

Chapter 18

Temperature, heat, and the first law of thermodynamics

1 Temperature

Typically there are three kinds of temperature units.

C (Celsius), F (Fahrenheit), and K (Kelvin).

$$T_C = \frac{5}{9}(T_F - 32)$$

$$T_K = T_C + 273.15$$

T_C :

The steam point of water is defined to be 100°C. The ice point of water is defined to be 0°C. The length of the column between these two points is divided into 100 equal segments, called degrees. T_C is the temperature in Celsius

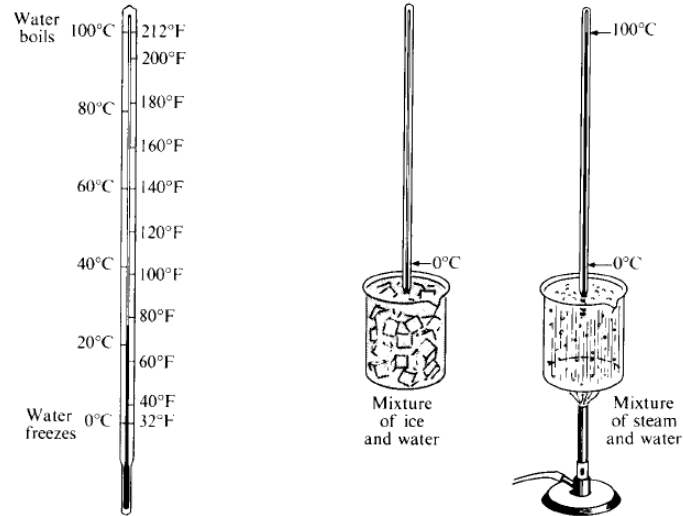
T_K :

Absolute zero is used as the basis of the absolute temperature scale. The size of the degree on the absolute scale is the same as the size of the degree on the Celsius scale. T is the Kelvin (absolute) temperature

T_F :

Ice point temperatures: 0°C = 273.15 K = 32°F. Steam point temperatures: 100°C = 373.15 K = 212°F.

Note that 212 F – 32 F = 180 F.



2 Thermal expansion

Thermal expansion is the increase in the size of an object with an increase in its temperature. Thermal expansion is a consequence of the change in the average separation between the atoms in an object. If the expansion is small relative to the original dimensions of the object, the change in any dimension is, to a good approximation, proportional to the first power of the change in temperature.

(a) Linear expansion

$$L = L_0(1 + \alpha\Delta T)$$

where α is the coefficient of linear expansion.

(b) Volume expansion

$$\begin{aligned} V &= L^3 = L_0^3(1 + \alpha\Delta T)^3 \\ &= L_0^3(1 + 3\alpha\Delta T + \dots) \\ &\approx V_0(1 + \beta\Delta T) \end{aligned}$$

with

$$\beta = 3\alpha$$

where β is the coefficient of volume expansion.

3 Unit of Heat

Heat is the energy transferred between a system and its surroundings because of their temperature difference. Heat flow Q is positive when energy flows into a system and negative when heat flows out. The heat is measured in the units of J (Joules), calories, and British thermal units (Btu).

1 cal = 4.186 J (see the Appendix).

1 Btu = 252 cal = 1054 J.

((Definition))

1 cal is defined as the heat necessary to raise the temperature of 1 g of water from 14.5 °C to 15.5 °C.

((Definition))

1 Btu is defined as the heat required to raise the temperature of 1 lb of water from 63 F to 64 F.

$$\Delta T_c = (5/9) \Delta T_F$$

$$1 \text{ lb} = 453 \text{ g}$$

$$1 \text{ Btu} = (453 \text{ g}) \times (5/9) \times 1 \text{ cal/g} = 251.67 \text{ cal}$$

4 Calorie

A calorie is a unit of measurement for energy. In most fields, it has been replaced by the joule, the SI unit of energy. However, the kilocalorie or calorie remains in common use for the amount of food energy. The calorie was first defined by Professor Nicolas Clément in 1824 as a kilogram-calorie and this definition entered French and English dictionaries between 1842 and 1867.

The calorie was never an SI unit. Modern definitions for calorie fall into two classes. The small calorie or gram calorie approximates the energy needed to increase the temperature of 1 gram of water by 1°C. This is about 4.186 joules. The large calorie or kilogram calorie approximates the energy needed to increase the temperature of 1 kg of water by 1°C. This is about 4.186 kJ, and exactly 1000 small calories. The conversion factor among calories and joules is numerically equivalent to the specific heat capacity of liquid water (in SI units).

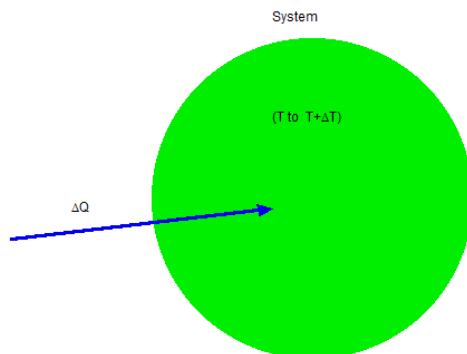
$$1 \text{ cal}_{IT} = 4.1868 \text{ J} \quad (1 \text{ J} = 0.23885 \text{ cal}_{IT}) \quad (\text{International Steam Table calorie, 1956})$$

$$1 \text{ cal}_{th} = 4.184 \text{ J} \quad (1 \text{ J} = 0.23901 \text{ cal}_{th}) \quad (\text{Thermochemical calorie})$$

$$1 \text{ cal}_{15} = 4.18580 \text{ J} \quad (1 \text{ J} = 0.23890 \text{ cal}_{15}) \quad (15^\circ\text{C calorie})$$

5 Heat and thermal energy

When heat is transferred to a system, the temperature of the system increases. When the heat is removed from the system, the temperature of the system decreases. The relationship between the heat absorbed (ΔQ) that is transferred and the change in temperature (ΔT) is



$$\Delta Q = C\Delta T.$$

The proportionality constant C is called the *heat capacity* (C). The heat capacity is the amount of heat required to raise the temperature of an object or substance one degree. The temperature change $\Delta T = T_f - T_i$ is the difference between the final temperature (T_f) and the initial temperature (T_i). The unit of C is J/K.

The heat capacity C_p at a constant pressure P is defined by

$$C_p = \left(\frac{\partial Q}{\partial T}\right)_P,$$

where Q is the heat absorbed as the temperature rises by ΔT . The heat capacity C_v at a constant volume V is defined by

$$C_v = \left(\frac{\partial Q}{\partial T}\right)_V.$$

((Note))

Typical dimensions of the quantities are as follows.

Q	heat absorbed	cal
C	heat capacity	cal/K
T	temperature	K

6 Specific heat

The heat required for the temperature change ΔT of mass m is given by

$$\Delta Q = mc\Delta T$$

where c is a quantity, different for different materials, called the **specific heat** of the material.

$$[c] = 1 \text{ cal/g K} = 4.186 \text{ J/g K} = 4.186 \times 10^3 \text{ J/kg K.}$$

((Note))

Typical dimensions of the quantities are as follows.

		cgs units	SI units
c	specific heat	cal/(g K)	J/(kg K)
Q	heat absorbed	cal	J
m	mass	g	kg

7 Molar specific heat

The heat required for temperature change of n mole is given by

$$\Delta Q = nc\Delta T$$

where c is the molar heat capacity. For water, we have

$$c = 18 \text{ cal/(mol K)} = 18 \times 4.186 \text{ J/(mol K)} = 75.3 \text{ J/(mol.K).}$$

((Note))

The molar mass of water is 18.0 g/mol. $[\text{cal/mol K}] = [\text{cal/g K}] [\text{g/mol}]$

8 Latent heat of water

To change 1 kg of ice at 0 °C to 1 kg of liquid water at 0 °C and normal atmospheric pressure requires 3.33×10^5 J/kg of heat. The heat required per unit mass is called the **latent heat of fusion**. The corresponding latent heat is given by

$$Q = mL_F$$

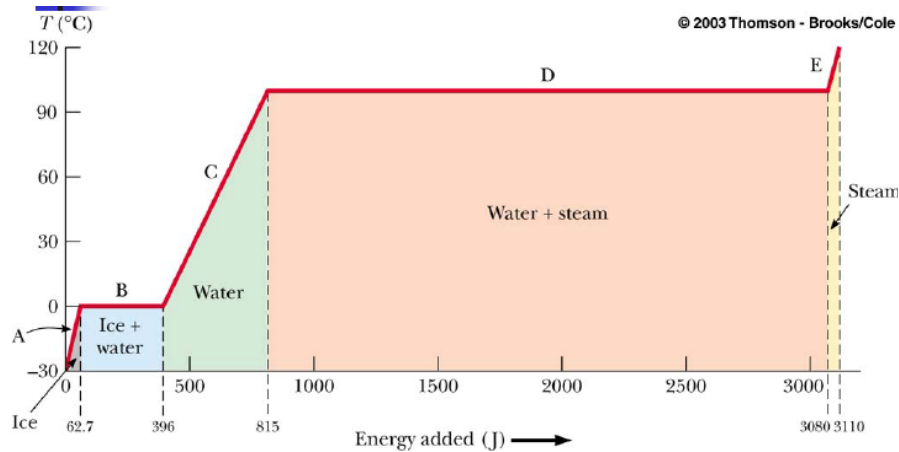
where L_F is the specific latent heat ($L_F = 3.33 \times 10^5 \text{ J/kg} = 79.6 \text{ cal/g} = 143 \text{ Btu/lb}$) and m is the mass.

To change 1 kg of liquid water at 100 °C to 1 kg of water vapor at 100 °C and normal atmospheric pressure requires 2.256×10^6 J/kg of heat. The heat required per

unit mass is called the **latent heat of vaporization**. The equation of the corresponding latent heat is given by

$$Q = mL_V$$

where $L_V = 2.256 \times 10^6 \text{ J/kg} = 539 \text{ cal/g} = 970 \text{ Btu/lb}$.



((Note))

We use

$$C_{\text{water}} = 4180 \text{ J/kg.K} = 4.180 \text{ J/g.K}$$

$$C_{\text{ice}} = 2220 \text{ J/kg.K} = 2.22 \text{ J/g.K}$$

in solving the problems.

9. van der Waals gas

(a) Definition

An equation of state called the van der Waals equation describes the behavior of many real gases over a wide range of pressures more accurately than does the ideal-gas equation of state ($PV = N_A k_B T = RT$). The van der Waals equation for 1 mole of gas is

$$\left(P + \frac{a}{V^2} \right) (V - b) = RT,$$

where a and b are constants. Figure shows the PV isothermal curves for a substance at various temperatures. Except for the region where the liquid and vapor coexist, these curves are described by the van der Waals equation, and can be used to determine the constants a and b .

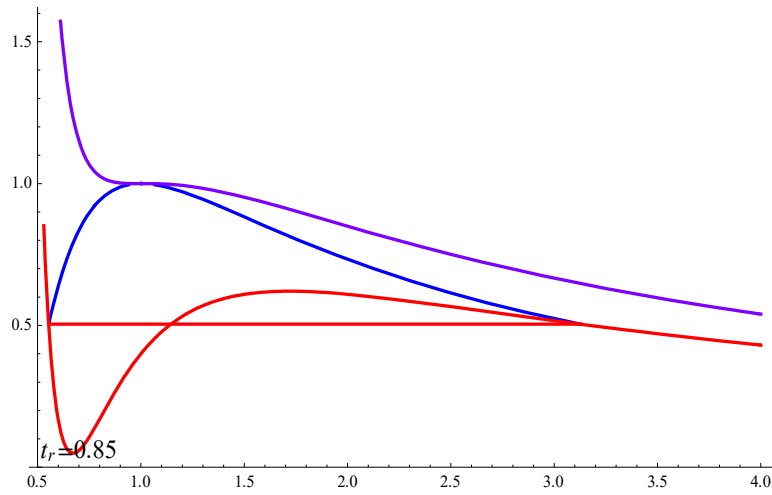
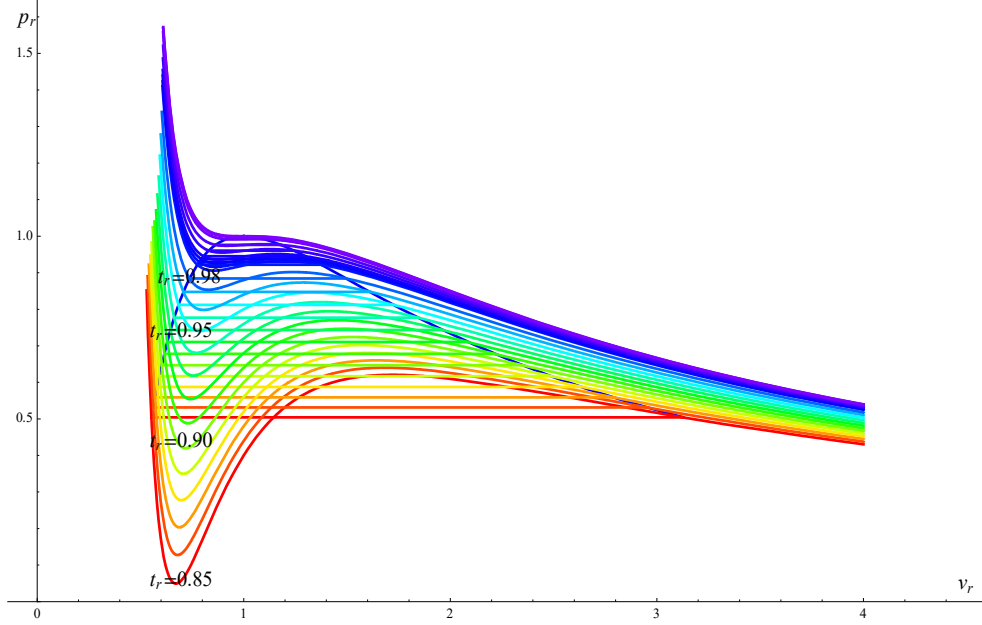


Fig. P - V phase diagram for the van der Waals gas with constant temperature (isotherm) exhibiting an unstable portion where $\partial P / \partial V < 0$. $t_r = T / T_c$. $t_r = 0.85$ (red) and 1.0 (blue).

(b) Critical points for the van der Waals gas

$$\left(\frac{\partial P}{\partial V} \right)_T = 0, \quad \left(\frac{\partial^2 P}{\partial V^2} \right)_T = 0 \quad (1)$$

From the condition that $\left(\frac{\partial P}{\partial V} \right)_T = 0$, we get

$$\frac{2a}{V^3} - \frac{RT}{(V-b)^2} = 0. \quad (2)$$

From the condition that $\left(\frac{\partial^2 P}{\partial V^2}\right)_T = 0$, we get

$$\frac{-3a}{V^4} + \frac{RT}{(V-b)^3} = 0. \quad (3)$$

From Eqs.(1) - (3), we have

$$P_c = \frac{a}{27b^2}, \quad V_c = 3b, \quad T_c = \frac{8a}{27Rb}$$

Note that

$$P_c V_c = \frac{3}{8} RT_c$$

$$a = \frac{27R^2 T_c^2}{64P_c} \quad b = \frac{RT_c}{8P_c}$$

Here we define the dimensionless variables by

$$p_r = \frac{P}{P_c}, \quad v_r = \frac{V}{V_c}, \quad t_r = \frac{T}{T_c},$$

and

$$\left(p_r + \frac{3}{v_r^2}\right)\left(v_r - \frac{1}{3}\right) = \frac{8}{3} t_r \quad \Rightarrow \text{Law of corresponding states.}$$

In terms of p_r , v_r , t_r , all gases look like - if they obey the van der Waals equation.

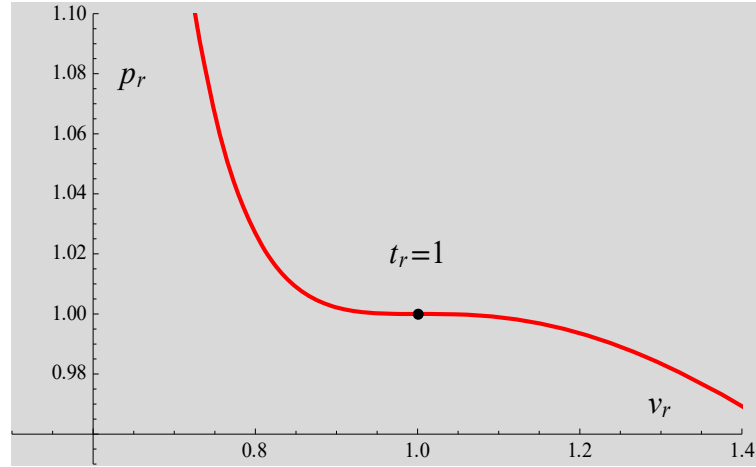


Fig. p_r vs v_r at $t_r = 1$.

