40 t]

```
Plot[f1[t], {t, 0, 2 π}]
```




```
list1 = Table[f1[T/N1 n], {n, 0, 2048}] // N; list2 = Fourier[list1] // Chop;
ListPlot[Abs[list2]^2 / 100, Joined → True, PlotRange → {{0, 300}, {0, 600}}]
```

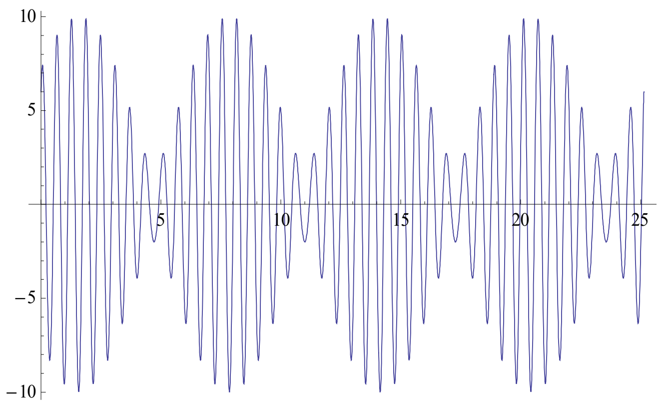


((**Example-2**)) FFT analysis of beat signal

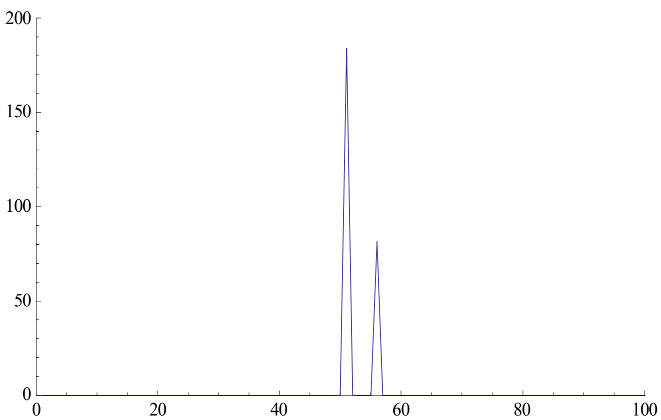
```
g1[t_] = 6 Cos[10 t] + 4 Sin[11 t]
```

```
6 Cos[10 t] + 4 Sin[11 t]
```

```
Plot[g1[t], {t, 0, 8 π}]
```

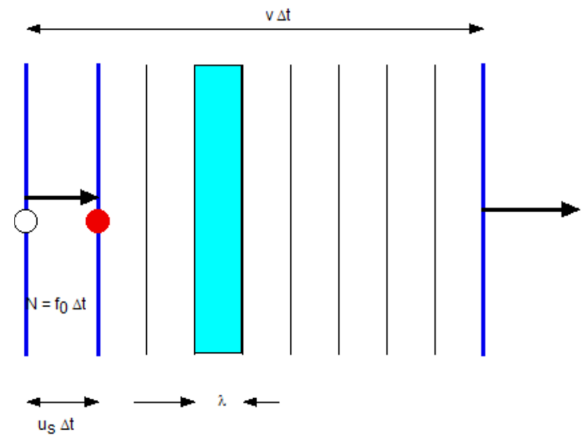



```
list1 = Table[g1[T/N1 n], {n, 0, 2048}] // N; list2 = Fourier[list1] // Chop;
ListPlot[Abs[list2]^2 / 100, Joined → True, PlotRange → {{0, 100}, {0, 200}}]
```

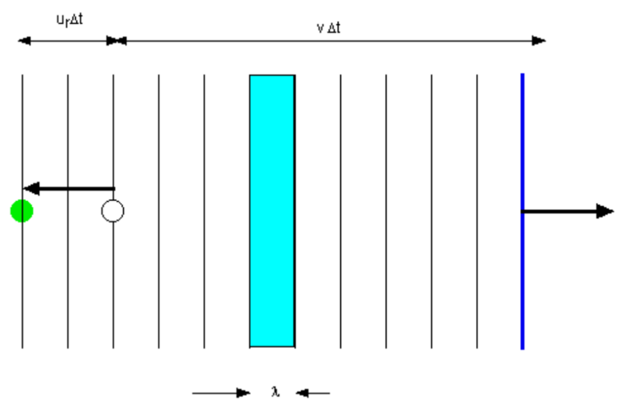


11. Doppler effect

The Doppler effect is the apparent change in frequency (or wavelength) that occurs because of motion of the source or observer of a wave. When the motion of the source or the observer is toward the other, the frequency appears to increase. When the motion of the source or observer is away from the other, the frequency appears to decrease



$$\lambda = \frac{(v-u_s)\Delta t}{N} = \frac{(v-u_s)\Delta t}{f_0 \Delta t} = \frac{v-u_s}{f_0}$$



$$f\Delta t = \frac{(u_r + v)\Delta t}{\lambda}$$

$$f = \frac{(u_r + v)}{\lambda} = \left(\frac{v+u_r}{v-u_s}\right) f_0$$

u_s (>0) is the velocity of sender approaching the receiver.
u_r (>0) is the velocity of the receiver approaching the sender.
f_0 is the frequency of the sender and f is the frequency of the receiver.

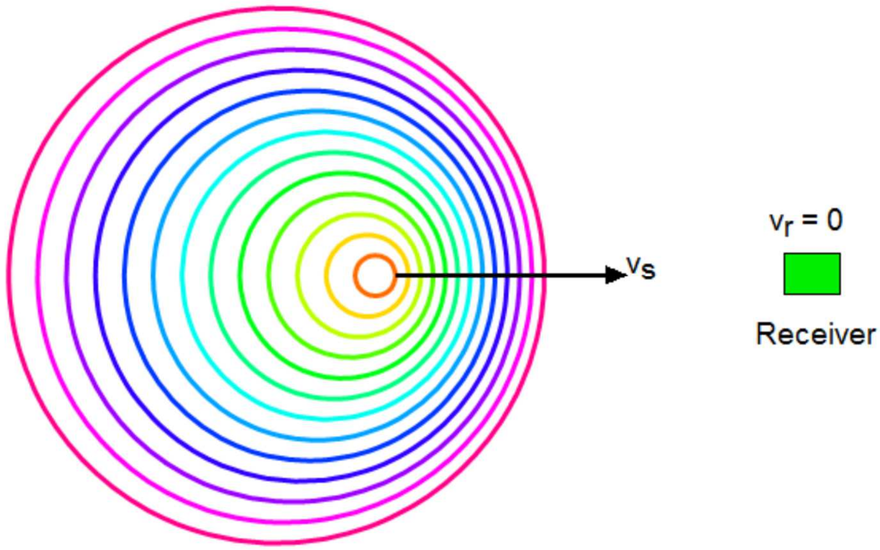


Fig. A receiver is stationary and a source is moving toward the receiver at the velocity v_s. v is the velocity of sound. $v_s < v$.

12. Example

12.1 Pie eater problem to understand the Doppler effect

A conveyor belt moves to the right with a speed v = 300 m min^{-1}. A very fast piemaker puts pies on the belt at a rate of 20 per minute, and they are received at the other end by a pie eater.

(a) If the piemaker is stationary, find the spacing λ between the pies and the frequency f with which they are received by the stationary pie eater.

(b) The piemaker now walks with a speed of 30 m min^{-1} toward the receiver while continuing to put pies on the belt at 20 per minute. Find the spacing of the pies and the frequency with which they are received by the stationary pie eater.

(c) Repeat your calculations for a stationary piemaker and a pie eater who moves toward the piemaker at 30 m min^{-1}.

((**Solution**))

(a) Since 20 pies per minute are placed on the belt, 20 pies per minute will be received. The time between the placing of the pies is 0.05 min, and during that time the belt moves 300 x 0.05 m = 15 m. Consequently, the spacing between the pies is λ = 15 m.

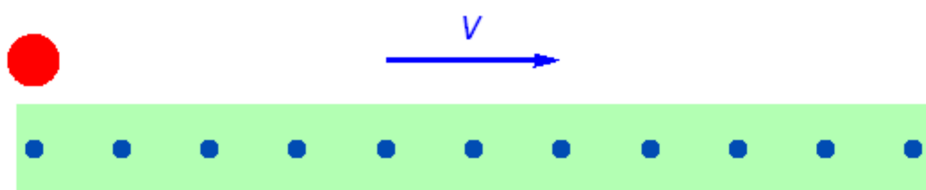
(b) Relative to the piemaker, the belt moves at 270 m min^{-1}. Consequently, λ = 270 x 0.05 m = 13.5 m. Since the pies are traveling toward the receiver at 300 m min^{-1}, the number received per minute, i.e., the frequency, is given by f = 300/13.5 min^{-1} = 22.2 min^{-1}.