(c) Critical behavior

To examine the critical behavior, we write

$$p = \frac{P}{P_c} = 1 + \pi, \quad v_r = \frac{v}{v_c} = 1 + \omega, \quad t = \frac{T}{T_c} = 1 + \tau$$

where π , ω , and τ can be regarded as small. We obtain the universal equation

$$1 + \pi = -\frac{3}{(1 + \omega)^2} + \frac{8(1 + \tau)}{2 + 3\omega}$$

or expanding

$$\begin{aligned} \pi = & 4\tau - 6\tau\omega + 9\tau\omega^2 - \left(\frac{3}{2} + \frac{27}{2}\tau\right)\omega^3 + \left(\frac{21}{4} + \frac{81}{4}\tau\right)\omega^4 - \left(\frac{99}{8} + \frac{243}{8}\tau\right)\omega^5 \\ & + \left(\frac{393}{16} + \frac{729}{16}\tau\right)\omega^5 + \dots \end{aligned}$$

REFERENCE

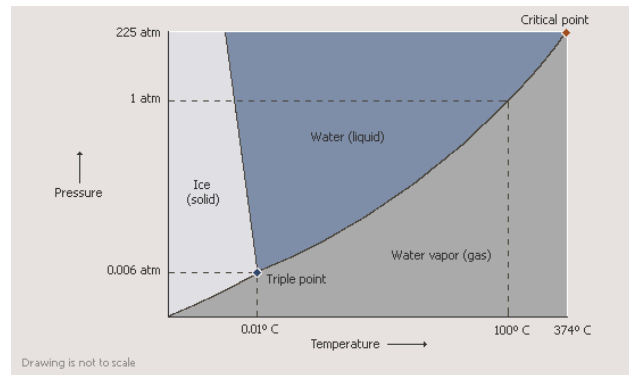
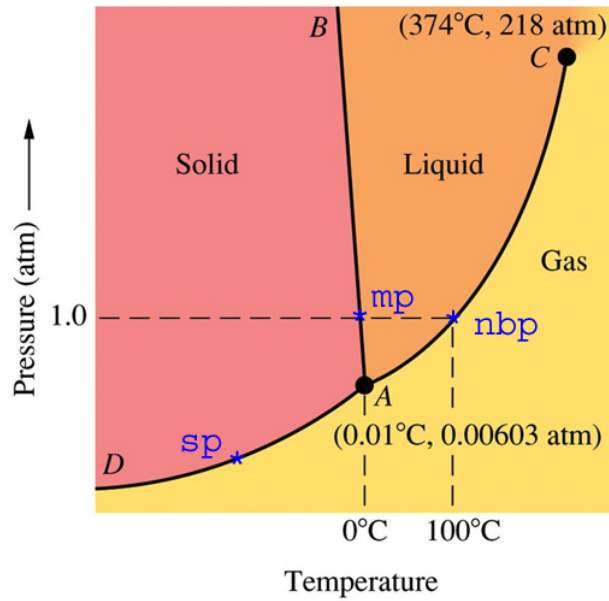
C. Domb, *The Critical Point: A historical introduction to the modern theory of critical phenomena* (Taylor & Francis, 1996).

10 The change of phase for water

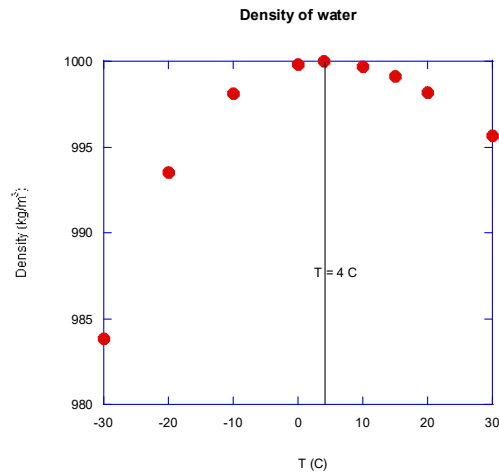
The water consists of the solid phase, the liquid phase, and the gas phase.

- (1) The first-order phase transition between the solid phase (ice) and the liquid phase (water), with latent heat L_F .

- (2) The first-order phase transition between the liquid phase (water) and the vapor phase, with the latent heat L_V .



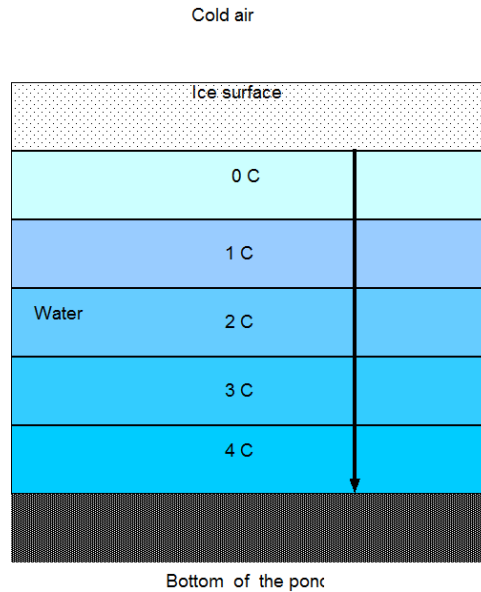
11 Unusual behavior of water around 4 °C



At room temperature, liquid water becomes denser with lowering temperature, just like other substances. *But at 4 °C, just above freezing, water reaches its maximum density*, and as water cools further toward its freezing point, the liquid water, under standard conditions, expands to become *less dense*. The physical reason for this is related to the crystal structure of ordinary ice, known as *hexagonal ice Ih*. The reason that the common form of ice is less dense than water is a bit non-intuitive and relies heavily on the unusual properties inherent to the hydrogen bond.

Generally, water expands when it freezes because of its molecular structure, in tandem with the unusual elasticity of the hydrogen bond and the particular lowest energy hexagonal crystal conformation that it adopts under standard conditions. That is, when water cools, it tries to stack in a crystalline lattice configuration that stretches the rotational and vibrational components of the bond, so that the effect is that each molecule of water is pushed further from each of its neighboring molecules. This effectively reduces the density ρ of water when ice is formed under standard conditions.

The unusual expansion of freezing water (in ordinary *natural* settings in relevant biological systems), due to the hydrogen bond, from 4 °C above freezing to the freezing point offers an important advantage for freshwater life in winter. Water chilled at the surface increases in density and sinks, forming convection currents that cool the whole water body, but when the temperature of the lake water reaches 4 °C, water on the surface decreases in density as it chills further and remains as a surface layer which eventually freezes and forms ice. Since downward convection of colder water is blocked by the density change, any large body of fresh water frozen in winter will have the coldest water near the surface, away from the riverbed or lakebed.



12 Thermal diffusion, heat transfer

There are various mechanisms responsible for the transfer. The mechanisms include

(a) **Conduction**

It is an energy transfer via the conducting material.

(b) **Convection**

It occurs when temperature differences cause an energy transfer by motion within a fluid.

(c) **Radiation**

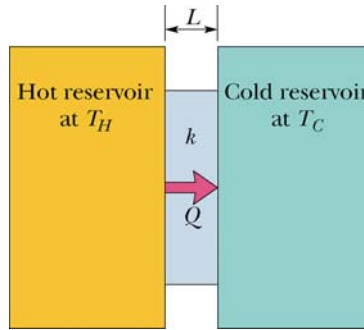
It is an energy transfer via the emission of electromagnetic energy.

12 Thermal conductivity

12.1 Formulation

Consider a slab of face area A and thickness L , whose faces are maintained at temperatures T_H and T_C by a hot reservoir and a cold reservoir. Let Q be the energy that is transferred as heat through the slab, from its hot face to its cold face, in time t . The conduction rate ϕ (the amount of energy transferred per unit time) is

$$\phi = \frac{dQ}{dt} = -\kappa A \frac{dT}{dx} = \kappa A \left(\frac{T_H - T_C}{L} \right)$$



where κ is a thermal conductivity. The unit of κ is W/m K. The unit of ϕ is W (= J/s).

12.2 Series connection

Two slabs of thickness L_1 and L_2 and thermal conductivities κ_1 and κ_2 are in thermal contact with each other. The temperatures of their outer surfaces are T_H and T_C respectively, and $T_H > T_C$. Determine the temperature at the interface and the rate of energy transfer by conduction through the slabs in the steady-state condition.

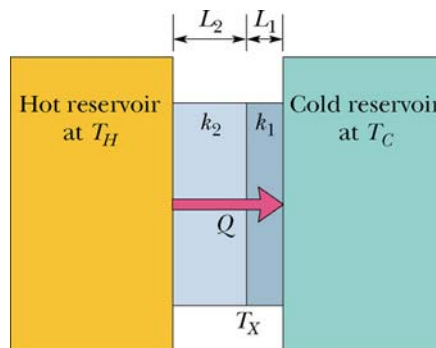
$$\phi = \frac{dQ}{dt} = -\kappa_2 A \frac{dT}{dx} = \kappa_2 A \frac{(T_H - T_i)}{L_2}$$

$$\phi = \frac{dQ}{dt} = -\kappa_1 A \frac{dT}{dx} = \kappa_1 A \frac{(T_i - T_C)}{L_1}$$

or

$$T_i = \frac{\kappa_2 L_1 T_H + \kappa_1 L_2 T_C}{\kappa_2 L_1 + \kappa_1 L_2}$$

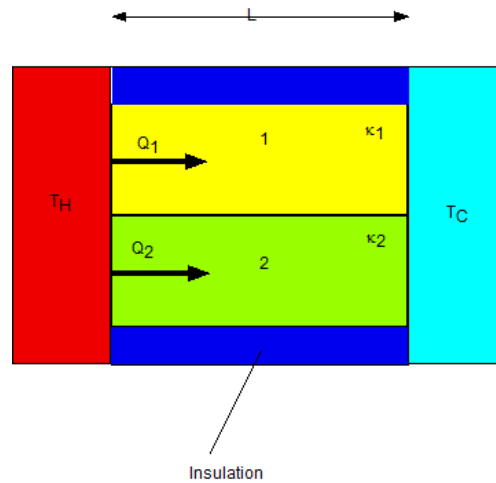
$$\phi = \frac{A \kappa_1 \kappa_2 (T_H - T_C)}{\kappa_2 L_1 + \kappa_1 L_2} = \frac{A(T_H - T_C)}{\frac{L_1}{\kappa_1} + \frac{L_2}{\kappa_2}}$$



We can extend this equation to any number n of materials making up a slab,

$$\varphi = \frac{A(T_H - T_C)}{\sum_{k=1}^n \frac{L_k}{\kappa_k}}$$

12.3 Parallel connection



$$\varphi_1 = \frac{dQ_1}{dt} = -\kappa_1 A_1 \frac{dT}{dx}$$

$$\varphi_2 = \frac{dQ_2}{dt} = -\kappa_2 A_2 \frac{dT}{dx}$$

or

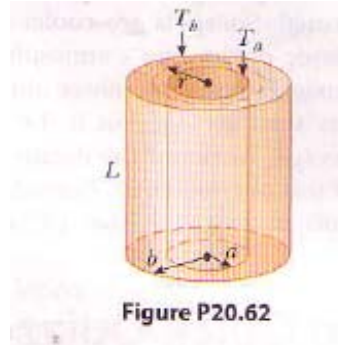
$$\varphi = \frac{dQ}{dt} = \varphi_1 + \varphi_2 = \frac{dQ_1}{dt} + \frac{dQ_2}{dt} = -(\kappa_1 A_1 + \kappa_2 A_2) \frac{dT}{dx}$$

13 Typical Example of thermal conductivity

13.1 Cylinders

The inside of a hollow cylinder is maintained at a temperature T_a , while the out side is at a lower temperature T_b . The wall of the cylinder has a thermal conductivity κ . Ignoring end effects, the rate of energy conduction from the inner to the outer surface in the radial direction is

$$\phi = \frac{dQ}{dt} = 2\pi L \kappa \frac{T_a - T_b}{\ln\left(\frac{b}{a}\right)}$$



((Proof))

$$\phi = \frac{dQ}{dt} = -\kappa A \frac{dT}{dr} = -2\pi r L \kappa \frac{dT}{dr}$$

$$\phi \int_a^b \frac{dr}{r} = -2\pi L \kappa \int_{T_a}^{T_b} dT = -2\pi L \kappa (T_b - T_a)$$

or

$$\phi = \frac{dQ}{dt} = \frac{2\pi L \kappa (T_a - T_b)}{\ln\left(\frac{b}{a}\right)}$$

13.2 Example-2: spherical shells

A spherical shell has inner radius r , inner radius a , outer radius b . It is made of material with thermal conductivity κ . The interior is maintained at T_a and the exterior at T_b . After an interval of time, the shell reaches a steady state with the temperature at each point within it remaining constant in time. The rate of energy conduction from the inner to the outer spherical shell in the radial direction is

$$\phi = -4\pi \kappa r^2 \frac{dT}{dr}$$

From this, we have

$$\int_{T_a}^{T_b} dT = -\frac{\wp}{4\pi\kappa} \int_a^b \frac{1}{r^2} dr$$

or

$$T_b - T_a = \frac{\wp}{4\pi\kappa} \left[\frac{1}{r} \right]_a^b = \frac{\wp}{4\pi\kappa} \left(\frac{1}{b} - \frac{1}{a} \right) = -\frac{\wp}{4\pi\kappa} \left(\frac{b-a}{ab} \right)$$

or

$$\wp = 4\pi\kappa ab \frac{T_a - T_b}{b - a}$$

14 Radiation

Radiation does not require physical contact. All objects radiate energy continuously in the form of electromagnetic waves due to thermal vibrations of the molecules.

((Example of the radiation))

(H.E. White, Classic and Modern Physics, 1940).

When the sun comes over the horizon in the early morning the heat can be felt as soon as the sun becomes visible. This heat, called radiation, travels with the speed of light. In fact, heat waves are light waves of a little longer wave-length, having all the general properties known to visible light. The essential difference between the two is that heat rays, sometimes called infrared rays, are not visible to the human eyes.

A demonstration of the reflection of infrared rays is diagramed in Fig. A candle flame acting as a source at F emits light and heat rays in all directions. Of these rays only the one travelling in the direction of the concave mirror M1 are reflected into a parallel beam. Arriving at the second concave mirror M2, these rays are again reflected, being brought together to a focus on the exposed junctions of a thermo-pile T. As the junctions of the thermopile warm up and a electric current is produced, causing the ammeter pointer P to move to the right. When the candle is removed the pointer returns to zero

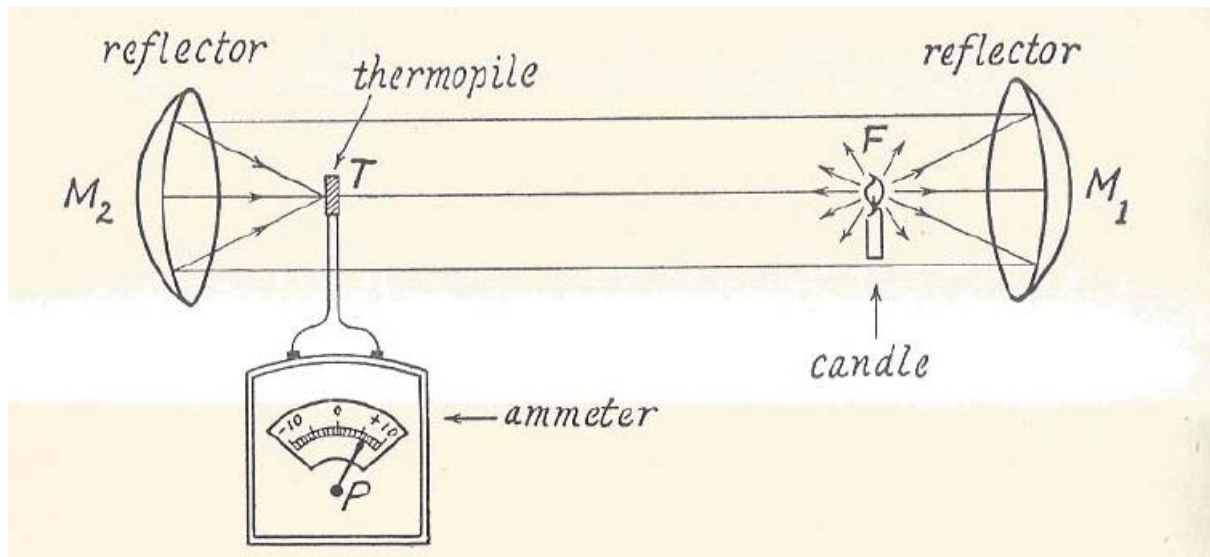


Fig. Reflection of heat rays by concave mirrors.

14.1 Black body

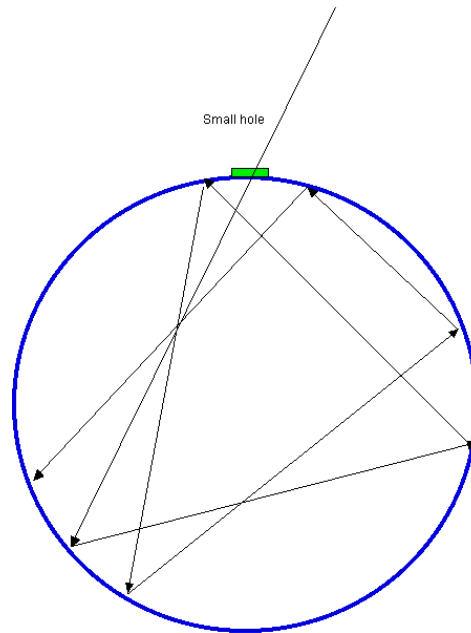


Fig. A photon entering a cavity through a small hole is effectively absorbed, so that the cavity represents a blackbody.

((Atkins Quanta))

A black body is one that absorbs all the radiation incident upon it. A practical example is a container completely sealed except for a tiny pinhole: this hole behaves

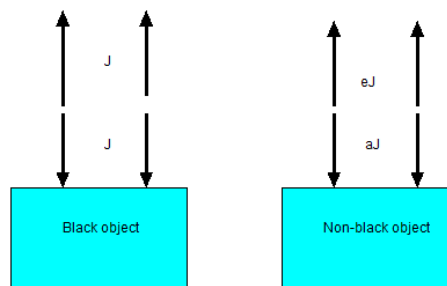
as a black body because all light incident on it from outside passes through and, once in, cannot escape through the vanishingly small hole. Inside it experiences an indefinitely large number of reflections before it is absorbed, and these reflections have the result that the radiation comes into thermal equilibrium with the wall. Within the cavity we can imagine the electromagnetic field as having a distribution of frequencies characteristic of the temperature of the walls.

The presence of the hole enables a small proportion of this equilibrium radiation to seep out and be detected, and the distribution of wavelength in the black-body radiation is the same as the distribution within the equilibrium enclosure because the pinhole is a negligible perturbation.

The radiant energy flux density from a black surface at a temperature T is equal to the radiant energy density emitted from a small hole in a cavity at the same temperature T .

14.2 Emission and absorption: Kirchhoff law

If a non-black object at T absorbs a fraction a of the radiation incident upon it, the radiation flux emitted by the object will be a times the radiation flux emitted by a black-body at the same temperature. The object must emit at the same rate as it absorbs if equilibrium is to be maintained: $a = e$.



((Example))

There exists a very close connection between the emissivity e and the absorptivity a of a body. A good emitter of radiation is also a good absorber of radiation, and vice versa. This is a qualitative statement of Kirchhoff's law.

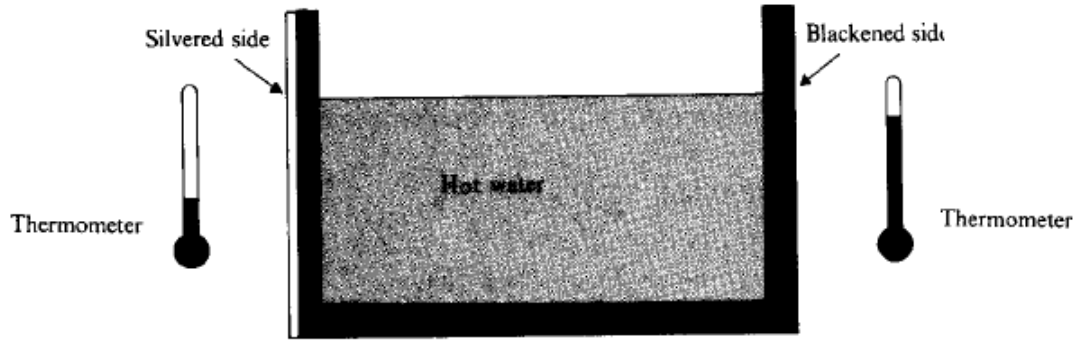


Fig. A classical experiment illustrating Kirchhoff's law. The container is filled with hot water. Its left side is silvered on the outside so that it is a poor absorber. Its right side is blackened so that it is a good absorber. Since the left side is then a poorer emitter of radiation than the right side, the thermometer on the left is found to indicate a lower temperature than the one on the right.

14.3 Stefan-Boltzman law

The Stefan–Boltzmann law, also known as Stefan's law, states that the total energy radiated per unit surface area of a black body in unit time (known variously as the black-body irradiance, energy flux density, radiant flux, or the emissive power), J , is directly proportional to the fourth power of the black body's thermodynamic temperature T (also called absolute temperature):

$$J = \varepsilon\sigma T^4 \quad (\text{W/m}^2)$$

where J has dimensions of energy per time per unit area and ε is the emissivity of the blackbody. If it is a perfect blackbody, $\varepsilon = 1$. The constant of proportionality σ , called the Stefan–Boltzmann constant or Stefan's constant. The value of the constant is

$$\sigma = \frac{2\pi^5 k_B^4}{15c^2 h^3} = \frac{\pi^2 k_B^4}{60c^2 h^3} = 5.670400 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4},$$

where k_B is the Boltzmann constant, h is Planck's constant ($h = 2\pi\hbar$) and c is the speed of light in a vacuum. The rate of energy transfer is given by Stefan's law

$$P = AJ = \sigma A\varepsilon T^4 \quad (\text{W})$$

where A is the surface area of the object.

((Mathematica))

$$\text{Physconst} = \{ \mathbf{kB} \rightarrow 1.3806504 \cdot 10^{-23}, \mathbf{c} \rightarrow 2.99792458 \cdot 10^8, \mathbf{h} \rightarrow 6.62606896 \cdot 10^{-34}, \\ \mathbf{\hbar} \rightarrow 1.05457162853 \cdot 10^{-34} \}$$

$$\{ \mathbf{kB} \rightarrow 1.38065 \cdot 10^{-23}, \mathbf{c} \rightarrow 2.99792 \cdot 10^8, \mathbf{h} \rightarrow 6.62607 \cdot 10^{-34}, \mathbf{\hbar} \rightarrow 1.05457 \cdot 10^{-34} \}$$

$$\sigma = \frac{2 \pi^5 \mathbf{kB}^4}{15 \mathbf{c}^2 \mathbf{h}^3} /. \mathbf{h} \rightarrow 2 \pi \mathbf{\hbar} // \text{Simplify}$$

$$\frac{\mathbf{kB}^4 \pi^2}{60 \mathbf{c}^2 \mathbf{\hbar}^3}$$

$$\sigma /. \text{Physconst}$$

$$5.6704 \cdot 10^{-8}$$

14.4 Energy absorption and emission by radiation

With its surroundings, the rate at which the object at temperature T with surroundings at T_0 radiates is

$$P_{net} = \sigma A e (T^4 - T_0^4)$$

When an object is in equilibrium with its surroundings, it radiates and absorbs at the same rate. Its temperature will not change.

14.5 The temperature at the surface of sun

One can measure the average flux of solar energy arriving at the Earth. This value is called the **solar constant** and is equal to **1370 W/m²**. Imagine a huge sphere with a radius of $a = 1 \text{ AU}$ ($=1.49597870 \times 10^{11} \text{ m}$) with the sun at its center. Each square meter of that sphere receives

$$p_0 = 1370 \quad \text{W/m}^2$$

of power from the sun.

((Note)) The order of the energy (per second) transferred from the sun to the Earth is roughly estimated as

$$\frac{\pi R_E^2 \times p_0}{4.1858} = 4.18 \times 10^{16} \text{ cal/s}$$

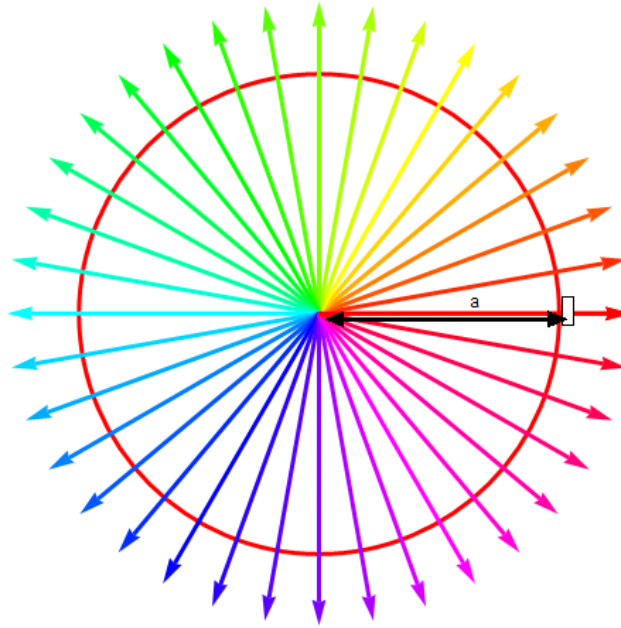


Fig. Luminosity of the sun. $a = 1$ AU. Sun is located at the center of sphere with radius a ($=1$ AU).

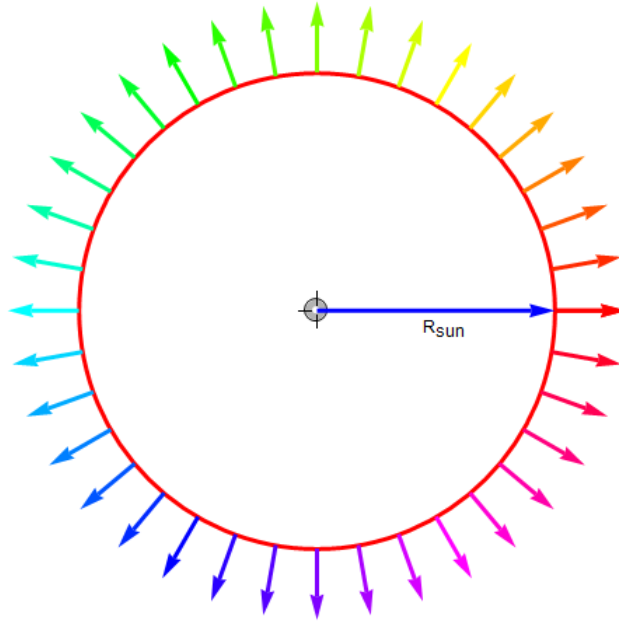
So we can calculate the total energy output of the sun by multiplying the solar constant by the sphere's area with a radius a ($= 1$ AU). The result, called the **luminosity of the sun**, is

$$L_{sun} = 4\pi a^2 p_0 = 3.853 \times 10^{26} \text{ (W)}$$

where $a = 1$ AU.

Using the Stefan-Boltzmann law, we can estimate the surface temperature of the sun.

$$L_{sun} = \sigma A_{sun} T_{sun}^4 = \sigma (4\pi R_{sun}^2) T_{sun}^4 \text{ (W)}$$



or

$$T_{sun} = \left[\left(\frac{a}{R_{sun}} \right)^2 \frac{P_0}{\sigma} \right]^{1/4} = 5780K$$

14.6 Temperature of the Earth

We can use the Stefan-Boltzmann law to estimate the temperature of the Earth from first principles. The Sun is a ball of glowing gas of radius $R_{Sun} \approx 6.9599 \times 10^5$ km and surface temperature $T_{Sun} \approx 5780$ K. Its luminosity is

$$L_{Sun} = 4\pi R_{Sun}^2 \sigma T_{Sun}^4 \quad (\text{W})$$

according to the Stefan-Boltzmann law. The Earth is a globe of radius $R_E \approx 6372$ km located an average distance $a = 1.49597870 \times 10^8$ km (= 1 AU) from the Sun. The Earth intercepts an amount of energy

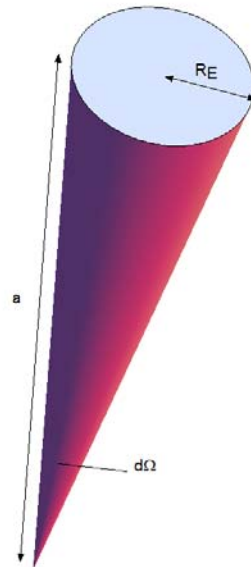
$$\Delta P_{intercept} = \pi R_E^2 P_0 \quad (\text{W})$$

or

$$\Delta P_{\text{intercept}} = L_{\text{Sun}} \frac{\Delta\Omega}{4\pi} = L_{\text{Sun}} \frac{\pi R_E^2}{4\pi a^2} = \sigma T_{\text{Sun}}^4 \frac{\pi R_{\text{Sun}}^2 R_E^2}{a^2} \quad (\text{W})$$

per second from the Sun's radiative output: *i.e.*, the power output of the Sun reduced by the ratio of the solid angle subtended by the Earth at the Sun to the total solid angle 4π .

$$\pi R_E^2 = a^2 \Delta\Omega,$$



The Earth absorbs this energy, and then re-radiates it at longer wavelengths (the Kirchhoff's law). The luminosity of the Earth is

$$L_E = \pi R_E^2 p_0 = 4\pi R_E^2 \sigma T_E^4, \quad (\text{W})$$

according to the Stefan-Boltzmann law, where T_E is the average temperature of the Earth's surface. Here, we are ignoring any surface temperature variations between polar and equatorial regions, or between day and night. In steady-state, the luminosity of the Earth must balance the radiative power input from the Sun, we arrive at

$$\Delta P_{\text{intercept}} = L_E, \text{ or } \frac{T_{\text{Sun}}}{T_E} = \sqrt{2} \frac{\sqrt{a}}{\sqrt{R_{\text{Sun}}}}$$

or

$$T_E = \frac{1}{\sqrt{2}} \frac{\sqrt{R_{\text{Sun}}}}{\sqrt{a}} T_{\text{Sun}}$$

Remarkably, the ratio of the Earth's surface temperature to that of the Sun depends only on the Earth-Sun distance and the radius of the Sun. The above expression yields $T_E = 278.78$ K. This is slightly on the cold side, by a few degrees, because of the greenhouse action of the Earth's atmosphere, which was neglected in our calculation. Nevertheless, it is quite encouraging that such a crude calculation comes so close to the correct answer.

14.7 Evaluation of the average surface temperature of our solar system

The average surface temperature of the planet may be expressed by

$$T_{\text{av}} = \frac{1}{\sqrt{2}} \frac{\sqrt{R_{\text{Sun}}}}{\sqrt{d}} T_{\text{Sun}} = \frac{278.774}{\sqrt{d(\text{AU})}} \text{ [K]}$$

where d is the mean distance from the Sun and $d(\text{AU})$ is the same distance in units of AU. The values of d , the calculated surface temperature T_{av} , and the reported surface temperature T_{obs} for each planet are listed in Table.

Planet	d (AU)	T_{av} [K]	T_{obs} [K]
Mercury	0.24	569.0	700
Venus	0.61	356.9	740
Earth	1	278.8	287.2
Mars	1.52	226.1	227
Jupiter	5.20	122.3	165 (1 bar level) 112 (0.1 bar level)

Saturn	9.53	90.3	134 (1 bar level) 84 (0.1 bar level)
Uranus	19.19	63.6	76 (1 bar level) 53 (0.1 bar level)
Neptune	30.06	50.8	72 (1 bar level) 55 (0.1 bar level)
Pluto	39.53	44.3 K	44

((Mathematica))

$cal=4.19 \text{ J}$,
 σ_{SB} =Stefan-Boltzmann constant ($\text{W}/\text{m}^2 \text{ K}^4$),
 $Mea = 5.9736 \times 10^{24} \text{ kg}$; Mass of the earth,
 $Rea=6372.797 \text{ km}$, radius of the earth,
 M_{sun} =mass of sun (kg) =Solar mass
 R_{sun} =radius of Sun (m)=Solar radius
light year=a distance light travels in a vacuum in one year= $9.4605 \times 10^{15} \text{ m}$,
Parsec (pc) = a unit of distance = 3.26 light yeras = $30.857 \times 10^{15} \text{ m}$,
AU = astronomical unit = average distance between the Earth and the Sun = $1.49597870 \times 10^{11} \text{ m}$
 p_0 = solar constant (W/m^2)

Physconst = { cal → 4.19, Mea → 5.9736 10²⁴,
σ_{SB} → 5.670400 10⁻⁸, Rea → 6.372 10⁶, Msun → 1.988435 10³⁰,
Rsun → 6.9599 10⁸, ly → 9.4605 10¹⁵, pc → 30.857 10¹⁵,
AU → 1.49597870 10¹¹, p0 → 1370 }

{ cal → 4.19, Mea → 5.9736 × 10²⁴, σ_{SB} → 5.6704 × 10⁻⁸,
Rea → 6.372 × 10⁶, Msun → 1.98844 × 10³⁰, Rsun → 6.9599 × 10⁸,
Mmoon → 7.3483 × 10²², ly → 9.4605 × 10¹⁵,
pc → 3.0857 × 10¹⁶, AU → 1.49598 × 10¹¹, p0 → 1370 }

Total heat (cal) per sec to the Earth from the Sun

$$\frac{\pi \text{Rea}^2 p_0}{\text{cal}} / . \text{Physconst}$$

$$4.17069 \times 10^{16}$$

Luminosity of Sun

$$L_{sun} = 4 \pi (\text{AU})^2 p_0 / . \text{Physconst}$$

$$3.85284 \times 10^{26}$$

Temperature of Sun

$$\text{eq1} = \text{Solve} \left[\left(\left(4 \pi \text{Rsun}^2 \sigma_{SB} \text{Tsun}^4 \right) / . \text{Physconst} \right) = L_{sun}, \text{Tsun} \right];$$

$$\text{Tsun} / . \text{eq1}[[4]]$$

$$5780.13$$

Luminosity of Earth

$$Lea = p_0 (\pi \text{Rea}^2) / . \text{Physconst}$$

$$1.74752 \times 10^{17}$$

Temperature of Earth

$$\text{eq2} = \text{Solve} \left[\left(\left(4 \pi \text{Rea}^2 \sigma_{SB} \text{Tea}^4 \right) / . \text{Physconst} \right) = Lea, \text{Tea} \right];$$

$$\text{Tea} / . \text{eq2}[[4]]$$

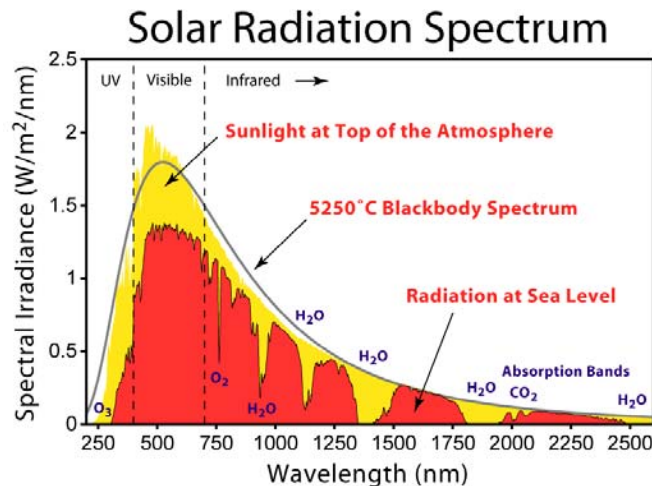
$$278.78$$

14.7 Radiation spectrum of the sun

We have a useful relation for the blackbody radiation, which is called a Wien's displacement law

$$\lambda_{\max} = \frac{2.897768551}{T} \times 10^6 \text{ (nm)}.$$

where T is the absolute temperature [K] and λ_{\max} is the wavelength (nm) at which the radiation intensity maximum. This law is directly derived from the Planck's law of black-body radiation. The peak of the sun's spectrum is 484 nm.