(c) If the receiver moves toward the piemaker, the spacing of the pies on the belt is, as in (a), 15 m. However, the speed of the belt relative to the receiver is 330 m min^{-1}, so the frequency f = 330/15 min^{-1} = 22 min^{-1}.

((**Note**))

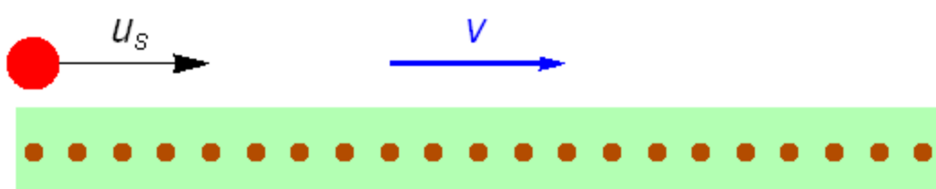
The red circle (the pie maker). The blue circle (the pies). The green band (the belt conveyor). The wavelength (the separation distance between pies) depends on the velocity of the belt conveyor and the velocity of the pie maker. It has nothing to do with the movement of the pie eater.

(a) The pie maker (at rest)



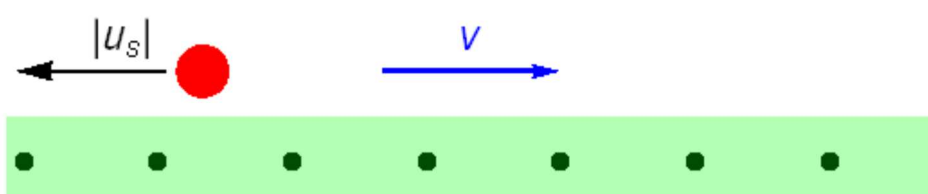
$$\lambda' = \lambda_0 = \frac{v}{f_0}$$ for $u_s = 0$ (pie maker at rest).

(b) The pie maker moving along the +*x* axis



$$\lambda' = \frac{v - u_s}{f_0} < \frac{v}{f_0}$$ for $u_s > 0$ (sender moving to the +*x* axis).

(c) The pie maker moving along the (-*x*) axis

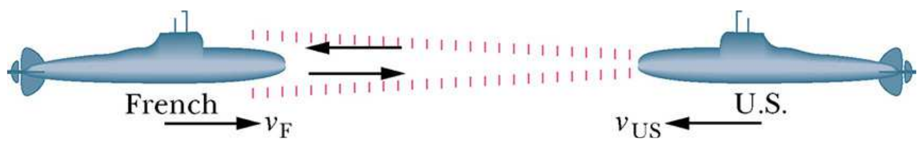


$$\lambda' = \frac{v + |u_s|}{f_0} > \frac{v}{f_0}$$ for $u_s < 0$ (sender moving to the -x axis).

12.2 Example-2

Problem 17-61 (SP-17)

In Fig., a French submarine and a U.S. submarine move toward each other during maneuvers in motionless water in the North Antarctic. The French sub moves at speed v_F = 50.00 km/h, and the U.S. sub at v_{US} = 70.00 km/h. The French sub sends a sonar signal (sound wave in water) at 1.000 x 10^3 Hz. Sonar waves travel at 5470 km/h. (a) What is the signal's frequency as detected by the U.S. sub? (b) What frequency is detected by the French sub in the signal reflected back to it by the U.S. sub?



((**Solution**))

v_F = 50 km/h

v_{US} = 70 km/h

f = 1 kHz

v = 5470 km/h

$$f_1 = \left(\frac{v + u_{US}}{v - v_F}\right) f = 1.022 kHz$$

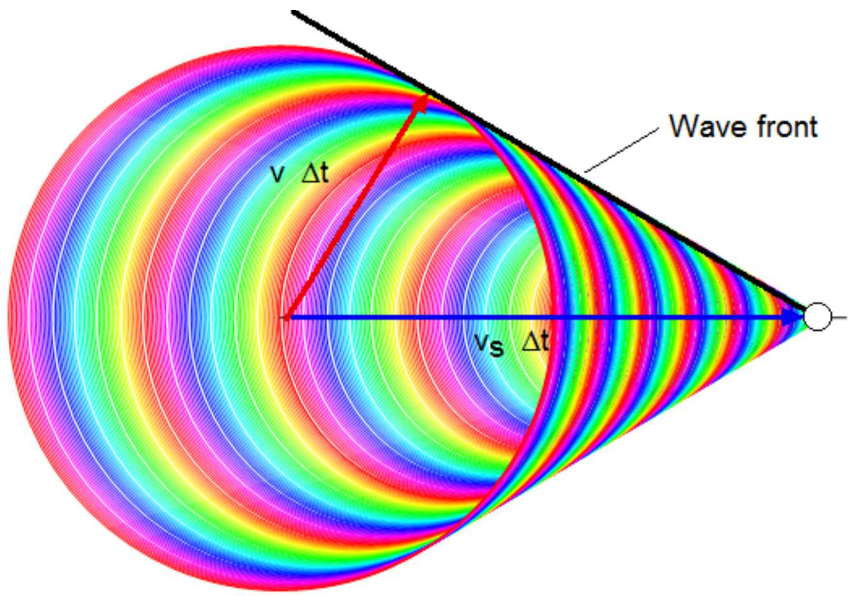
$$f_2 = \left(\frac{v + v_F}{v - u_{US}}\right) f_1 = 1.0447 kHz$$

13. Shock waves

If the speed of a source relative to the medium exceeds the speed of sound in the medium, the Doppler equation no longer applies. In such a case, shock waves result. We have a series of wave circles with a common tangent line which goes through the center of the source. These series of circle waves form a wavefront which forms a cone in 3D or a pair of lines in 2D. The half-angle θ of the cone is given by

$$\sin\theta = \frac{v}{v_s}$$

where v is the velocity of sound and v_s is the velocity of the object (source). The ratio v_s/v is called the Mach number. The concentration of energy in wavefront of the source results in a shock wave.



((**Note**))

(a) v_s/v = 1.0 (critical case)

Fig. A receiver is stationary and a source is moving toward the receiver at the velocity v_s. v is the velocity of sound. $v_s = v$.

(b) v_s/v = 1.5

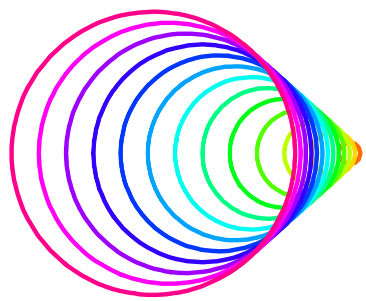


Fig. A receiver is stationary and a source is moving toward the receiver at the velocity v_s. v is the velocity of sound. $v_s / v = 1.5$

((**Note**))

1 Mach = 768 miles/h = 340 m/s

14. Sonic boom

A sonic boom is the sound associated with the shock waves created whenever an object traveling through the air travels faster than the speed of sound. Sonic booms generate enormous amounts of sound energy, sounding similar to an explosion or a thunderclap to the human ear. Sonic booms due to large supersonic aircraft can be particularly loud and startling, tend to awaken people, and may cause minor damage to some structures. A sonic boom does not occur only at the moment an object crosses the speed of sound; and neither is it heard in all directions emanating from the speeding object. Rather the boom is a continuous effect that occurs while the object is travelling at supersonic speeds. But it only affects observers that are positioned at a point that intersects a region in the shape of geometrical cone behind the object. As the object moves, this conical region also moves behind it and when the cone passes over the observer, they will briefly experience the boom.

((**Note**))
The velocity of sound is given by

$$v_{sound} = 340 \text{ m/s} = 760.56 \text{ miles/h}$$

$$\sin\alpha = \frac{v_{sound}}{v_{plane}} = \frac{1}{Ma} = \sin(20°) = 0.342031$$

$v_{plane} = 2223.73$ miles/h or $v_{plane} = 994.1$ m/s

where Ma is the Mach number and is defined by

$$Ma = \frac{v_{plane}}{v_{sound}}$$

((**Example-1**))
Twin sonic booms from space shuttle Atlantis 2008
https://www.youtube.com/watch?v=lNL4HHFG8H4

Space Shuttle Discovery Landing (STS-131)
https://www.youtube.com/watch?v=6k70hn4-ffc

The space shuttle faster than the speed of sound create two sonic booms. The first one is created at the front of the plane, where the nose presses on the air it runs into. The second is made at the rear, where the tail leaves an empty space behind it. At each end, the air pressure is strongly changed by the plane, creating sound waves. The wave fronts come from the nose and tail as being two cones, separated by the length of the plane. The time between two sonic booms is the time it takes for the plane to fly its own length.