Using the Wien's displacement law, the temperature of the sun can be estimated as

$$T_{\text{sun}} = \frac{2.89776851 \times 10^6}{484} \approx 5987K$$

14.7 Cosmic background radiation; direct evidence of Big Bang

Arno Penzias and Robert Wilson (Bell Lab, 1965)

No matter where in the sky they pointed their antenna, they detected faint background noise. They had discovered the cooled-down cosmic background radiation left over from the hot **Big Bang**. The spectrum of the cosmic microwave background shows a peak at $\lambda_{\max} = 1.063$ mm. Using the Wien's displacement law, the temperature of the cosmic background radiation is estimated as

$$\begin{aligned}
 T &= \frac{2.89776851 \times 10^6}{\lambda_{\max} (nm)} \\
 &= \frac{2.89776851 \times 10^6}{1.063 \times 10^6} && \text{(Wien's displacement law)} \\
 &= 2.726K
 \end{aligned}$$

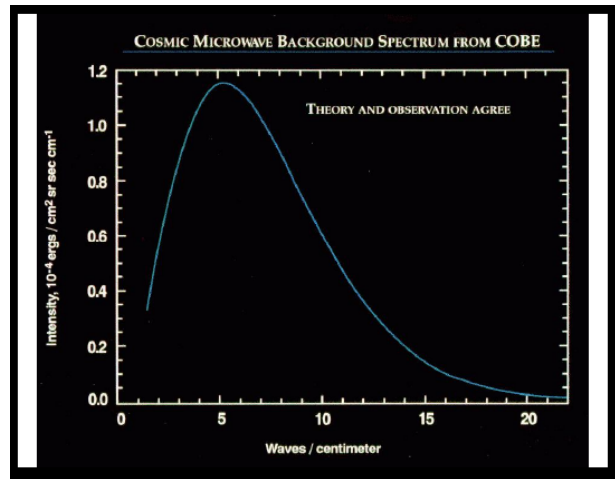


Fig. Experimental measurements of the spectrum of the cosmic blackbody radiation. The x axis is the wavenumber k defined by $k = 2\pi/\lambda$.

Dicke and Peebles (1960)

Early universe had been at least as hot as the Sun center. The hot early universe must therefore have been filled with many high-energy, short-wavelength photons, which formed a radiation field with that can be given by Planck's blackbody law. The universe has expanded so much since those ancient times that all those short-wavelength photons have their wavelengths stretched by a tremendous factor. As a result, they have becomes low-energy, long-wavelength photons.

The detail of the Hubble law is discussed in the Appendix and Chapter 17. (Chapter 17 will be taught in Phys.132).

15 Zeroth law of thermodynamics

If objects A and B are separately in thermal equilibrium with a third object C, then A and B are in thermal equilibrium with each other.

Two objects in thermal equilibrium with each other are at the same temperature.

16 First law of thermodynamics

If one has a system and puts heat into it, and does work on it, then its energy is increased by the heat put in and the work done.

First law of thermodynamics is about the relationship of heat, work, and energy.

1. Internal energy is a function of state.
2. Heat is a form of energy.

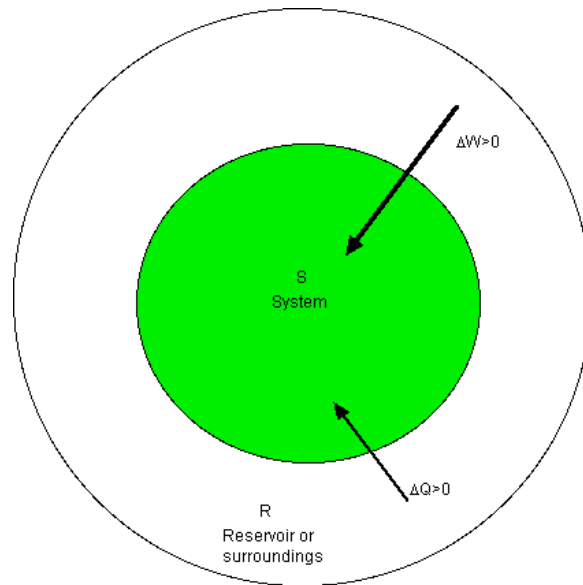
$$\Delta E = \Delta W + \Delta Q \quad (1)$$

where

ΔE : change in the internal energy

ΔW : work done on the system

ΔQ : heat which flows into the system.



For terms on the right hand side of Eq.(1) the sign is taken as positive when energy is added to the system. In other words, if heat flows from system to surroundings, ΔQ is negative. Similarly, if the system does work on the surroundings, ΔW is negative.

Work W

We call the transfer of energy “work”, if it involves an ordered process like a force pushing a piston.

Heat Q

We call it “heat”, if it involves energy in a disordered form like the thermal motions of atoms in a solid or of molecules in a gas.

Internal energy of the system E

17 The first law of thermodynamics (II)

If the surroundings exert a constant pressure P , the work done on the system is

$$\Delta W = -P\Delta V$$

where ΔV is the change of volume of the system. Then we have

$$\Delta Q = \Delta E + P\Delta V$$

In the case of C_V (heat capacity at constant volume), $\Delta V = 0$,

$$C_V = \left(\frac{\partial Q}{\partial T}\right)_V = \left(\frac{\partial E}{\partial T}\right)_V$$

In the case of C_P (heat capacity at constant pressure), $\Delta P = 0$,

$$C_P = \left(\frac{\partial Q}{\partial T}\right)_P = \left(\frac{\partial E}{\partial T}\right)_P + P\left(\frac{\partial V}{\partial T}\right)_P$$

The term $P\left(\frac{\partial V}{\partial T}\right)_P$ is the extra energy required to push back surroundings.

18. Ideal gas

The ideal gas is one which obeys Boyle’s and Joule’s laws at all temperatures and pressures. Real gases do not obey these laws at all temperatures and pressures (van der Waals).

(1)

In particular, an ideal gas does not condense to a liquid, because there is no attractive force between its molecules to make them come together, where all real gases condense at sufficiently low temperatures.

(2)

On the other hand, if T is not too low and the pressure not too high, real gases do obey the Boyle’s and Joule’s laws quite well.

(a) Boyle's law

If T of a given mass of gas is kept constant, then the product PV is constant.

$$PV = Nk_B T = \frac{N}{N_A} N_A k_B T = nRT$$

where N_A is the Avogadro number;

$$N_A = 6.02214179 \times 10^{23}/\text{mol}$$

k_B is the Boltzmann constant,

$$k_B = 1.3806488 \times 10^{-23} \text{ J/K}$$

and n is the number of moles, $n = N/N_A$. R is the gas constant.

$$R = N_A k_B = 8.3144621 \text{ J/mol K}$$

((Example))

$P = 1 \text{ atm} = 1.01325 \times 10^5 \text{ Pa}$. $T = 273 \text{ K}$. $n = 1 \text{ mole}$.

$$V = \frac{nRT}{P} = \frac{8.314427 \times 273}{1.01325 \times 10^5} = 22.4 \times 10^{-3} \text{ m}^3 = 22.4 \text{ liters},$$

which is independent of the kinds of gases

(b) Joule's law

The internal energy of an ideal gas depends only on its temperature.

$$E = E(T)$$

This is derived from the 2nd law of thermodynamics. In general, the change of E when the temperature changes from T to $T + dT$ and the volume changes from V to $V + dV$, is expressed by

$$dE = \left(\frac{\partial E}{\partial T} \right)_V dT + \left(\frac{\partial E}{\partial V} \right)_T dV$$

Suppose that E depends only on T . Then we have

$$dE = \left(\frac{\partial E}{\partial T} \right)_V dT$$

If we warm the gas an infinitesimally small amount dT at a constant volume V ,

$$dE = dQ + dW = dQ$$

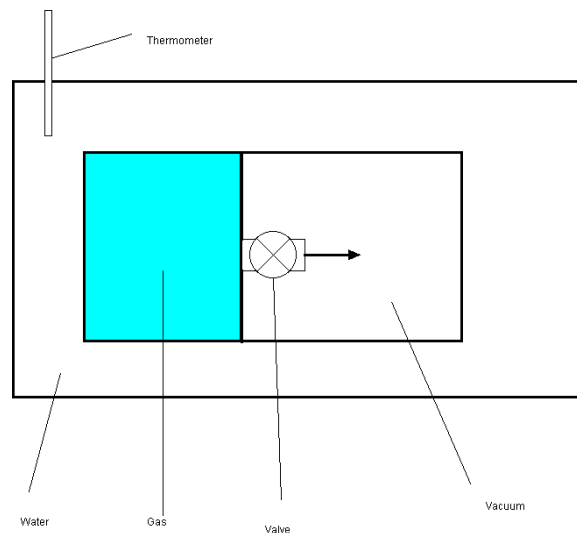
since $dW = -PdV = 0$. So that changes in E will always be given by

$$dE = \left(\frac{\partial Q}{\partial T} \right)_V dT = C_V dT \quad (\text{Joule's law})$$

This is a very important law. We use this equation hereafter.

(c) Historical remark on the free expansion experiment (Joule's law)

The fact that the internal energy E of a gas does not depend on its volume (if the gas is sufficiently dilute that it can be considered ideal) was verified in a classical experiment by Joule. He made use of the free expansion of an ideal gas as illustrated in Fig.



A container consisting of two compartments separated by a valve is immersed in water. Initially, the valve is closed and one compartment is filled with the gas under investigation, while the other compartment is evacuated. Suppose that the valve is

now opened so that the gas is free to expand and fill both compartments. In this process no work gets done by the system consisting of the gas and container. (The container walls are rigid and nothing moves.). Hence one can say, by the first law, that the heat absorbed by this system equals its increase in internal energy.

$$\Delta Q = \Delta E$$

Assume that the internal energy change of the (thin-walled) container is negligibly small. Then ΔE measures simply the energy change of the gas. Joule found that the temperature of the water did not change in this experiment. (Because of the large heat capacity of the water, any anticipated temperature change is, however, quite small; Joule's actual sensitivity of temperature measurement was, in retrospect, rather inadequate.). Thus the water absorbed no heat from the gas; consequently, the heat absorbed by the gas also vanished. All that happens in the experiment is that the temperature of the gas remains unchanged with its volume changes from its initial value V_i to its final value V_f . Since $\Delta Q = 0$, Joule's experiment leads by virtue of $\Delta Q = \Delta E$ to the conclusion

$$E_i(T, V_i) = E(T, V_f)$$

19 Isothermal process

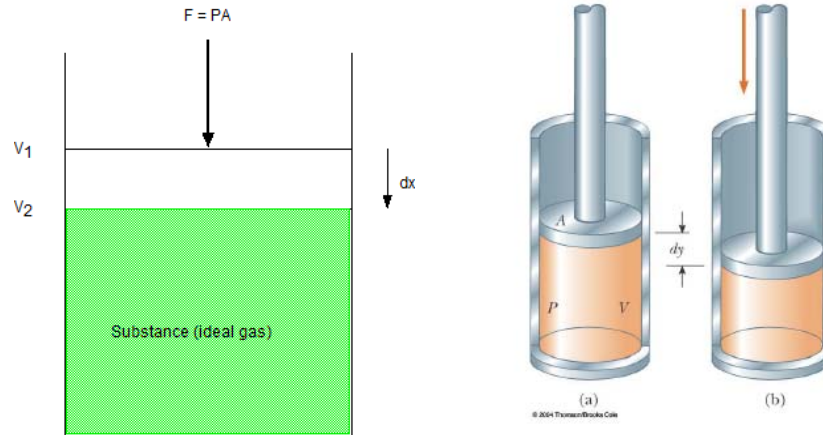
An isothermal process occurs at a constant temperature. An example would be to have a system immersed in a large constant-temperature bath. Any work energy performed by the system will be lost to the bath, but its temperature will remain constant. In other words, the system is thermally connected, by a thermally conductive boundary to a constant-temperature reservoir.

The expression for the work done by a force can be used to derive that for work done by hydrostatic pressure. We consider a substance in a cylinder with a frictionless piston of area A . The work done dW on the system is

$$dW = Fdx = PAdx = -PdV$$

The negative sign is present because when the piston moves in (dx positive), the volume decreases (dV negative). So

$$dV = -Adx$$



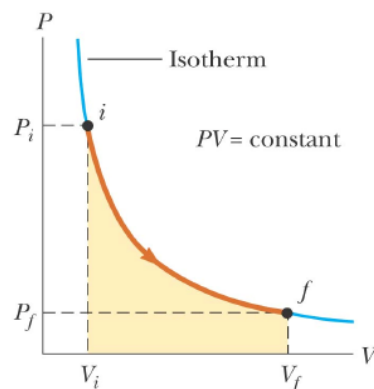
For an ideal gas

$$PV = Nk_B T = \frac{N}{N_A} N_A k_B T = nRT$$

where N_A is the Avogadro number, k_B is the Boltzmann constant, and n is the number of moles. The work done on the system (gas) is given by

$$dW = -PdV = -nRT \frac{dV}{V}$$

$$W = -\int_{V_i}^{V_f} PdV = -\int_{V_i}^{V_f} nRT \frac{dV}{V} = -nRT \ln\left(\frac{V_f}{V_i}\right) = nRT \ln\left(\frac{V_i}{V_f}\right)$$



where V_i and V_f are the initial and final volumes. The work done in the expansion is the area under the curve in a PV diagram. $W > 0$ for compression and $W < 0$ for expansion.

((Note))

$\int_{V_i}^{V_f} PdV$ is the area enclosed by the P vs V curve and $P = 0$ between V_i and V_f

PdV : Work done by the system (gas)

$-PdV$: work done on the system (gas)

In order to avoid the confusion, it is preferable to use W_s for the work done by the system,

$$W_s = -W$$

where W is the work done on the system.