((**Example 2**))

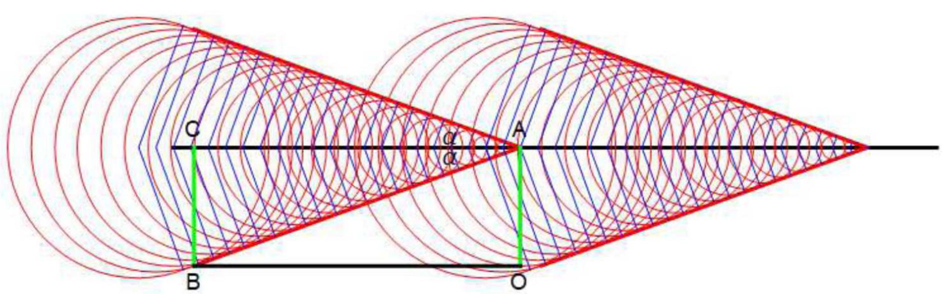


Fig. Sonic boom. The observer is at the point O. $\overline{OA} = h$, $\overline{OB} = \dfrac{h}{\tan\alpha}$, and $\overline{AB} = \dfrac{h}{\sin\alpha}$.

Suppose that an airplane passed at the height h = 5000 m, just above an observer (at the point O), with the velocity of plane v_{plane}, where the sound velocity is $v_{sound} = 340$ m/s and the angle α (=20°).

$$\sin\alpha = \frac{v_{sound}}{v_{plane}} = \frac{340}{994.1} = 0.3420 = \frac{1}{k}$$

Thus the airplane flies at the velocity of Mach (k =2.924); $v_{plane} = 994$ m/s. The wave front of the sonic boom will arrive at the observer (at the point O) at the time

$$\Delta t = \frac{h}{v_{plane}\tan\alpha} = 13.8 \text{ s},$$

after the airplane passes just above the observer (at the point O).

((Note))

$$\overline{AB} = \frac{h}{\sin\alpha} = 14.62 \text{ km.} \qquad \overline{OB} = \frac{h}{\tan\alpha} = 13.74 \text{ km}$$

15. Red shift and Hubble's law (special relativity)

We suppose that a source is located at the origin of the reference frame S. An observer moves relative to S at velocity v. So that he is at rest in S' (in Fig we use S_1 instead of S' for convenience). According to the special relativity, we obtain the Doppler effect for the light as

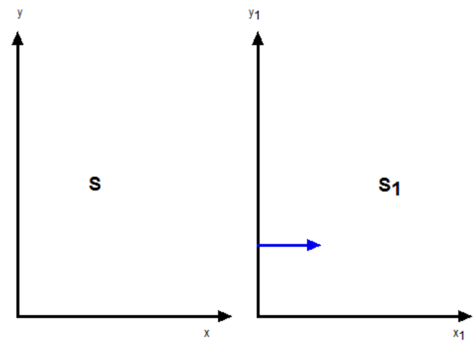
$$\lambda' = \frac{\lambda}{\gamma(1-\frac{v}{c})} = \lambda\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}} \approx \lambda(1+\frac{v}{c}+\frac{v^2}{2c^2}+...)$$

and

$$f' = f\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}}$$

where c is the speed of light and $\lambda f = \lambda' f' = c$. This means that a spectral line that normally has a wavelength λ is observed at a longer wavelength λ'. Note that

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

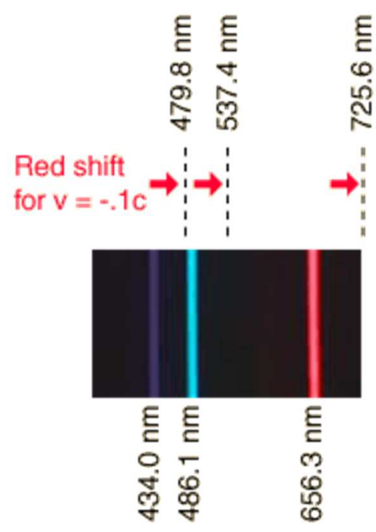


The spectral line is shifted by an amount of $\Delta\lambda = \lambda' - \lambda$. The red shift of the galaxy (usually noted by *z*) is given by

$$z = \frac{\lambda' - \lambda}{\lambda} \approx \frac{v}{c}$$

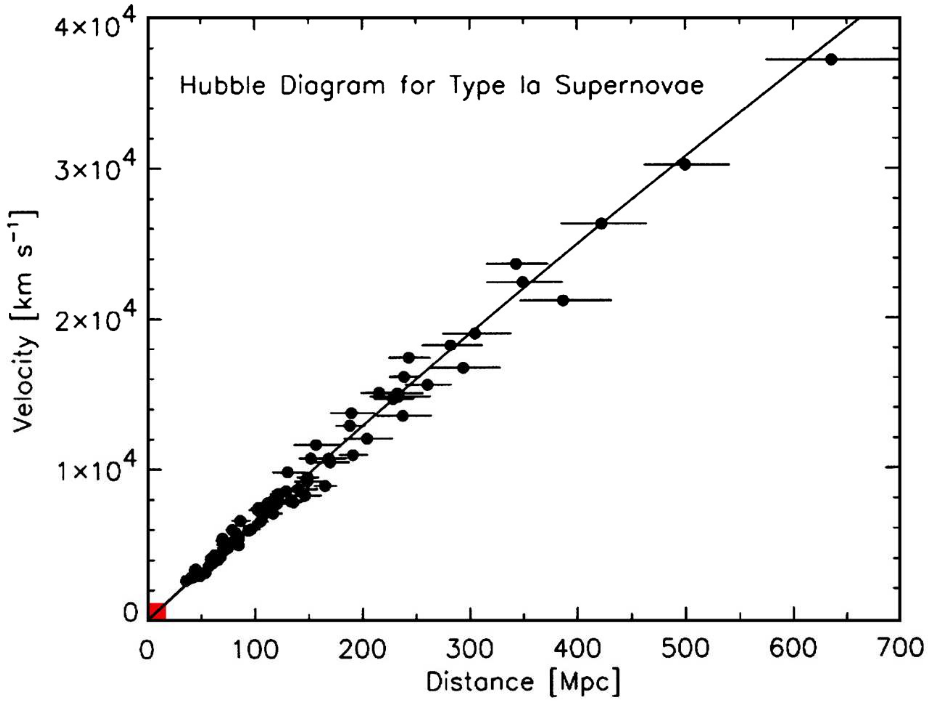
(a) The red shift

The light from distant stars and more distant galaxies is not featureless, but has distinct spectral features characteristic of the atoms in the gases around the stars. When these spectra are examined, they are found to be shifted toward the red end of the spectrum. This shift is apparently a Doppler shift and indicates that essentially all of the galaxies are moving away from us. The measured red shifts are usually stated in terms of a *z* parameter. The largest measured *z* values are associated with the quasars.



Hydrogen red-shift example

(b) Hubble's law



The relationship between the distances to galaxies and the red shift is one of the most important astronomical discoveries of the twentieth century. This relation tells us that we are living in an expanding universe. In 1929, Hubble published this discovery. According to the Hubble's law, the recessional velocity v of a galaxy is related to its distance r from the Earth by

$$v = H_0 r,$$

where H_0 is constant commonly called the Hubble constant and $H_0 = 75$ km s^{-1} Mpc^{-1}. Here Mpc is a megaparsecs (parsec, 1 pc = 3.262 ly; light year, 1 ly = 9.462 x 10^{15} m = 63,240 AU).

pc = 3.262 x 9.462 x 10^{15} m = 3.0857 x 10^{16} m.
1 Mpc = 3.0857 x 10^{22} m.= 3.0857 x 10^{19} km

Since $v = zc$ and $v = H_0 r$, then H_0r = zc. Thus the distance to a galaxy is related to its red shift by

$$r = \frac{zc}{H_0}$$

(c) Big Bang

How long ago did the Big Bang take place?

$$T_0 = \frac{r}{v} = \frac{1}{H_0}$$

where T_0 is the same for all galaxies.