20 Adiabatic process

An **adiabatic** process is a process in which there is no energy added or subtracted from the system by heating or cooling. For a reversible process, this is identical to an isentropic process. We may say that the system is thermally insulated from its environment and that its boundary is a thermal insulator. For a adiabatic change,

$$dQ = 0,$$

we have

$$dE = dW = -PdV = C_v dT$$

Differentiating $PV = RT$ ($n = 1$), we get another formula for dT

$$RdT = PdV + VdP$$

Multiplying C_v on both sides

$$RC_v dT = C_v PdV + C_v VdP$$

Eliminating dT ,

$$R(-PdV) = C_v PdV + C_v VdP$$

Since $C_p - C_v = R$ (Mayer's relation, see Sec. 21), we have

$$(C_p - C_v)(-PdV) = C_v PdV + C_v VdP$$

or

$$\frac{C_p}{C_v} \frac{dV}{V} = -\frac{1}{P} dP$$

Here we define

$$\gamma = \frac{C_p}{C_v}$$

Then we can get

$$\gamma \int \frac{dV}{V} = -\int \frac{1}{P} dP$$

or

$$\gamma \ln(V) = -\ln(P) + \text{const}$$

or

$$PV^\gamma = \text{const} \quad (\text{Poisson})$$

Since $PV = RT$,

$$TV^{\gamma-1} = \text{const}$$

The work done in expanding from a state (P_i, V_i) to a state (P_f, V_f) is given by

$$\begin{aligned}
W &= -\int_{V_i}^{V_f} P dV = -P_i V_i^\gamma \int_{V_i}^{V_f} V^{-\gamma} dV = -\frac{P_i V_i^\gamma}{1-\gamma} [V^{1-\gamma}]_{V_i}^{V_f} = -\frac{P_i V_i^\gamma}{1-\gamma} (V_f^{1-\gamma} - V_i^{1-\gamma}) \\
&= \frac{1}{\gamma-1} (P_f V_f^\gamma V_f^{1-\gamma} - P_i V_i^\gamma V_i^{1-\gamma}) \\
&= \frac{1}{\gamma-1} (P_f V_f - P_i V_i)
\end{aligned}$$

or

$$W = \frac{1}{\gamma-1} (P_f V_f - P_i V_i) = \frac{R}{\gamma-1} (T_f - T_i) = C_V (T_f - T_i)$$

where

$$PV^\gamma = P_i V_i^\gamma = P_f V_f^\gamma.$$

21 Mayer's relation

We derive a formula for the difference of C_p and C_v . If we warm a gas ($n = 1$ mol) slightly at constant P , we have

$$\begin{aligned}
dQ &= C_p dT \\
dW &= -PdV = -RdT \\
dE &= C_v dT
\end{aligned}$$

or

$$\begin{aligned}
dE &= dQ + dW = C_p dT - RdT = (C_p - R)dT \\
&= C_v dT
\end{aligned}$$

Then we have

$$C_p - C_v = R \quad (\text{Mayer's relation})$$

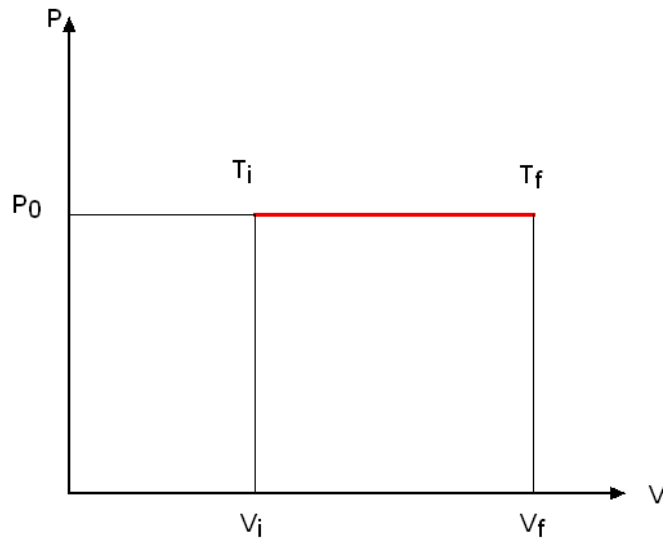
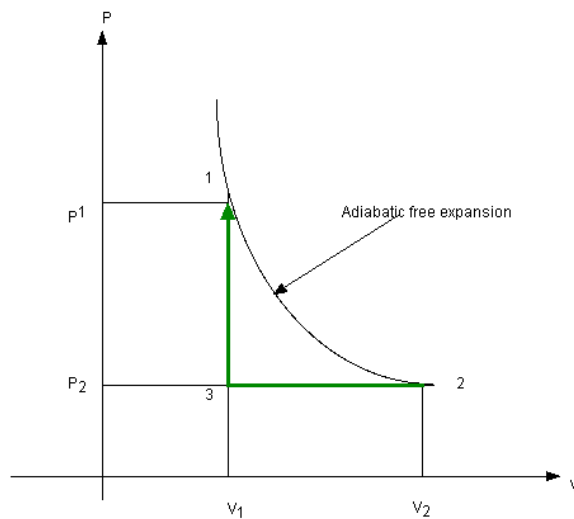


Fig. Change of volume for gas at constant P_0 . $\Delta W = -\int_{V_i}^{V_f} P dV = -P_0 \Delta V = -R \Delta T$.
 $\Delta E = C_v \Delta T$. $\Delta Q = C_p \Delta T$

22 Mayer's cycle

Using the Mayer's cycle, we show the Mayer's relation,



- | | | |
|----|--------------------|-------------------------------|
| 1: | $(P_1, V_1, T_1),$ | $P_1 V_1 = RT_1$ |
| 2: | $(P_2, V_2, T_2),$ | $P_2 V_2 = RT_2 (T_2 = T_1).$ |
| 3: | (P_2, V_1, T_3) | $P_2 V_1 = RT_3$ |

- (1) The process 1→2 (*adiabatic free expansion, irreversible process*)

$$\Delta Q_{12} = 0, \quad \Delta W_{12} = 0.$$

So we have

$$\Delta E_{12} = 0 \quad (T_1 = T_2).$$

(2) The process 2→3 (pressure constant)

$$\Delta Q_{23} = C_p(T_3 - T_2)$$

$$\Delta W_{23} = -\int_{V_2}^{V_1} P_2 dV = -P_2(V_1 - V_2) = -R(T_3 - T_1)$$

$$\Delta E_{23} = \Delta Q_{23} + \Delta W_{23} = C_p(T_3 - T_2) - R(T_3 - T_1)$$

(3) The process 3→1 (volume constant)

$$\Delta Q_{31} = C_v(T_1 - T_3)$$

$$\Delta W_{31} = 0$$

$$\Delta E_{31} = \Delta Q_{31} + \Delta W_{31} = C_v(T_1 - T_3)$$

Then we have

$$\Delta E = \Delta E_{12} + \Delta E_{23} + \Delta E_{31} = C_p(T_3 - T_2) - R(T_3 - T_1) - C_v(T_3 - T_1) = 0$$

From this we have the Mayer's relation; $C_p - C_v = R$.

((Note)) **Adiabatic free expansion**

An adiabatic free expansion of an ideal gas, where a greater volume suddenly becomes available to the gas, is an irreversible process which proceeds through a chaotic non-equilibrium path. Nonetheless we can characterize the beginning and end points and the net values of relevant changes in energy. Since the gas expands against a vacuum it does no work and thus

$$\Delta W = 0.$$

Combining this with our requirement that the process is adiabatic we have

$$\Delta E = \Delta Q + \Delta W = 0$$

((Wikipedia) Adiabatic free expansion

Free expansion is an irreversible process in which a gas expands into an insulated evacuated chamber. Real gases experience a temperature change during free expansion. For an ideal gas, the temperature doesn't change, and the conditions before and after adiabatic free expansion satisfy $P_i V_i = P_f V_f$, where P is the pressure, V is the volume, and i and f refer to the initial and final states.

During free expansion, no work is done by the gas. The gas goes through states of no thermodynamic equilibrium before reaching its final state, which implies that one cannot define thermodynamic parameters as values of the gas as a whole. For example, the pressure changes locally from point to point, and the volume occupied by the gas (which is formed of particles) is not a well defined quantity.

23 Constant γ

Since $C_V = 3R/2$ for a monatomic ideal gas (see Chapter 19), Mayer's relation predicts

$$C_P = C_V + R = \frac{5R}{2}$$

Then the ratio γ is

$$\gamma = \frac{C_P}{C_V} = \frac{5}{3} = 1.67$$

When f is the degree of freedom (in general), the heat capacity C_V is described by

$$C_V = \frac{f}{2}R.$$

Since $C_P = C_V + R = \frac{(f+2)R}{2}$, the ratio γ is obtained as

$$\gamma = \frac{C_P}{C_V} = \frac{f+2}{f} = 1 + \frac{2}{f}$$

Experimentally, one can determine the degree of freedom from the value of γ .

TABLE 17.3 Molar Specific Heats of Various Gases

Gas	Molar Specific Heat ^a (J/mol·K)			$\gamma = C_P/C_V$
	C_P	C_V	$C_P - C_V$	
Monatomic Gases				
He	20.8	12.5	8.33	1.67
Ar	20.8	12.5	8.33	1.67
Ne	20.8	12.7	8.12	1.64
Kr	20.8	12.3	8.49	1.69
Diatomic Gases				
H ₂	28.8	20.4	8.33	1.41
N ₂	29.1	20.8	8.33	1.40
O ₂	29.4	21.1	8.33	1.40
CO	29.3	21.0	8.33	1.40
Cl ₂	34.7	25.7	8.96	1.35
Polyatomic Gases				
CO ₂	37.0	28.5	8.50	1.30
SO ₂	40.4	31.4	9.00	1.29
H ₂ O	35.4	27.0	8.37	1.30
CH ₄	35.5	27.1	8.41	1.31

^aAll values except that for water were obtained at 300 K.

24 Physical constants (NIST)

N_A : Avogadro number

$$N_A = 6.02214179 \times 10^{23} \text{ mol}^{-1}$$

k_B : Boltzmann constant

$$k_B = 1.3806488 \times 10^{-23} \text{ J K}^{-1}$$

R : gas constant

$$R = N_A k_B = 8.3144621 \text{ J mol}^{-1} \text{ K}^{-1} = 1.9862 \text{ cal mol}^{-1} \text{ K}^{-1}$$

σ : Stefan-Boltzmann constant

$$\sigma = 5.670400 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

\hbar : Planck's constant

$$\hbar (= h/2\pi) = 1.05457162853 \times 10^{-34} \text{ J s}$$

25. Summary of units and physics constants

$$1 \text{ cal} = 4.186 \text{ J}$$

$$1 \text{ Btu} = 252 \text{ cal} = 1054 \text{ J}$$

$$L_F = 3.33 \times 10^5 \text{ J/kg} = 79.6 \text{ cal/g} = 143 \text{ Btu/lb}$$

$$L_V = 2.256 \times 10^6 \text{ J/kg} = 539 \text{ cal/g} = 970 \text{ Btu/lb}$$

$$C_{\text{water}} = 4180 \text{ J/kg.K} = 4.180 \text{ J/g.K}$$

$$C_{\text{ice}} = 2220 \text{ J/kg.K} = 2.22 \text{ J/g.K}$$

$$p_0 = 1370 \text{ W (solar constant)}$$

$$L_{\text{sun}} = 4\pi a^2 p_0 = 3.853 \times 10^{26} \text{ W (luminosity of the Sun, } a = 1 \text{ AU)}$$

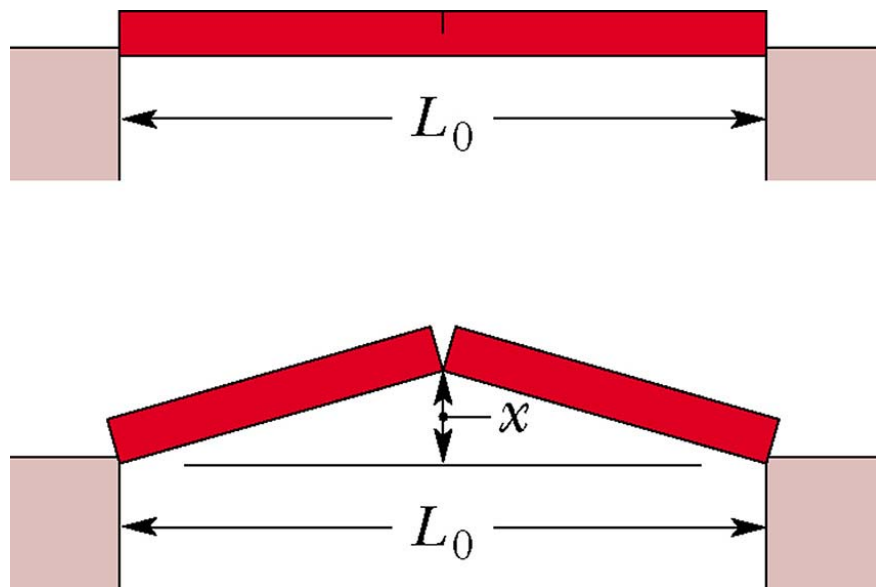
26. Problems related to the coefficient of thermal expansion, heat capacity, and thermal conductivity

26.1

Problem 18-21***(SP-18)

(10-th edition)

As a result of a temperature rise of 32°C , a bar with a crack at its center buckles upward (Fig.). If the fixed distance L_0 is 3.77 m and the coefficient of linear expansion of the bar is $25 \times 10^{-6}/^{\circ}\text{C}$, find the rise x of the center.

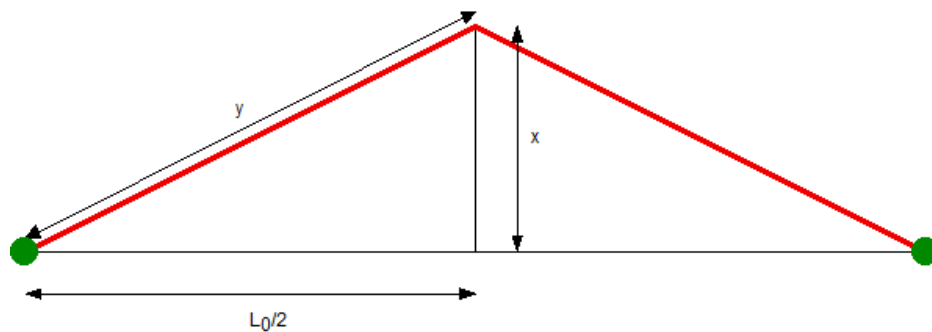


((Solution))

$L_0 = 3.77 \text{ m}$

$\alpha = 25 \times 10^{-6}/^{\circ}\text{C}$

$\Delta T = 32^{\circ}\text{C}$



In the above figure,

$$\begin{aligned}2y &= 2\sqrt{x^2 + \frac{L_0^2}{4}} = L_0\left(1 + \frac{4x^2}{L_0^2}\right)^{1/2} \\ &= L_0\left(1 + \frac{2x^2}{L_0^2}\right) = L_0(1 + \alpha\Delta T)\end{aligned}$$

or

$$\frac{2x^2}{L_0^2} = \alpha\Delta T$$

or

$$x = L_0\sqrt{\frac{\alpha}{2}\Delta T} = 3.77\sqrt{\frac{25}{2}\times 10^{-6}\times 32} = 0.075m$$

26.2

Problem 18-31 (HW-18)**

(10-th edition)

What mass of steam at 100°C must be mixed with 150 g of ice at its melting point, in a thermally insulated container, to produce liquid water at 50°C?

Use

$L_s = 2256 \times 10^3 \text{ J/kg}$ (latent heat of evaporation)

$C_w = 4180 \text{ J/kg K}$ (heat capacity of water)

$L_F = 333 \times 10^3 \text{ J/kg}$ (latent heat of fusion)

1 cal = 4.1868 J

26.3

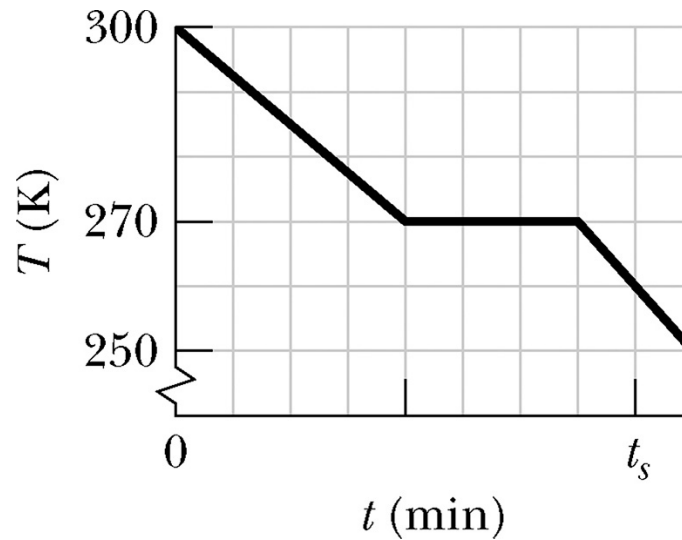
Problem 18-34 (SP-18)**

(8-th edition)

Problem 18-30 (SP-18)**

(9-th edition)

0.400 kg sample is placed in a cooling apparatus that removes energy as heat at a constant rate. Figure gives the temperature T of the sample versus time t ; the horizontal scale is set by $t_s = 80.0$ min. The sample freezes during the energy removal. The specific heat of the sample in its initial liquid phase is 3000 J/kg K. What are (a) the sample's heat of fusion and (b) its specific heat in the frozen phase?



((Solution))

$m = 0.4$ kg, $C_{\text{liq}} = 3000$ J/kg K (heat capacity of the liquid)

$$\frac{dQ}{dt} = mC_{\text{liq}} \frac{dT}{dt}$$

$mL_F =$ Latent heat

$$\frac{dQ}{dt} = mC_{\text{sol}} \frac{dT}{dt}$$

26.4

Problem 18-41*** (SP-18)

(10-th edition)

(a) Two 50 g ice cubes are dropped into 200 g of water in a thermally insulated container. If the water is initially at 25°C, and the ice comes directly from a freezer at -15°C, what is the final temperature at thermal equilibrium? (b) What is the final temperature if only one ice cube is used?

((Solution))

$$m_{\text{water}} = 200 \text{ g at } 25^\circ\text{C}$$

$$m_{\text{ice}} = 50 \text{ g at } -15^\circ\text{C}$$

$$L_f = 333 \times 10^3 \text{ J/kg} = 333 \text{ J/g}$$

$$C_{\text{water}} = 4180 \text{ J/kg.K} = 4.180 \text{ J/g.K}$$

$$C_{\text{ice}} = 2220 \text{ J/kg.K} = 2.22 \text{ J/g.K}$$

(a)

(i) The case-1

We assume that two ices melt and that the equilibrium temperature is positive; $T_e > 0$.

$$\Delta Q_1 = 2m_{\text{ice}}[C_{\text{ice}}(15) + L_f + C_{\text{water}}(T_e - 0)]$$

$$\Delta Q_2 = m_{\text{water}}(25 - T_e)C_{\text{water}}$$

The solution of $\Delta Q_1 = \Delta Q_2$ leads to $T_e = -12.54 < 0$. So this is not the case.

(ii) The case-2

We assume that the ice is partially melted.