$T_0 = 1/(75 \text{ km s}^{-1} \text{ MPc}^{-1})) = \frac{1}{75}\frac{MPc \cdot s}{km} = \frac{1}{75} \cdot \frac{3.09 \times 10^{19}}{3.156 \times 10^{7}} year = 1.3 \times 10^{10}$ years=13 billion years

Note that 1 year=365 x 24 x 60 x 60 = $3.156 \text{x} 10^7$ sec. The age of the solar system is 4.5 billion years.

(d). Dicke and Peebles (1960)

Early universe had been at least as hot as the Sun center, where He is currently produced. The hot early universe must therefore have been filled with many high-energy, short-wavelength photons, which formed a radiation field with that can be given by Planck's blackbody law. The universe has expanded so much since those ancient times that all those short-wavelength photons have their wavelengths stretched by a tremendous factor. As a result, they have become low-energy, long-wavelength photons.

The temperature of this cosmic radiation field is now quite low, only a few degrees above 0 K.

(e). Arno Penzias and Robert Wilson

No matter where in the sky they pointed their antenna, they detected faint background noise.

They had discovered the cooled-down cosmic background radiation left over from the hot Big Bang.

(f). Cosmic Background

Left over from the hot Big Bang.

T = 2.726 K, cosmic microwave background

$$\lambda_{\max} = \frac{0.0029}{T[K]}[m] \text{ (Wien's displacement law)}$$

When T = 2.726 K, $\lambda_{\max} = 1.06[mm]$.

The spectrum of the cosmic microwave background is given by COBE (Cosmic Background Explorer).

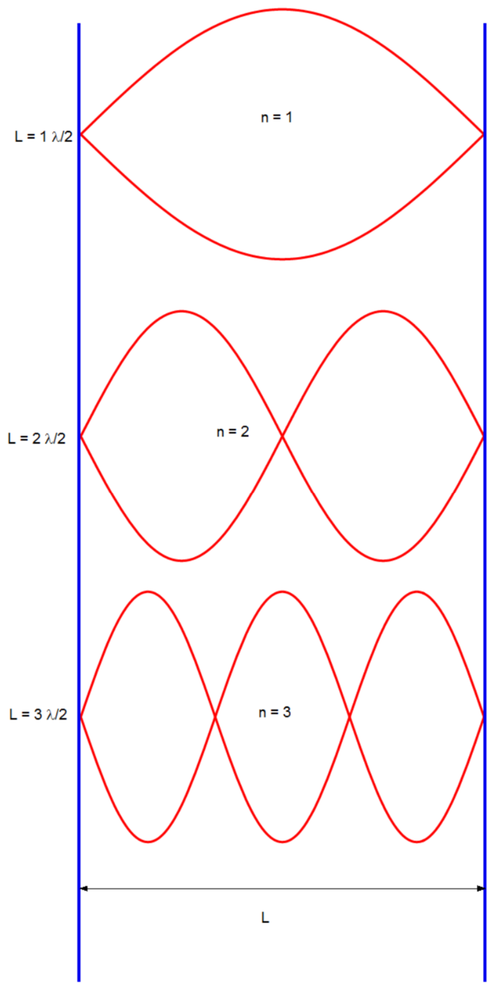
16. Typical problems

16.1 Problem 17-47 (SP-17)

A violin string 30.0 cm long with linear density 0.659 g/m is placed near a loudspeaker that is fed by an audio oscillator of variable frequency. It is found that the string is set into oscillation only at the frequencies 880 and 1320 Hz as the frequency of the oscillator is varied over the range 500 – 1500 Hz. What is the tension in the string?

((**Solution**))
$\mu = 0.650 \times 10^{-3}$ kg/m
f_s= 880 Hz and 1320 Hz
f_{osc} = 500 – 1500 Hz
L = 0.30 m

$$v_s = \sqrt{\frac{T_s}{\mu}}$$



For the *n*-th harmonics,

$$\frac{\lambda}{2} n = L$$

$$\lambda = \frac{2L}{n}$$

The frequency of the string (violin) is

$$f_s = \frac{v_s}{\lambda} = \frac{v_s}{\frac{2L}{n}} = \frac{v_s}{2L}n = \frac{n}{2L}\sqrt{\frac{T_s}{\mu}}$$

$$\Delta f_s = \frac{1}{2L}\sqrt{\frac{T_s}{\mu}} = 1320 - 880 = 440$$

Then we have

$T_s = 45.3$ N

16.2 Problem 17-48

A tube 1.20 m long is closed at one end. A stretched wire is placed near the open end. The wire is 0.330 m long and has a mass of 9.60 g. It is fixed at both ends and oscillates in its fundamental mode. By resonance, it sets the air column in the tube into oscillation at that column's fundamental frequency. Find (a) that frequency and (b) the tension in the wire.

((**Solution**))

$L = 1.20$ m	(tube length)
$L_0 = 0.330$ m	(wire length)
$m = 9.60$ g $= 9.60 \times 10^{-3}$ kg	(mass of wire)

The fundamental frequency of the wire whose ends are fixed.

The wavelength:

$$L_0 = \frac{\lambda_0}{2} \qquad \lambda_0 = 2L_0 = 0.66 \text{ m}$$

The velocity:

$$v_0 = \sqrt{\frac{T_s}{\mu}}$$

where T_s is the tension and μ is the mass per unit length,

$$\mu = \frac{m}{L_0} = \frac{9.60 \times 10^{-3} kg}{0.330 m} = 29.09 \times 10^{-3} kg/m$$

Then the fundamental frequency f_0 is

$$f_0 = \frac{v_0}{\lambda_0} = \sqrt{\frac{T_s}{\mu}}\frac{1}{2L_0}$$

Column's fundamental frequency

$$L = \frac{\lambda}{4} \qquad \lambda = 4L$$

$$f = \frac{v}{\lambda} = \frac{v}{4L}$$

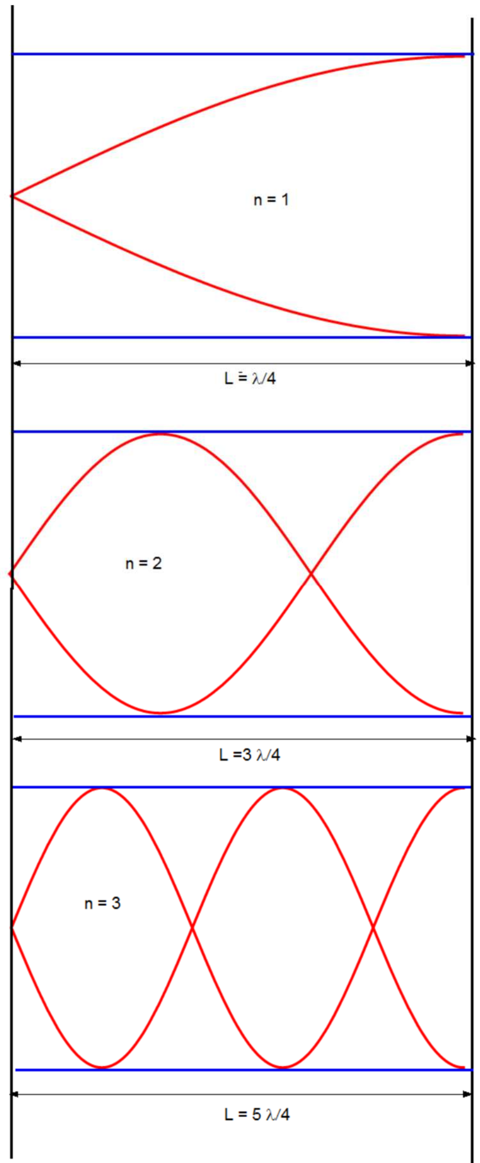
where v (= 343 m/s) is the velocity of sound.

In resonance, we have $f = f_0$.

$$\frac{v}{4L} = \sqrt{\frac{T_s}{\mu}}\frac{1}{2L_0}$$

or

T_s = 65.0 N.



16.4 Problem 17-34 (SP-17)

Party hearing. As the number of people at a party increases, you must raise your voice for a listener to hear you against the background noise of the other partygoers. However, once you reach the level of yelling, the only way you can be heard is if you move closer to your listener, into the listener's "personal space." Model the situation by replacing you with an isotropic point source of fixed power P and replacing your listener with a point that absorbs part of your sound waves. These points are initially separated by r_i = 1.20 m. If the background noise increases by $\Delta\beta$ = 5 dB, the sound level at your listener must also increase. What separation r_f is then required?

((**Solution**))

r_i = 1.20 m

$I_0 = 10^{-12}$ W/m^2

The intensity I_i is given by

$$I_i = \frac{P}{4\pi r_i^2}$$

$$\beta_i = 10\log_{10}\frac{I_i}{I_0} \text{ (dB)}$$

For $r = r_i$,

$$\beta_i = 10\log_{10}(\frac{1}{I_0}\frac{P}{4\pi r_i^2}) = 10\log_{10}(\frac{1}{I_0}\frac{P}{4\pi}) - 20\log_{10} r_i$$

For $r = r_f$,

$$\beta_f = 10\log_{10}(\frac{1}{I_0}\frac{P}{4\pi r_f^2}) = 10\log_{10}(\frac{1}{I_0}\frac{P}{4\pi}) - 20\log_{10} r_f$$

Then we have

$$\Delta\beta = \beta_f - \beta_i = 20\log_{10}\frac{r_i}{r_f} = 5 \text{ dB}$$

or

$$\log_{10}\frac{r_i}{r_f} = \frac{5}{20} = 0.25$$

$$\frac{r_i}{r_f} = 10^{0.25}$$

Then we get

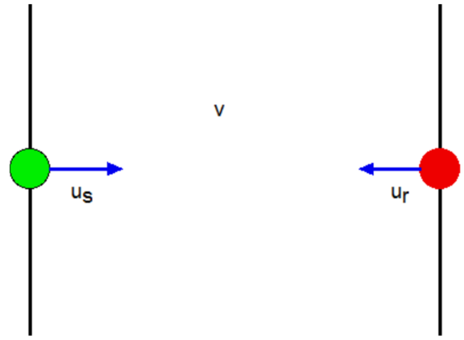
$$r_f = 10^{-0.25} r_i = 0.675m$$

16.5 Problem 17-66 (SP-17); Doppler effect

Two trains are travelling toward each other at 30.5 m/s relative to the ground. One train is blowing a whistle at 500 Hz. (a) What frequency is heard on the other train in still air? (b)What frequency is heard on the other train if the wind is blowing at 30.5 m/s toward the whistle and away from the listener? (c) What frequency is heard if the wind direction is reversed?

((**Solution**))

$v = 343$ m/s
$f = 500$ Hz
$u_r = u_s = 30.5$ m/s



(a)

$$f_1 = \left(\frac{v+u_r}{v-u_s}\right) f = 597.6 Hz$$

(b) $v_1 = v - 30.5 = 312.5$ m/s

$$f_2 = \left(\frac{v_1+u_r}{v_1-u_s}\right) f = 608.2 Hz$$

(c) $v_2 = v + 30.5 = 373.5$ m/s

$$f_2 = \left(\frac{v_2+u_r}{v_2-u_s}\right) f = 588.9 Hz$$

APPENDIX-1 Geometry of phasor diagram

Proof of the geometry for the phasor diagram

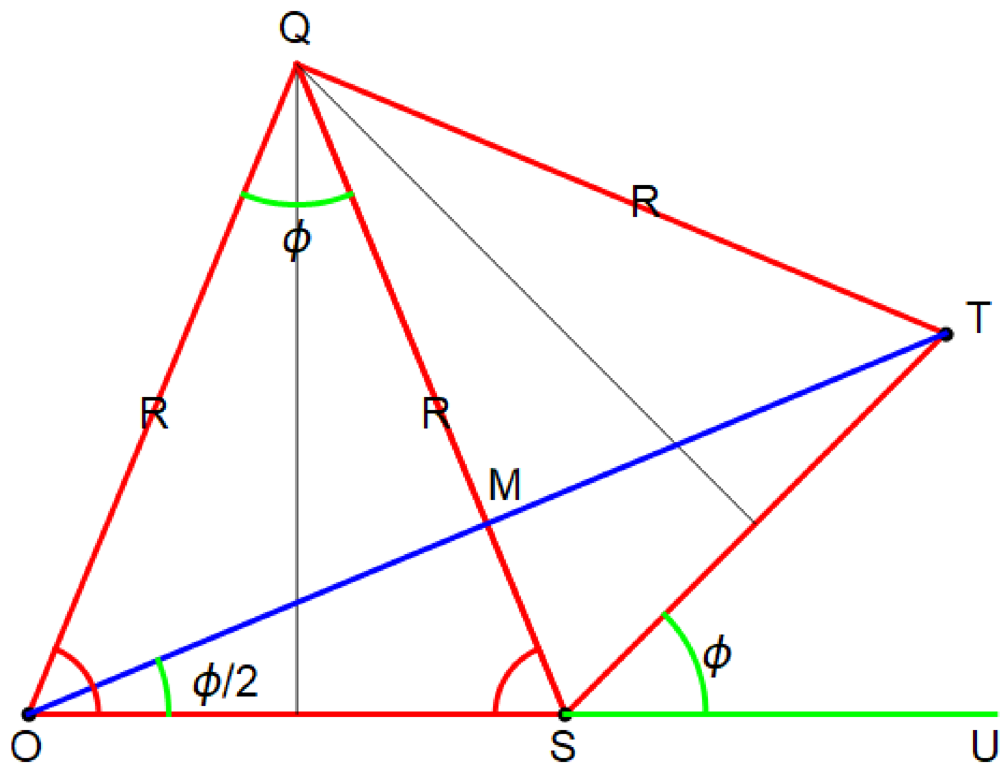


Fig. $\theta = \phi$. $\alpha = \dfrac{\pi - \phi}{2}$. $\overline{OS} = 2R\sin\dfrac{\phi}{2}$. $\overline{OT} = 2\overline{OM} = 2\overline{OS}\sin\dfrac{\phi}{2} = 4R\sin^2\dfrac{\phi}{2}$

$\angle SOM = \angle STM = \dfrac{\phi}{2}$.

We consider two isosceles triangles.

$$\overline{QO} = \overline{QS} = \overline{QT} = R$$
$$\angle QOS = \angle QSO = \angle QST = \angle QTS = \alpha$$

In other words, points O, S, and T lie on a circle with radius R (the value of R will be specified later). We assume that

$$\angle TSU = \phi$$

From the geometry shown in the Fig, we have

$$\theta + 2\alpha = \pi$$
$$2\alpha + \phi = \pi$$

Then we get

$$\theta = \phi$$

The radius R and the side $\overline{OS}$ are related as

$$\overline{OS} = 2R\sin\frac{\theta}{2} = 2R\sin\frac{\phi}{2}.$$

This means that R is uniquely determined when the angle ϕ and $\overline{OS}$ are given. In Fig., ΔOST is a isosceles triangle with $\overline{OS} = \overline{ST}$. When M is the midpoint of the side $\overline{OT}$, it is found that $\overline{SM}$ is perpendicular to $\overline{OT}$. We also get

$$\angle SOM = \angle STM = \frac{\phi}{2}$$

Finally, we have

$$\overline{OT} = 2\overline{OM} = 2\overline{OS}\sin\frac{\phi}{2} = 4R\sin^2\frac{\phi}{2}$$

APPENDIX-2 Laser cooling of alkali metal atom

In one experiment by Cornell and Wieman, a Bose-Einstein condensate contained 2000 ^{87}Rb atoms within a volume of about 10^{-15} m^3. Estimate the temperature at which Bose-Einstein condensation should have occurred.

For Rb atom

m = 85.4678 u,

N = 2000, $V = 10^{-15}$ m^3.

Then T_E can be evaluated as

T_E = 29.8441 nK.

The lase cooling of alkali metal atoms consists of radiation pressure and the doppler effect. The radiation pressure arises from the spontaneous emission. The velocity of atoms is decreased by the radiation pressure. The velocity of the atoms is uniquely determined from the Doppler effect. As the velocity is decreased due to the radiation pressure, the frequency of the laser should be changed appropriately. Again, from the Doppler effect again, the specified velocity is uniquely selected. The velocity is decreased due to the radiation pressure from the laser.

(A) Radiation pressure

How can we get such a very low temperature? In order to achieve the lowest temperature, we use the laser cooling techniques. The temperature of the atoms is linearly proportional to the kinetic energy of atoms. So we need to reduce the velocity of atoms.