In other words, x g become water at 0°C and $(200 - x)$ g is still ice at 0°C.]

$$\Delta Q_3 = 2m_{\text{ice}}C_{\text{ice}}(15) + xL_f$$

$$\Delta Q_4 = m_{\text{water}}(25)C_{\text{water}}$$

So we get $x = 52.76 \text{ g}$

(b)

We assume that a ice (50 g) melt and that the equilibrium temperature is positive; $T_e > 0$.

$$\Delta Q_1 = m_{\text{ice}}[C_{\text{ice}}(15) + L_f + C_{\text{water}}(T_e - 0)]$$

$$\Delta Q_2 = m_{\text{water}}(25 - T_e)C_{\text{water}}$$

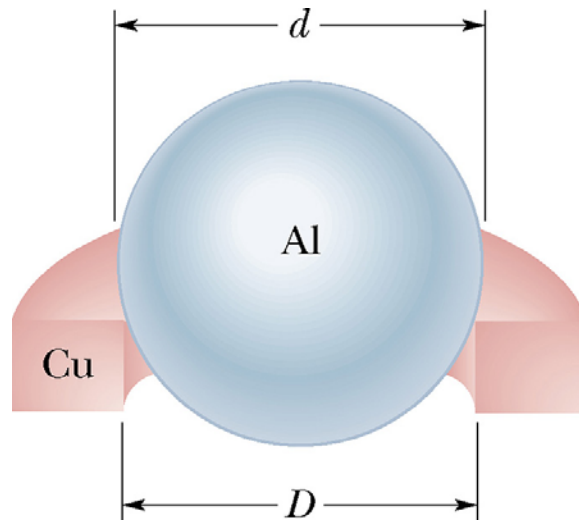
The solution of $\Delta Q_1 = \Delta Q_2$ leads to $T_e = 2.47^\circ\text{C}$. So this is the case.

26.5

Problem 18-42*** (HW-18)

(10-th edition)

A 20.0 g copper ring at 0.000°C has an inner diameter of $D = 2.54000$ cm. An aluminum sphere at 100.0°C has a diameter of $d = 2.54508$ cm. The sphere is placed on top of the ring (Fig.), and the two are allowed to come to thermal equilibrium, with no heat lost to the surroundings. The sphere just passes through the ring at the equilibrium temperature. What is the mass of the sphere?

**((Solution))**

$$d = 2.54508 \text{ cm at } 100^\circ\text{C (Al)}$$

$$D_0 = 2.54000 \text{ cm at } 0^\circ\text{C (Cu)}$$

$$m(\text{Cu}) = 20 \text{ g}$$

$$T(\text{Cu}) = 0^\circ\text{C}$$

$$T(\text{Al}) = 100^\circ\text{C}$$

$$C(\text{Cu}) = 386 \text{ J/kg.K}$$

$$C(\text{Al}) = 900 \text{ J/kg.K}$$

$$\alpha(\text{Cu}) = 17 \times 10^{-6}$$

$$\alpha(\text{Al}) = 23 \times 10^{-6}$$

$$D = D_0[1 + \alpha(\text{Cu})[T_f - T(\text{Cu})]]$$

$$d = d_0[1 + \alpha(\text{Al})[T(\text{Al}) - T_f]]$$

$$\Delta Q(Al) = m(Al)C(Al)[T_f - T(Cu)]$$

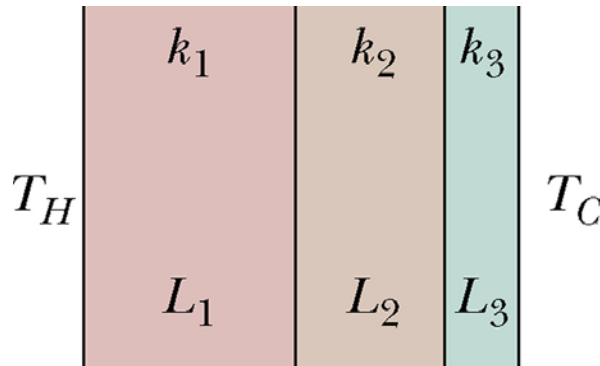
$$\Delta Q(Cu) = m(Cu)C(Cu)[T(AL) - T_f]$$

26.6

Problem 18-60 (SP-18)**

(10-th edition)

Figure shows the cross section of a wall made of three layers. The thickness of the layers are L_1 , $L_2 = 0.700 L_1$, and $L_3 = 0.350 L_1$. The thermal conductivities are κ_1 , $\kappa_2 = 0.900 \kappa_1$, and $\kappa_3 = 0.800 \kappa_1$. The temperatures at the left and right sides of the wall are $T_H = 30.0^\circ\text{C}$ and $T_C = -15.0^\circ\text{C}$, respectively. Thermal conduction is steady. (a) What is the temperature difference ΔT_2 across layer 2 (between the left and right sides of the layer)? If κ_2 were, instead, equal to $1.1 \kappa_1$, (b) would the rate at which energy is conducted through the wall be greater than, less than, or the same as previously, and (c) what would be the value of ΔT_2 ?



((Solution))

$$T_H = 30^\circ\text{C}, \quad T_C = -15^\circ\text{C},$$

$$L_2 = 0.700 L_1, \quad L_3 = 0.350 L_1$$

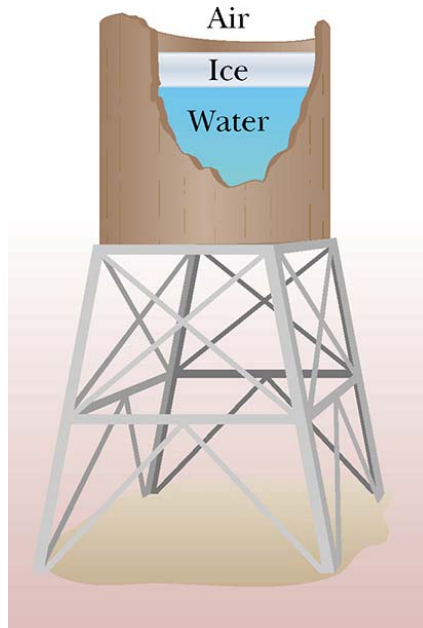
$$k_3 = 0.8 k_1.$$

$$\dot{Q} = \frac{dQ}{dt} = -kA \frac{dT}{dx} = k_1 A \left(\frac{T_H - T_{12}}{L_1} \right) = k_2 A \left(\frac{T_{12} - T_{23}}{L_2} \right) = k_3 A \left(\frac{T_{23} - T_C}{L_3} \right)$$

26.7

Problem 18-61 (HW-18)****(10-th edition)**

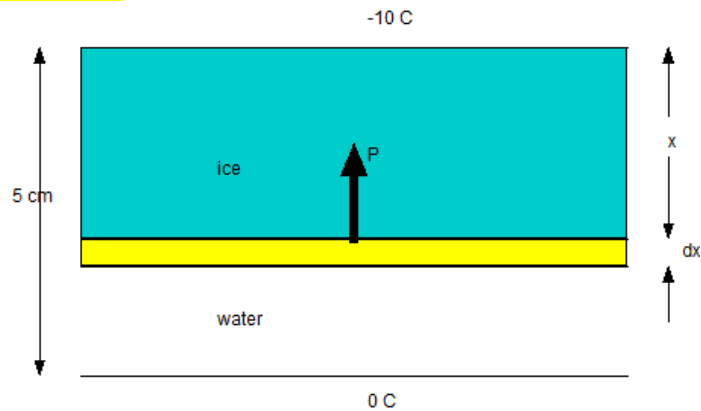
A tank of water has been outdoors in cold weather, and a slab of ice 5.0 cm thick has formed on its surface (Fig.). The air above the ice is at -10°C . Calculate the rate of ice formation (in cm/h) on the ice slab. Take the thermal conductivity of ice to be $0.004 \text{ cal/s cm } ^{\circ}\text{C}$ and its density to be 0.92 g/cm^3 . Assume no energy transfer through the tank.

**((Solution))**

$$\kappa_{\text{ice}} = 0.0040 \text{ cal/s cm } ^{\circ}\text{C}, \quad L_f = 3.33 \times 10^5 \text{ J/kg} = 79.50 \text{ cal/g}$$

$$\rho_{\text{ice}} = 0.92 \text{ g/cm}^3$$

$$T_C = -10^{\circ}\text{C}, \quad T_H = 0^{\circ}\text{C}.$$



$$\begin{aligned}\Delta Q &= L_f (A dx) \rho_{ice} \\ \phi &= \frac{dQ}{dt} = -k_{ice} A \frac{dT}{dx} = k_{ice} A \frac{\Delta T}{x} \\ &= L_f A \rho_{ice} \frac{dx}{dt}\end{aligned}$$

26.8 Serway

A pound of water at 0°C is covered with a layer of ice 4.0 cm thick. If the air temperature stays constant at -10°C, how long does it take for the ice thickness to increase to 8.0 cm?

((Solution))

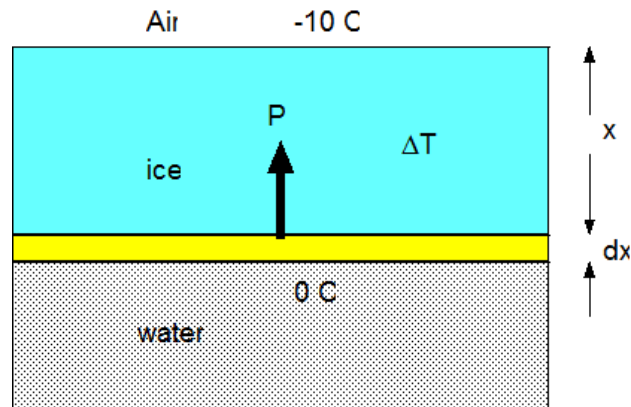
$$x = x_0 = 4.0 \text{ cm at } t = 0$$

$$x = x_f = 8.0 \text{ cm at } t = t_f.$$

$$\rho = 0.917 \times 10^3 \text{ kg/m}^3$$

$$L_F = 3.33 \times 10^5 \text{ J/kg}$$

$$\kappa = 2 \text{ W/(m K)}.$$



We use the equation in the form

$$\frac{dQ}{dt} = \kappa A \frac{\Delta T}{x} \quad (1)$$

where $\Delta T = 10 \text{ K}$, κ is the thermal conductivity of ice, $\kappa = 2 \text{ W/(m K)}$. We also note that the incremental energy dQ extracted from the water through the thickness x of ice is the amount required to freeze a thickness dx of ice. That is,

$$dQ = L_F \rho A dx \quad (2)$$

where ρ is the density of the ice ($\rho = 0.917 \times 10^3 \text{ kg/m}^3$), A is the area, and L_F is the Latent heat of fusion, $L_F = 3.33 \times 10^5 \text{ J/kg}$. From Eqs.(1) and (2), we have a first order differential equation for $x(t)$,

$$L_F \rho A \frac{dx}{dt} = \kappa A \frac{\Delta T}{x}$$

or

$$\frac{dx}{dt} = \left(\frac{\kappa \Delta T}{L_F \rho} \right) \frac{1}{x} = \frac{\alpha}{x} \quad (3)$$

where

$$\alpha = \frac{\kappa \Delta T}{L_F \rho} = 6.55 \times 10^{-8} [\text{m}^2/\text{s}] = 6.55 \times 10^{-4} [\text{cm}^2/\text{s}]$$

The solution of the differential equation Eq.(3) is as follows.

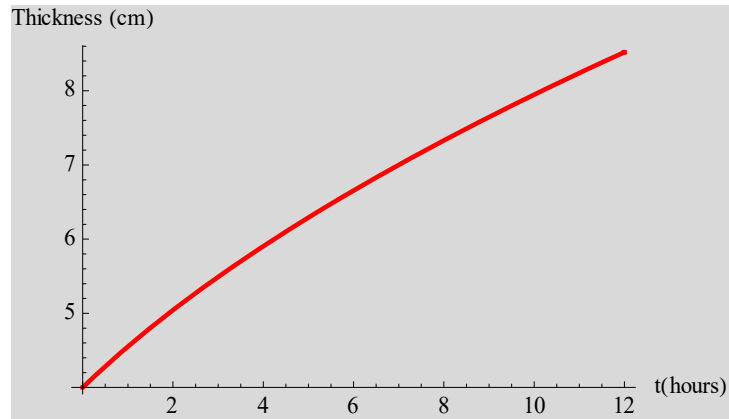
$$\int_{x_0}^x x dx = \int_0^t \alpha dt$$

or

$$x = \sqrt{x_0^2 + 2\alpha t}$$

where $x = x_0$ at $t = 0$ (initial condition).

It takes $t = 10.1787 \text{ h}$ for $x_f = 8.0 \text{ cm}$.

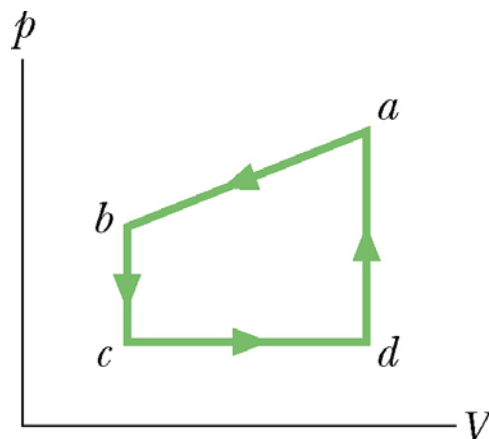


27 Problems related to the P - V phase diagram

27.1

Problem 18-49 (HW-18) (10-th edition)

Figure displays a closed cycle for a gas (the figure is not drawn to scale). The change in the internal energy of the gas as it moves from a to c along the path abc is -200 J. As it moves from c to d , 180 J must be transferred to it as heat. An additional transfer of 80 J to it as heat is needed as it moves from d to a . How much work is done on the gas as it moves from c to d ?



((Solution))

$$E_{abc} = -200 \text{ J}$$

$$Q_{cd} = 180 \text{ J}$$

$$Q_{da} = 80 \text{ J}$$

$$W_{da} = 0$$

$$E_{da} = 80 \text{ J}$$

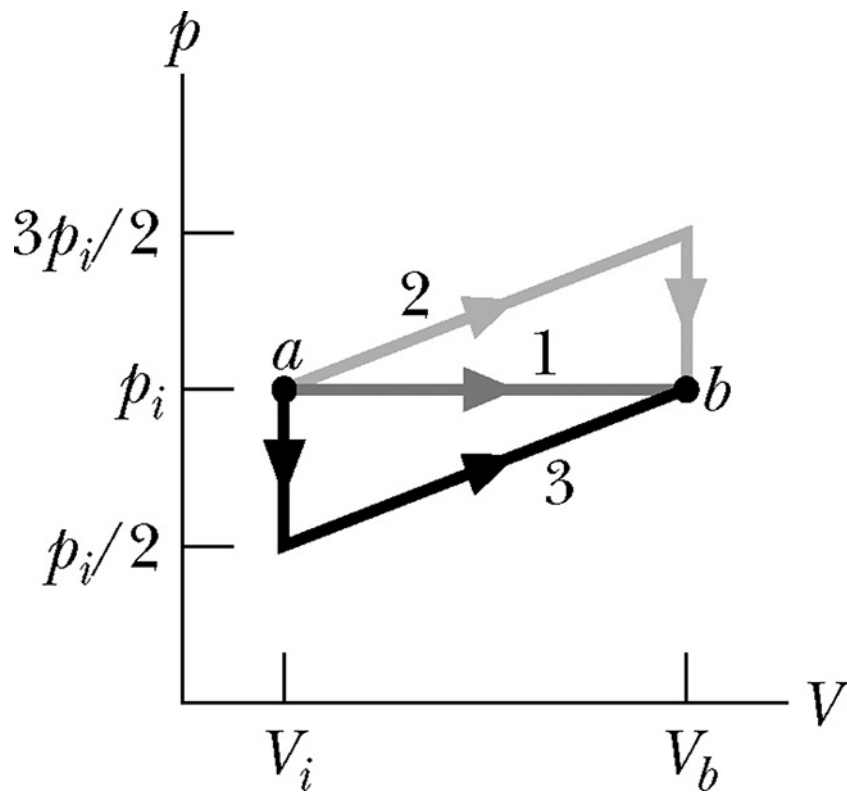
$$E_{cda} = -E_{abc} = 200 \text{ J} = E_{cd} + E_{da} = E_{cd} + 80 \text{ J}$$

27.2

Problem 18-81 (SP-18)

(10-th edition)

A sample of gas undergoes a transition from an initial state a to a final state b by three different paths (processes), as shown in the P - V diagram in Fig., where $V_b = 5.00 V_i$. The energy transferred to the gas as heat in process 1 is $10 P_i V_i$. In terms of $P_i V_i$, what are (a) the energy transferred to the gas as heat in process 2 and (b) the change in internal energy that the gas undergoes in process 3?