((Example))
Rb atom

$v_{\rm rms}$ = 295.89 m/s at T = 300 K
$v_{\rm rms}$ = 17.083 m/s at T = 1 K
$v_{\rm rms}$ = 0.54 m/s at T = 1 mK
$v_{\rm rms}$ = 0.017 m/s at T = 1 μ K

The force that light could exert on matter are well understood. Maxwell's calculation of the momentum flux density of light, and the laboratory observation of light pressure on macroscopic object by Lebedxev and by Nichols and Hull provided the first quantitative understanding of how light could exert forces on material object. Einstein pointed out the quantum nature of this force: an atom that absorbs a photon of energy $h\nu$ will receive a momentum

$$G = \frac{h\nu}{c},$$

along the direction of incoming photon. If the atom emits a photon with momentum, the atom will recoil in the opposite direction. Thus the atom experiences a net momentum change

$$\Delta \boldsymbol{p}_{atom} = \boldsymbol{P}_{in} - \boldsymbol{P}_{out},$$

due to the incoherent scattering process. Since the scattered photon has no preferred direction, the net effect is due to the absorbed photon, resulting in scattering force,

$$\boldsymbol{F}_{scatt} = N\boldsymbol{p}_{in},$$

where N is the number of photons scattered per second. Typical scattering rates for atoms excited by a laser tuned to a strong resonance line are on the order of 10^7 - 10^8/s. The velocity of Na atom changes by 3cm/s.

$$m\frac{\Delta v}{\Delta t} = m\frac{\Delta v}{(1/N)} = p_{in} = \frac{h}{\lambda} \qquad \text{(per one photon).}$$

or

$$\Delta v = \frac{h}{Nm\lambda}.$$

After 1 sec, the velocity changes as

$$N\Delta v = \frac{h}{m\lambda} = 2.947 \text{ cm/s},$$

for the *N* photons, where *m* = 22.9897 u for Na atom and λ = 589.0 nm for the Na D line.

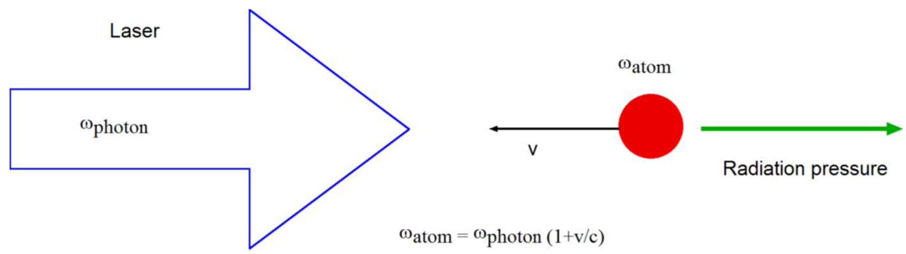


Fig. With the laser tuned to below the peak of atomic resonance. Due to the Doppler shift, atoms moving in the direction opposite the laser beam will scatter photons at a higher rate than those moving in the same direction as the beam. This leads to a larger force on the counter-propagating atoms.

((**Model**)) **Schematic explanation for the radiation pressure**

(a), (b), (c) (d), (e)

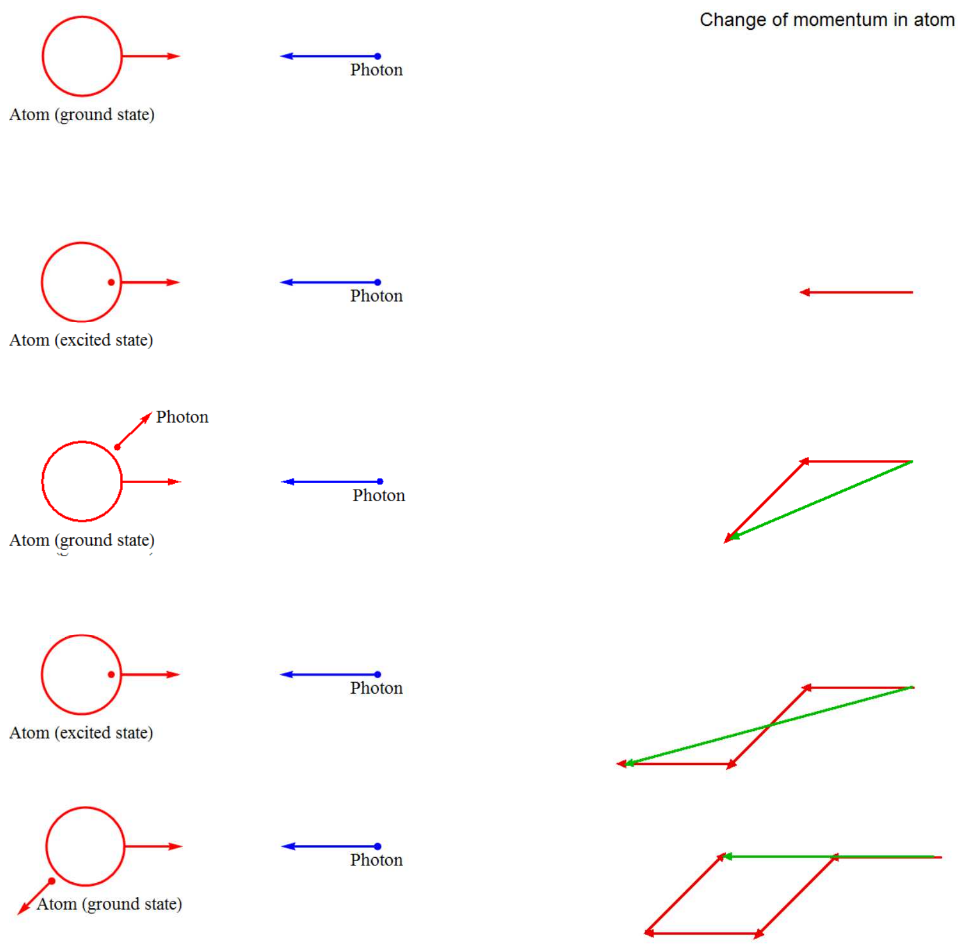


Fig. The change of linear momentum of atom due to the absorption and spontaneous emission of light.

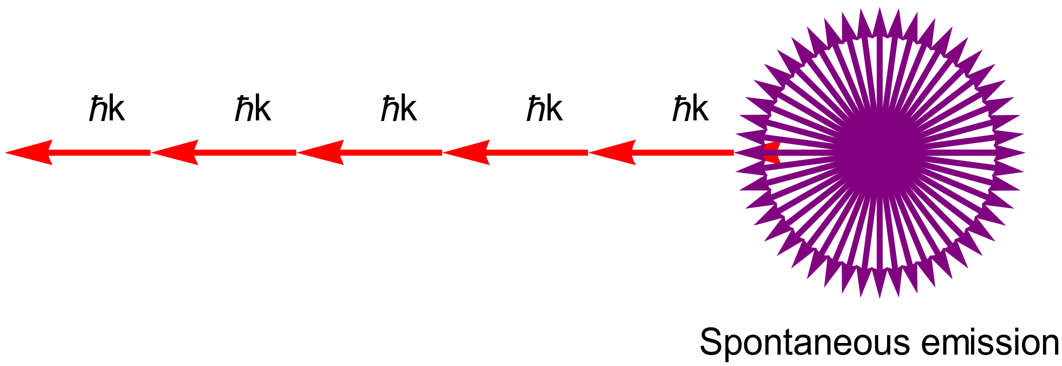
Fig.(a) Atom in the ground state. The laser beam comes in from the right side and is applied to the atom.

Fig.(b) The state of the atom changes from the ground state to the excited stated due to the absorption of laser light. The atom gets a momentum $\hbar k$ of photon.

Fig.(c) The atoms in the excited state return to the ground state due to the spontaneous emission. If the photon emits from the atom in the direction shown in Fig.(c), the atom receives momentum $\hbar k$ in the opposite direction of photon. This behavior is called as recoil. The momentum is called recoiled momentum. Subsequently the spontaneous emission occurs. After the spontaneous emission, the atom returns to the ground state,

Fig.(d) Since the atom is again in the ground state, the atom absorbs the laser light. The atom receives momentum $\hbar k$ of photon in the direction from to right to left side.

Fig.(e) Due to the spontaneous emission, again the atom returns to the ground state. Note that the direction of the spontaneous emission may be different from that of the emission shown in Fig.(c). So in this case, the atom receives recoiled momentum. Again the atom returns to the ground state.



In a series of such processes, whenever the atom absorbs the laser light, the atom receives momentum $\hbar k$. During the spontaneous emission the atom receives isotropic recoiled momentum, which become zero in momentum after many repeated spontaneous emission processes as shown in the above Fig. However, after each cycle of absorption and spontaneous emission, the atom receives linear momentum $\hbar k$ in the direction of laser light.

When the angular frequency of the laser light is nearly equal to the energy difference between the ground state and the excited state for the atom, the atom receive momentum. Correspondingly, we define the radiation pressure as

$$F = \frac{dp}{\Delta t} = \frac{\hbar \Delta k}{\Delta t}.$$

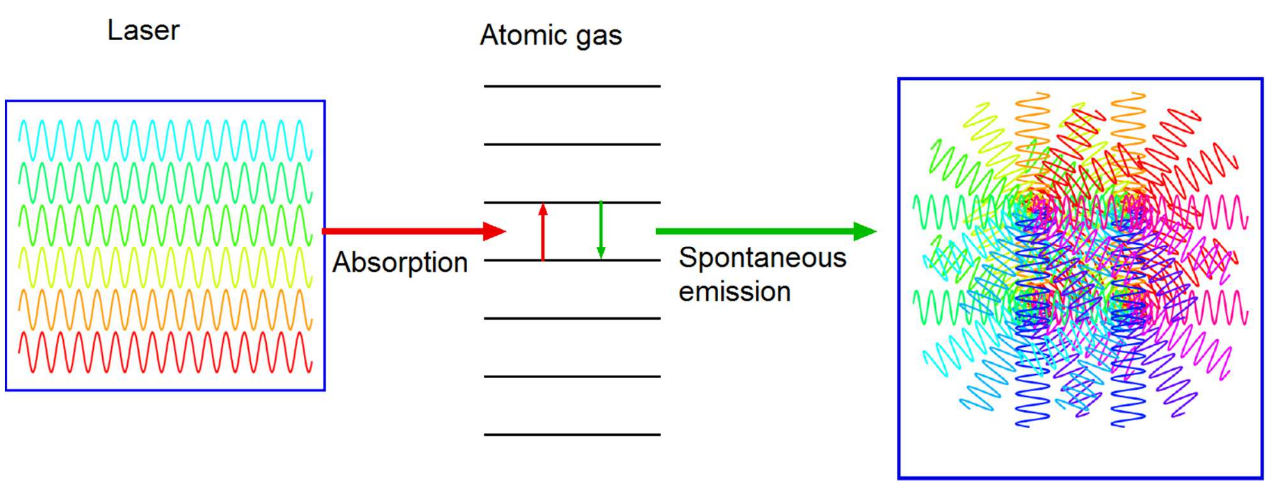


Fig. Spontaneous emission and absorption.

(B) Doppler effect

The primary force used in laser cooling and trapping is the recoil when the linear momentum is transferred from photons scattering off an atom. The momentum kick that the atom receives from each scattered photon is quite small; a typical velocity change is about 1 cm/s. However, by exciting a strong atomic transition, it is possible to scatter more than 10^7 photons per second and produce large accelerations. The radiation-pressure force is controlled in such a way that it brings the atoms in a sample to a velocity near zero ("cooling"), and holds them at a particular point in space ("trapping"). The cooling is achieved by making the photon scattering rate velocity-dependent using the Doppler effect.

The basic principle is illustrated below. If an atom is moving in a laser beam, it will see the laser angular frequency ω_{photon} shifted by an amount $\omega_{photon}(v/c)$ where v is the velocity of the atom along the opposite direction of the laser beam.

$$\omega_{atom} = \omega_{photon}\sqrt{\frac{1+\beta}{1-\beta}} \approx \omega_{photon}(1+\beta)^{1/2}(1-\beta)^{-1/2} \approx \omega_{photon}(1+\beta) .$$

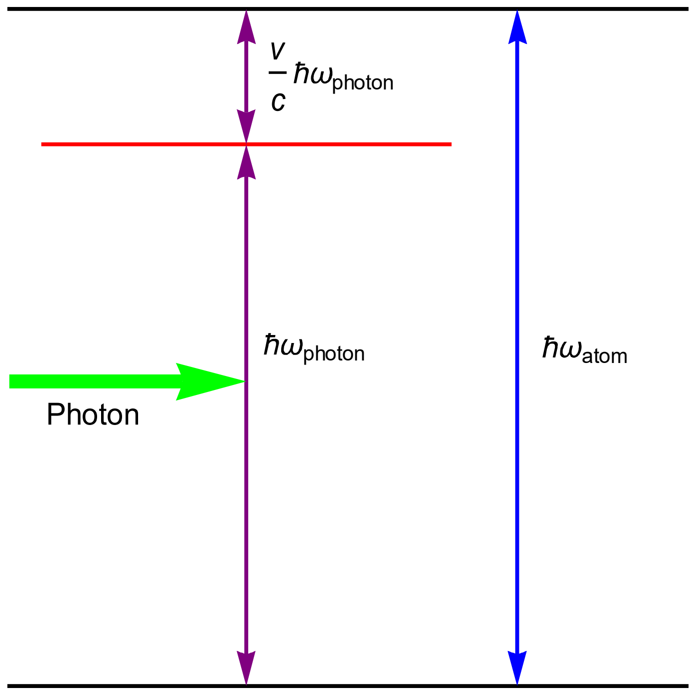
or

$$\omega_{atom} - \omega_{photon} = \omega_{photon}\beta = \omega_{photon}\frac{v}{c}$$

or

$$v = c(\frac{\omega_{atom} - \omega_{photon}}{\omega_{photon}})$$

If the laser frequency is below the atomic resonance frequency, the atom, as a result of this Doppler shift, will scatter photons at a higher rate if it is moving toward the laser beam than if it is moving away. If laser beams impinge on the atom from all six directions, the only remaining force on the atom is the velocity-dependent part, which opposes the motion of the atoms. This provides strong damping of any atomic motion and cools the atomic vapor This arrangement of laser fields is often known as "optical molasses". It will scatter photons at a higher rate than those moving in the same direction as the beam. This leads to a larger force on the counter propagating atoms.



(a)

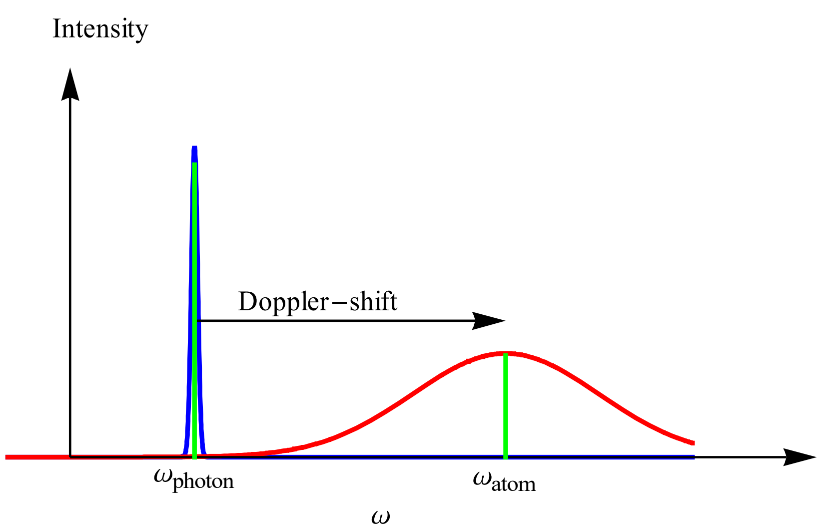


Fig. Doppler effect. $\omega_{atom} - \omega_{photon} = \omega_{photon}\frac{v}{c}$. The laser and the atom approach each other.