((Solution))

$$V_b = 5 V_i$$

Path 1

$$\begin{aligned}\Delta Q_1 &= 10P_iV_i \\ \Delta W_1 &= -P_i(5V_i - V_i) = -4P_iV_i \\ \Delta E_1 &= \Delta Q_1 + \Delta W_1 = 6P_iV_i\end{aligned}$$

Path 2

$$\begin{aligned}\Delta W_2 &= -\frac{1}{2}\left(P_i + \frac{3P_i}{2}\right)(5V_i - V_i) = -5P_iV_i \\ \Delta E_2 &= \Delta E_1 = 6P_iV_i \\ 6P_iV_i &= \Delta Q_2 + \Delta W_2 = \Delta Q_2 - 5P_iV_i \\ \Delta Q_2 &= 11P_iV_i\end{aligned}$$

Path 3

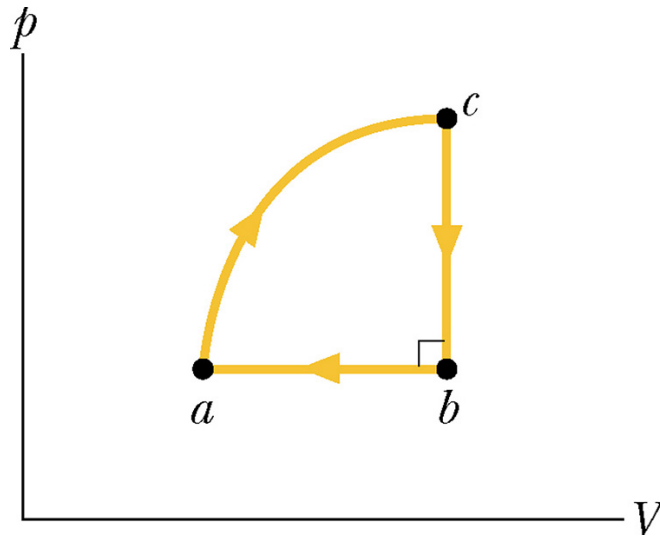
$$\Delta E_3 = \Delta E_1 = 6P_iV_i$$

27.3

Problem 18-75 (SP-18)

(10-th edition)

Figure displays a closed cycle for a gas. From c to b , 40 J is transferred from the gas as heat. From b to a , 130 J is transferred from the gas as heat, and the magnitude of the work done by the gas is 80 J. From a to c , 400 J is transferred to the gas as heat. What is the work done by the gas from a to c (Hint: you need to supply the plus and minus signs for the given data.)



((Solution))

$$W_{cb} = 0$$

$$Q_{cb} = -40\text{ J}$$

$$E_{cb} = -40\text{ J}$$

$$Q_{ba} = -130\text{ J}$$

$$W_{ba} = 80\text{ J}$$

$$E_{ba} = -50\text{ J}$$

$$Q_{ac} = 400\text{ J}$$

$$E_{ac} + E_{cb} + E_{ba} = 0$$

$$E_{ac} = -E_{cb} - E_{ba} = 40 + 50 = 90\text{ J}$$

$$E_{ac} = W_{ac} + Q_{ac}$$

$$W_{ac} = E_{ac} - Q_{ac} = 90 - 400 = -310\text{ J}$$

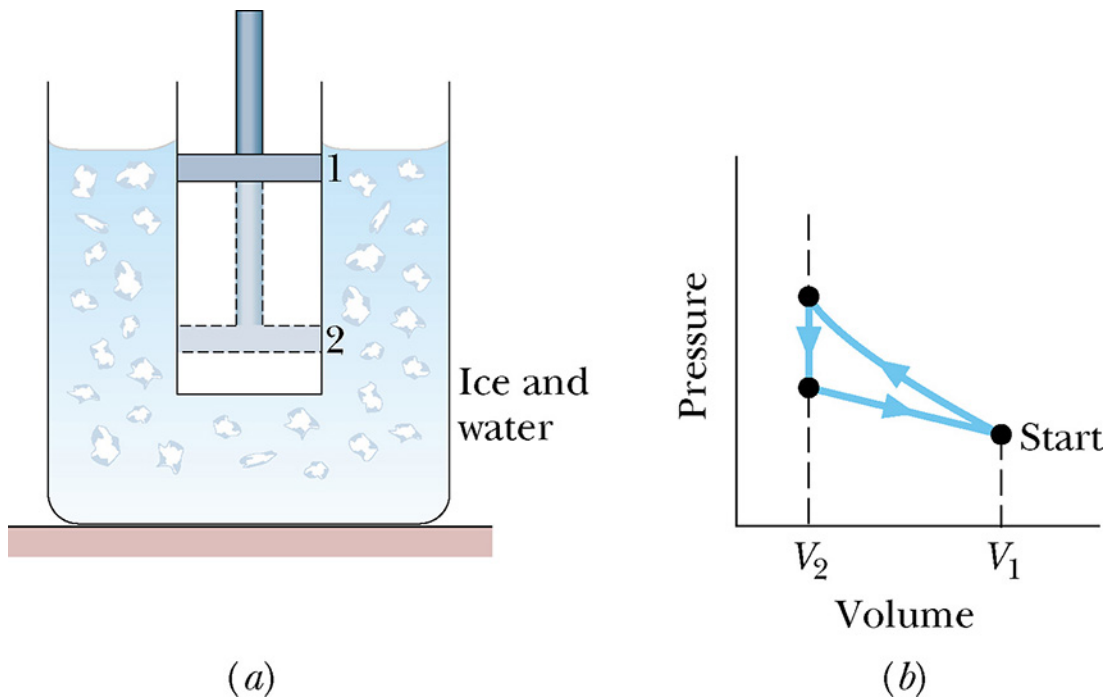
27.4

Problem 18-80 (SP-18)

(10-th edition)

Figure (a) shows a cylinder containing gas and closed by a movable piston. The cylinder is kept submerged in an ice-water mixture. The piston is quickly pushed down from position 1 to position 2 and then held at position 2 until the gas is again at the temperature of the ice-water mixture; it then is slowly raised **back to position 1**.

Figure (b) is a P - V diagram of the process. If 100 g of ice is melted during the cycle, how much work has been done on the gas?



((Solution))

$$L_F = 333 \times 10^3 \text{ J/kg}, \quad m = 0.10 \text{ kg.}$$

$$\Delta Q(\text{ice} - \text{melting}) = mL_F = 3.33 \times 10^4 \text{ J}$$

In the one cycle of the P - V diagram,

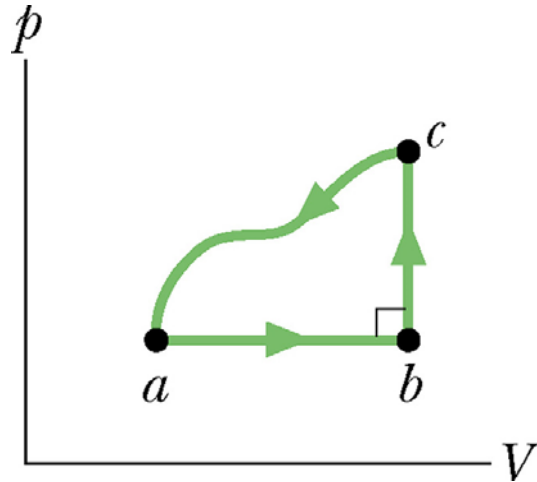
$$\Delta E = \Delta Q + \Delta W = 0$$

27.5

Problem 18-69 (SP-18)

(10-th edition)

Figure displays a closed cycle for a gas. The change in internal energy along ca is -160 J. The energy transferred to the gas is 200 J along path ab , and 40 J along path bc . How much work is done by the gas along (a) path abc and (b) path ab ?



((Solution))

$$E_{ca} = -160 \text{ J}, \quad Q_{ab} = 200 \text{ J}, \quad Q_{bc} = 40 \text{ J}, \quad W_{bc} = 0.$$

$$E_{bc} = Q_{bc} + W_{bc} = 40 \text{ J}$$

$$E_{ab} = -E_{bc} - E_{ca} = 120 \text{ J}$$

$$W_{ab} = E_{ab} - Q_{ab} = 120 - 200 = -80 \text{ J}$$

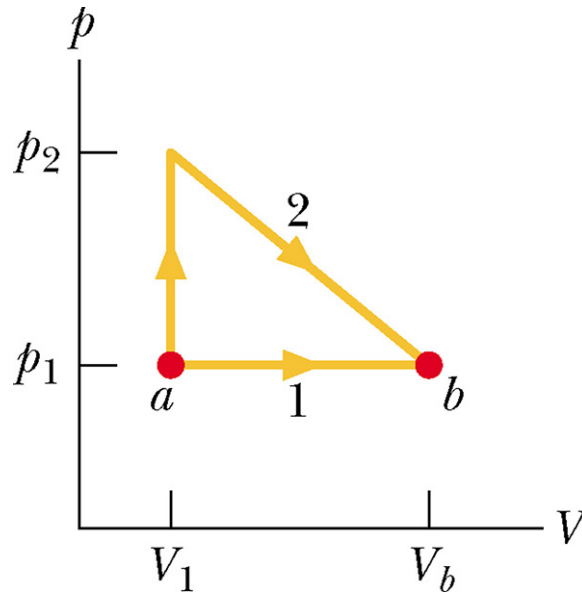
$$W_{abc} = W_{ab} + W_{bc} = -80 \text{ J}$$

27.6

Problem 18-98

(10-th edition)

The P - V diagram shows two paths along which a sample of gas can be taken from state a to state b , where $V_b = 3.0 V_1$. Path 1 requires that energy equal to $5.0 P_1 V_1$ be transferred to the gas as heat. Path 2 requires that energy equal to $5.5 P_1 V_1$ be transferred to the gas as heat. What is the ratio P_2/P_1 ?



((Solution))

For the path-1 (a→b).

$$Q_1 = 5.0P_1V_1$$

$$W_1 = -P_1(3V_1 - V_1) = -2.0P_1V_1$$

$$E_1 = Q_1 + W_1 = 3.0P_1V_1$$

For the path-2

$$Q_2 = 5.5P_1V_1$$

$$W_2 - W_1 = -\frac{2V_1}{2}(-P_1 + P_2) = P_1V_1 - P_2V_1$$

$$W_2 = -P_1V_1 - P_2V_1$$

$$E_2 = Q_2 + W_2 = 4.5P_1V_1 - P_2V_1$$

28. Link

Lecture Note (University of Rochester)

<http://teacher.pas.rochester.edu/phy121/LectureNotes/Contents.html>

Kirchhoff's law

http://en.wikipedia.org/wiki/Kirchhoff's_law_of_thermal_radiation

Stefan-Boltzmann law

http://en.wikipedia.org/wiki/Stefan-Boltzmann_law

Black body problem

<http://instruct1.cit.cornell.edu/Courses/astro101/lectures/lec09.htm>

Planck's law

http://en.wikipedia.org/wiki/Planck%27s_law_of_black_body_radiation

Hubble's law

http://en.wikipedia.org/wiki/Hubble%27s_constant

<http://hyperphysics.phy-astr.gsu.edu/hbase/astro/hubble.html>

Cosmic background radiation

http://en.wikipedia.org/wiki/Cosmic_microwave_background_radiation

<http://csep10.phys.utk.edu/astr162/lect/cosmology/cbr.html>

http://www.astro.ubc.ca/people/scott/cmb_intro.html

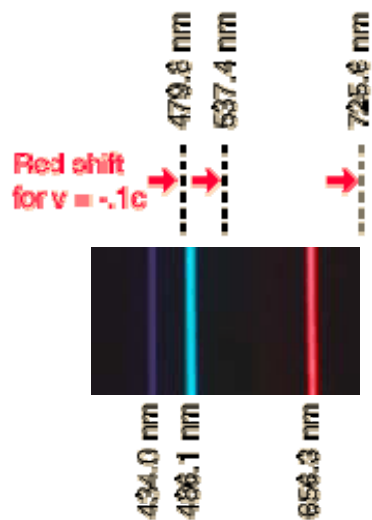
http://www.amtp.cam.ac.uk/user/gr/public/cmbr_home.html

APPENDIX

A. Expanding Universe

A.1. Hubble's law

Edwin Hubble (1920)



Redshift of a receding object,

$$\frac{\Delta\lambda}{\lambda_0} = \frac{v}{c} = z \quad (1)$$

with

$$\frac{\lambda'}{\lambda} = \sqrt{\frac{1 + \frac{v}{c}}{1 - \frac{v}{c}}} > 1 \quad (\text{red shift})$$

where z is a redshift of an object, and v is a recessional velocity.

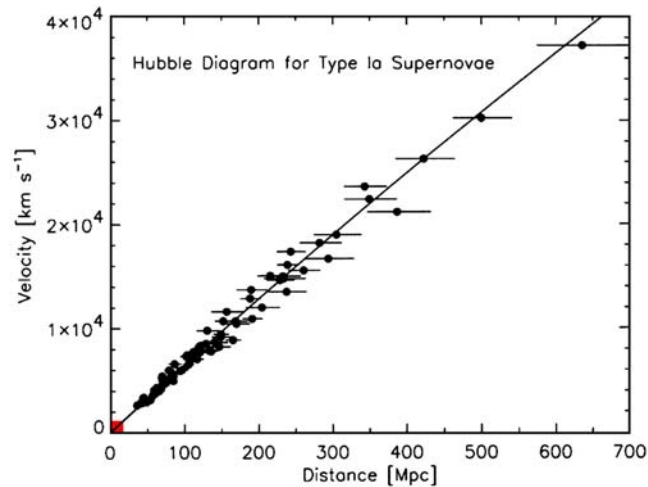
$$1 \text{ pc} = 3.09 \times 10^{13} \text{ km} = 3.26 \text{ ly}$$

$$1 \text{ Mpc} = 3.26 \times 10^6 \text{ ly.}$$

where pc is a parsec and ly is a light year.

Hubble's law

$$v = H_0 d \quad (2)$$



v is the recessional velocity of a galaxy, H_0 is a Hubble constant, d is the distance to the galaxy.

$$H_0 = 75 \text{ km/(s Mpc).}$$

Clusters of galaxies are getting farther and farther apart as time goes. That means the universe is expanding. The redshift caused by the expansion of the universe is properly called a cosmological redshift.

A2. Big Bang

How long ago did the Big Bang take place?

$$T_0 = \frac{d}{v} = \frac{1}{H_0}$$

where T_0 is the same for all galaxies.

$$T_0 = 1/(75 \text{ km}/(\text{s Mpc})) = \frac{1}{75} \frac{\text{Mpc} \cdot \text{sec}}{\text{km}} = \frac{1}{75} \cdot \frac{3.09 \times 10^{19}}{3.156 \times 10^7} \text{ year} = 1.3 \times 10^{10} \text{ years}$$

or

$$T_0 = 13 \text{ billion years}$$

Note that $1 \text{ year} = 365 \times 24 \times 60 \times 60 = 3.156 \times 10^7 \text{ sec}$. The age of the solar system is 4.5 billion years.

B. Blackbody problem

B.1 Planck's law

We now consider the photons inside the blackbody with a cube (side L). The system is in thermal equilibrium at a temperature T . The photon with the wave number k ($=2\pi/\lambda$) has an energy of $\epsilon_k = \hbar\omega = \hbar ck$. There are 2 states per $\left(\frac{\pi}{L}\right)^3$ for each wave number k ,

$$\omega = ck = c\sqrt{k_x^2 + k_y^2 + k_z^2}$$

or

$$\frac{\omega^2}{c^2} = k_x^2 + k_y^2 + k_z^2$$

The density of states (k to $k + dk$)

$$\rho_k dk = \frac{1}{8} \frac{4\pi k^2 dk}{\left(\frac{\pi}{L}\right)^3} 2 = \frac{Vk^2 dk}{\pi^2}$$

where $V = L^3$.

Since $\omega = ck$,

$$\rho_\omega d\omega = \frac{V\left(\frac{\omega}{c}\right)^2 d\frac{\omega}{c}}{\pi^2} = \frac{V\omega^2 d\omega}{\pi^2 c^3}$$

of modes having their frequencies between ω and $\omega + d\omega$.

$$\rho_\omega = \frac{V\omega^2}{\pi^2 c^3} \quad (\text{density of modes})$$

We have the following formula;

$$\sum_k \rightarrow \int \rho_k dk$$

or

$$\sum_k \rightarrow \int \rho_\omega d\omega$$

For single mode $|\mathbf{k}\rangle$, the energy is given by

$$E_{n,\mathbf{k}} = \left(n_k + \frac{1}{2}\right)\hbar\omega_{\mathbf{k}}$$

We use the Planck distribution. The total energy is given by

$$E_{tot} = \sum_{\mathbf{k}} n_{\mathbf{k}} \hbar\omega_{\mathbf{k}} = \int \frac{V\omega^2}{\pi^2 c^3} d\omega n_{\omega} \hbar\omega = V \int u(\omega) d\omega$$

The Planck's law for the radiation energy density is given by

$$u(\omega) = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} = \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp(x) - 1} = \frac{k_B^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1}$$

where

$$x = \frac{\hbar\omega}{k_B T}.$$

The experimentally observed spectral distribution of the black body radiation is very well fitted by the formula discovered by Planck.