(b)

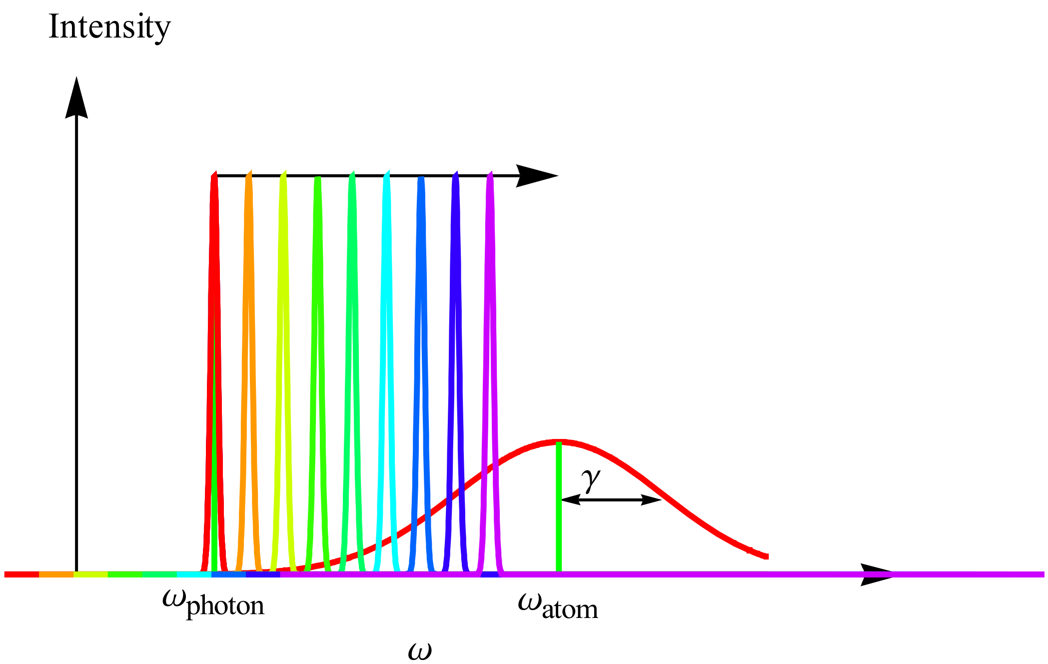


Fig. Doppler effect. $\omega_{atom} - \omega_{photon} = \omega_{photon}\dfrac{v}{c}$. The laser and the atom approach each other. As the velocity of the atom decreases, one needs to increase ω_{photon}.

((**Example**))
Rb atom

At 300 K, v_{rms}=295.89 m/s

$\lambda_{\mathrm{photon}}$ = 780 nm.

f_{photon} = 3.84349 x 10^{14} Hz

$$\Delta f = f_{atm} - f_{photon} = f_{photon}\frac{v}{c} = 309.0 \text{ MHz.}$$

APPENDIX-III Synchrotron: electron motion in the presence of magnetic field
It seems that the laser cooling of alkali atoms is similar to the synchrotron in a sense that the angular frequency is no longer constant but now depends on the velocity *v*.
((**Synchrotron**))
As we discussed in LN 28, the cyclotron motion (circular motion with radius *R*) is expressed by

$$R = \frac{v}{\omega} = \frac{vE}{c^2 qB} = \frac{vmc^2\gamma}{c^2 qB} = \frac{mv\gamma}{qB}$$

where ω is the angular frequency,

$$\omega = \frac{c^2 qB}{E} = \frac{qB}{m\gamma}$$

The angular frequency is no longer constant but now depends on the velocity v. The resonance between the circulating frequency and the oscillation frequency no longer occurs.

As the energy of the particles (E) increase, the strength of B must be changed with each turn to keep the particles moving in the same ring. The change in B must be carefully synchronized to the change in energy, or the beam will be lost (hence the name "*synchrotron*"). The range of energies over which particles can be accelerated in a single ring is determined by the range of field strength available with high precision from a particular set of magnets.

APPENDIX-IV frequency of the standing waves in tubes

(a) Open-open system

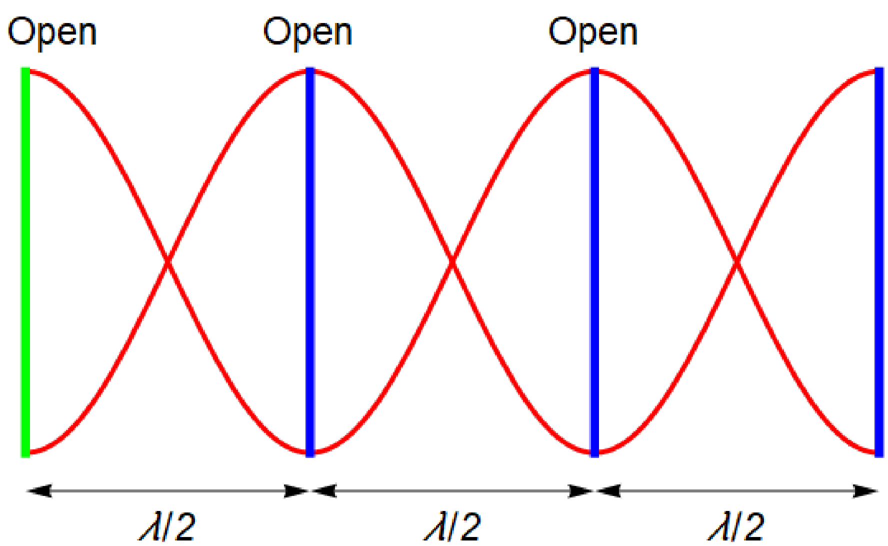


Fig. Open (green line)-open (blue line) system

The condition of the resonance:

$$L = \frac{\lambda}{2} m \qquad (m = 1, 2, 3, \ldots)$$

$$\lambda = \frac{2L}{m}, \qquad f = \frac{v}{\lambda} = \frac{v}{2L} m$$

where v is the velocity of the sound wav e.

(b) Closed-open system

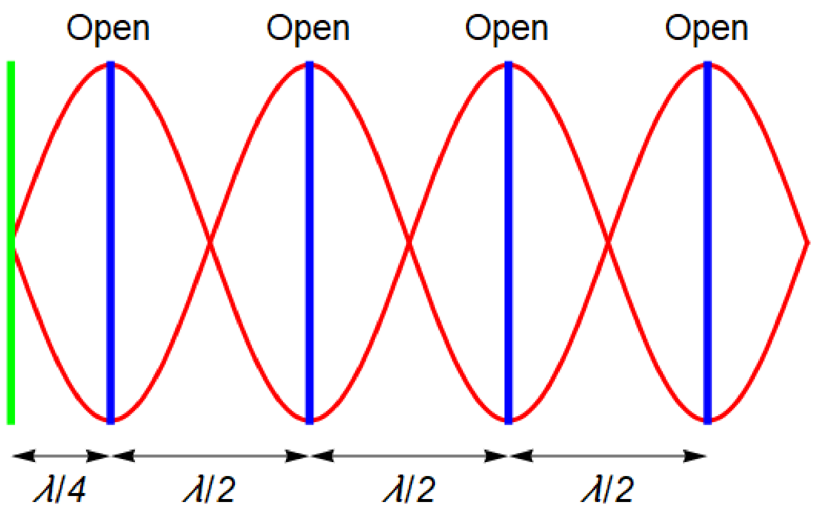


Fig. Closed (green line)-open (blue line) system

The condition for the resonance:

$$L = \frac{\lambda}{4} + m\frac{\lambda}{2} = \frac{\lambda}{4}(2m+1), \qquad (m = 0, 1, 2, \ldots)$$

$$f = \frac{v}{\lambda} = \frac{v}{4L}(2m+1) = \frac{v}{2L}(m + \frac{1}{2})$$

(c) Closed-closed system

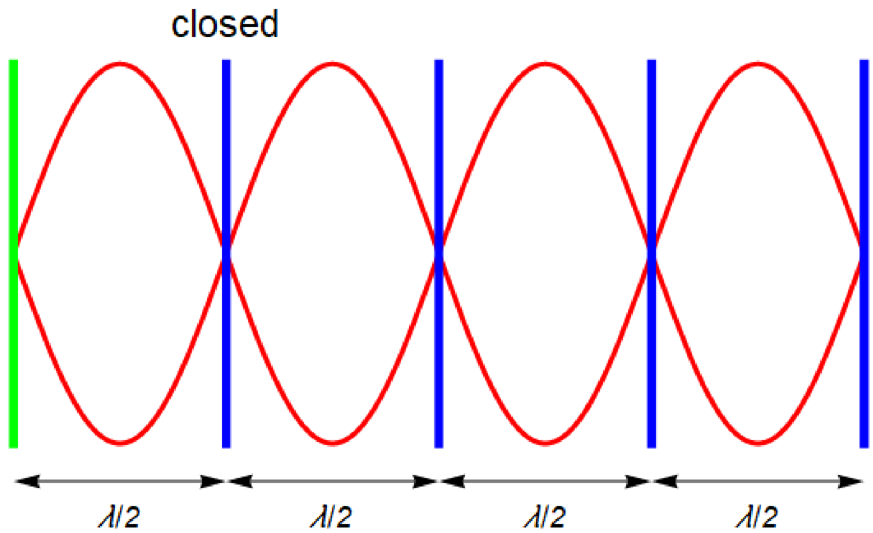


Fig. Closed (green line)-closed (blue line) system

The condition of the resonance:

$$L = \frac{\lambda}{2} m \qquad (m = 1, 2, 3, \ldots)$$

$$\lambda = \frac{2L}{m}, \qquad f = \frac{v}{\lambda} = \frac{v}{2L} m$$