(i) Region of Wien

$$x = \frac{\hbar\omega}{k_B T} \gg 1$$

$$u(\omega) = \frac{k_B^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1} \approx \frac{k_B^3 T^3}{\pi^2 \hbar^2 c^3} x^3 e^{-x}$$

(ii) Region of Rayleigh-Jeans

$$x = \frac{\hbar\omega}{k_B T} \gg 1$$

$$u(\omega) = \frac{k_B^3 T^3}{\pi^2 \hbar^2 c^3} \frac{x^3}{\exp(x) - 1} \approx \frac{k_B^3 T^3}{\pi^2 \hbar^2 c^3} x^2$$

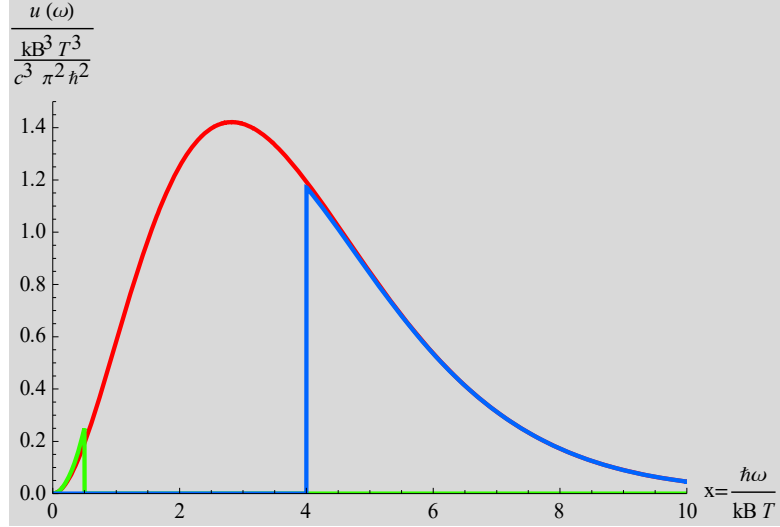


Fig. Planck's black body radiation spectrum (red) as a function of $x = \hbar\omega/k_B T$. Region of Wien (green; particle-like) and region of Rayleigh-Jean (blue; wave-like).

B2. Wien's displacement law

We start with

$$\int_0^{\infty} u(\omega) d\omega = \int_0^{\infty} \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} d\omega$$

Since $\omega = \frac{2\pi c}{\lambda}$, $d\omega = -2\pi c \frac{d\lambda}{\lambda^2}$

$$\int_0^{\infty} u(\omega) d\omega = \int_0^{\infty} \frac{\hbar\omega^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{\hbar\omega}{k_B T}\right) - 1} d\omega = \int_0^{\infty} \frac{\hbar\left(\frac{2\pi c}{\lambda}\right)^3}{\pi^2 c^3} \frac{1}{\exp\left(\frac{2\pi\hbar c}{\lambda k_B T}\right) - 1} 2\pi c \frac{d\lambda}{\lambda^2}$$

or

$$\int_0^{\infty} u(\omega) d\omega = \int_0^{\infty} u(\lambda) d\lambda = \int_0^{\infty} 16\pi^2 \hbar c \frac{1}{\lambda^5} \frac{1}{\exp\left(\frac{2\pi\hbar c}{\lambda k_B T}\right) - 1} d\lambda$$

$$u(\lambda) = 16\pi^2 \hbar c \frac{1}{\lambda^5} \frac{1}{\exp\left(\frac{2\pi\hbar c}{\lambda k_B T}\right) - 1}$$

This has a maximum at

$$\frac{2\pi\hbar c}{\lambda k_B T} = 4.96511.$$

The wavelength of the maximum intensity of radiation is given by the Wien displacement law,

$$\lambda = \frac{2.897768551}{T} \times 10^6 \text{ (nm)}.$$

3. Stefan-Boltzmann radiation law for a black body (1879).

The total energy per unit volume is given by

$$\varepsilon = \frac{E_{tot}}{V} = \int u(\omega) d\omega = \frac{(k_B T)^4}{\pi^2 \hbar^3 c^3} \int_0^\infty \frac{x^3 dx}{\exp(x) - 1} = \frac{\pi^2 (k_B T)^4}{15 \hbar^3 c^3}$$

((Mathematica))

$$\int_0^\infty \frac{x^3}{e^x - 1} dx = \frac{\pi^4}{15}$$

A spherical enclosure is in equilibrium at the temperature T with a radiation field that it contains. The power emitted through a hole of unit area in the wall of enclosure is

$$P = \frac{1}{4} c \varepsilon = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} T^4 \quad (\text{W/m}^2)$$

σ is the Stefan-Boltzmann constant and $1/4$ is a geometrical factor.

$$\sigma = \frac{\pi^2 k_B^4}{60 \hbar^3 c^2} = 0.567 \times 10^{-4} \text{ erg/s-cm}^2\text{-K}^4 = 5.67 \times 10^{-8} \text{ W m}^{-2} \text{ K}^{-4}$$

((Mathematica))

Planck's law for the radiative energy density

$$u = \frac{\hbar \omega^3}{\pi^2 c^3} \frac{1}{\text{Exp}\left[\frac{\hbar \omega}{k_B T}\right] - 1}$$

$$\frac{\omega^3 \hbar}{c^3 \left(-1 + e^{\frac{\omega \hbar}{k_B T}}\right) \pi^2}$$

$$\text{rule1} = \left\{ \omega \rightarrow \frac{x}{\hbar} k_B T \right\}$$

$$\left\{ \omega \rightarrow \frac{k_B T x}{\hbar} \right\}$$

`u1 = u /. rule1 // Simplify`

$$\frac{k_B^3 T^3 x^3}{c^3 (-1 + e^x) \pi^2 \hbar^2}$$

$$u2 = \frac{x^3}{e^x - 1}$$

$$\frac{x^3}{-1 + e^x}$$

Wien's formula ($x \gg 1$)

$$u2W = \text{If}[x > 4, x^3 e^{-x}, 0]$$

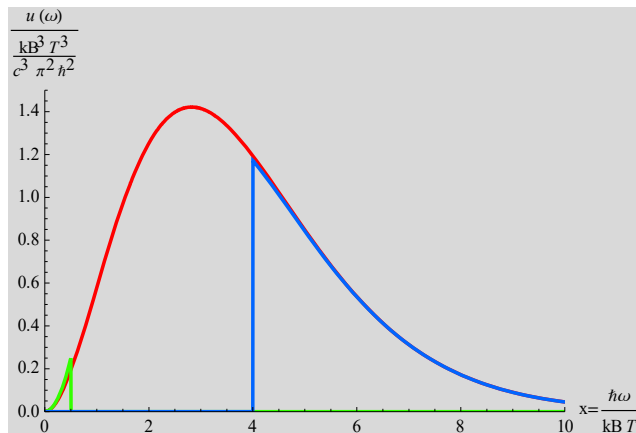
$$\text{If}[x > 4, x^3 e^{-x}, 0]$$

Rayleigh-Jean's formula ($x \ll 1$)

$$u2RJ = \text{If}[x \leq 0.5, x^2, 0]$$

$$\text{If}[x \leq 0.5, x^2, 0]$$

```
Plot[{u2, u2RJ, u2W}, {x, 0, 10},
PlotStyle -> Table[{Hue[0.3 i], Thick}, {i, 0, 2}],
AxesLabel -> {"x = \frac{\hbar \omega}{k_B T}", "\frac{u(\omega)}{k_B^3 T^3 / c^3 \pi^2 \hbar^2}"},
PlotRange -> {{0, 10}, {0, 1.5}},
Background -> LightGray]
```



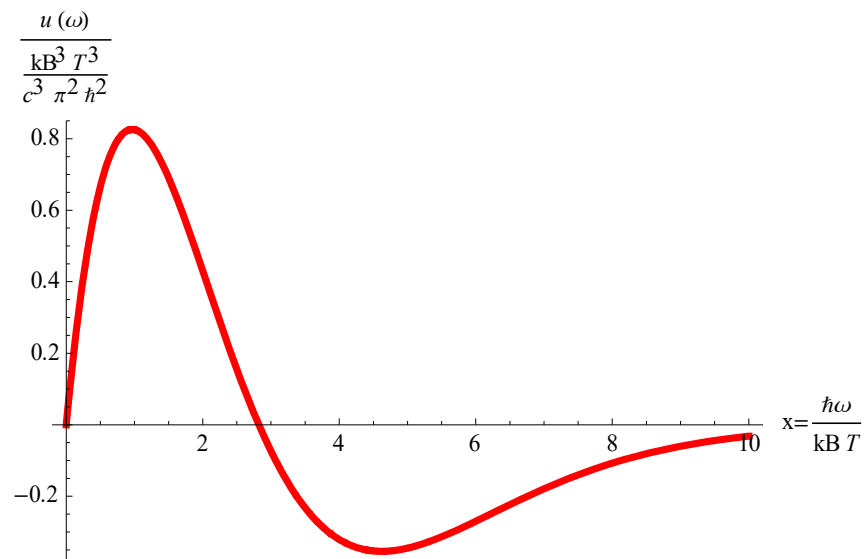
```
u3 = D[u2, x] // Simplify
```

$$-\frac{(3 + e^x (-3 + x)) x^2}{(-1 + e^x)^2}$$

```
Plot[u3, {x, 0, 10},
```

```
PlotStyle -> {Hue[0], Thickness[0.01]},
```

```
AxesLabel -> {"x =  $\frac{\hbar\omega}{k_B T}$ ", " $\frac{u(\omega)}{\frac{k_B^3 T^3}{c^3 \pi^2 \hbar^2}}$ "}]
```



```
FindMaximum[u2, {x, 0.1}]
```

```
{1.42144, {x -> 2.82144}}
```

```
FindRoot[u3, {x, 2, 4}]
```

```
{x -> 2.82144}
```

```
Series[u2, {x, 0, 5}]
```

$$x^2 - \frac{x^3}{2} + \frac{x^4}{12} + O[x]^6$$

Wien's displacement law

```
rule1 = { $\hbar \rightarrow 1.054571596 \cdot 10^{-27}$ ,  $k_B \rightarrow 1.380650324 \cdot 10^{-16}$ ,  
 $c \rightarrow 2.99792458 \cdot 10^{10}$ }
```

```
{ $\hbar \rightarrow 1.05457 \times 10^{-27}$ ,  
 $k_B \rightarrow 1.38065 \times 10^{-16}$ ,  $c \rightarrow 2.99792 \times 10^{10}$ }
```

$$u4 = \frac{16 \pi^2 \hbar c}{\lambda^5} \frac{1}{\text{Exp}\left[\frac{2 \pi \hbar c}{k_B T \lambda}\right] - 1}$$

$$\frac{16 c \pi^2 \hbar}{\left(-1 + e^{\frac{2 c \pi \hbar}{k_B T \lambda}}\right) \lambda^5}$$

$$u5 = \frac{1}{\lambda^5} \frac{1}{\text{Exp}\left[\frac{\alpha}{\lambda}\right] - 1}$$

$$\frac{1}{\left(-1 + e^{\alpha/\lambda}\right) \lambda^5}$$

`D[u5, λ] // Simplify`

$$\frac{e^{\alpha/\lambda} (\alpha - 5 \lambda) + 5 \lambda}{\left(-1 + e^{\alpha/\lambda}\right)^2 \lambda^7}$$

$$u6 = \left(e^{\alpha/\lambda} \left(\frac{\alpha}{\lambda} - 5 \right) + 5 \right) /. \{\alpha \rightarrow \lambda y\}$$

$$5 + e^y (-5 + y)$$

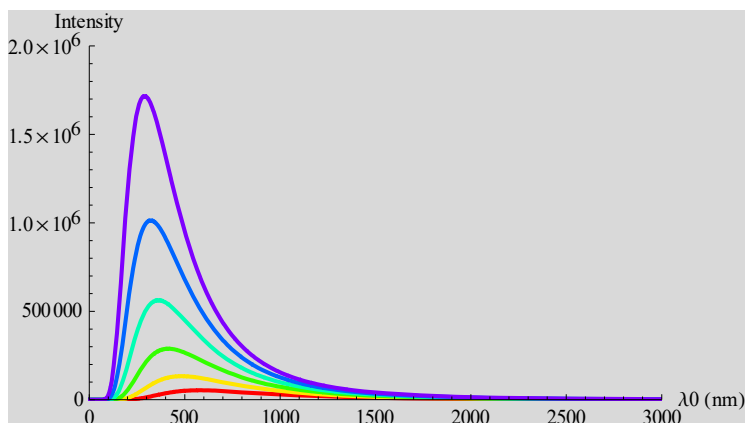
`FindRoot[u6 == 0, {y, 1, 5}]`

`{y → 4.96511}`

$$u6 = u4 /. \text{rule1} /. \lambda \rightarrow \lambda 0 10^{-7}$$

$$\frac{4.99248 \times 10^{20}}{\left(-1 + e^{\frac{1.43878 \times 10^7}{T \lambda 0}}\right) \lambda 0^5}$$

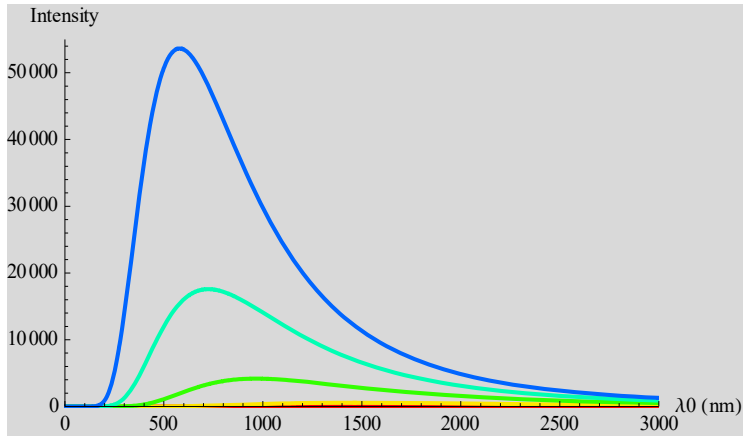
`Plot[Evaluate[Table[u6, {T, 5000, 10 000, 1000}]],
{λ0, 0, 3000}, PlotRange → {{0, 3000}, {0, 2 106}},
PlotStyle → Table[{Hue[0.15 i], Thick}, {i, 0, 5}],
Background → LightGray,
AxesLabel → {"λ0 (nm)", "Intensity"}]`



```

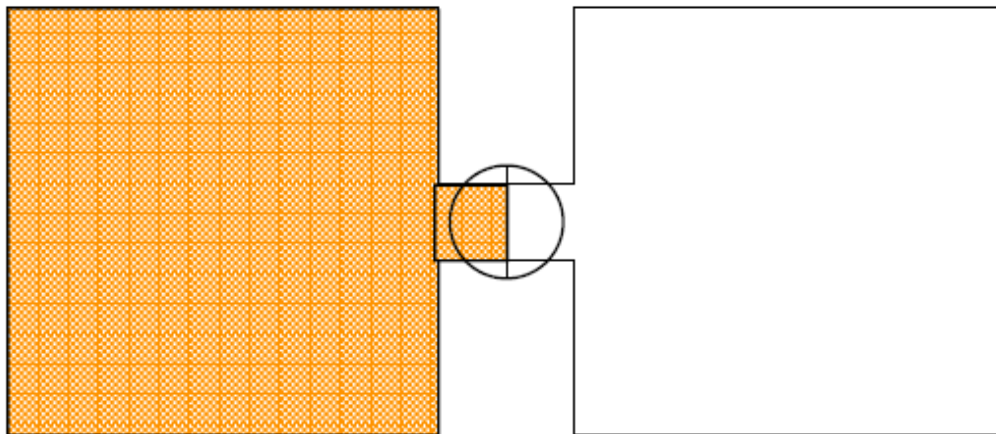
Plot[Evaluate[Table[u6, {T, 1000, 5000, 1000}]],
  {λ0, 0, 3000}, PlotRange → {{0, 3000}, {0, 5.5 104}},
  PlotStyle → Table[{Hue[0.15 i], Thick}, {i, 0, 5}],
  Background → LightGray,
  AxesLabel → {"λ0 (nm)", "Intensity"}]

```

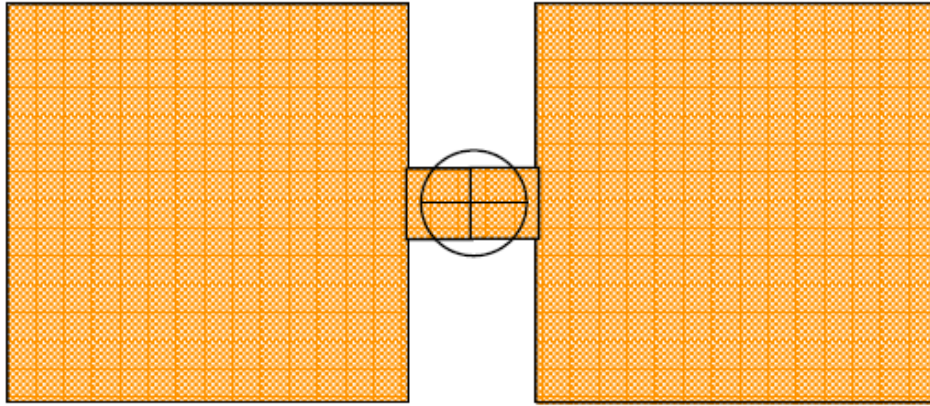


C. Adiabatic free expansion of an ideal gas (irreversible process): sudden expansion into vacuum

Two containers connected by stopcock. They are thermally insulated so no heat can flow in or out.



Initial: One container is evacuated. Gas is in volume V_i at temperature T_i .



Final: Stockcock opened, gas rushes into second chamber. Gas does not work (nothing to push against) and there is no heat transfer. So internal energy does not change. Final volume $V_f > V_i$ at temperatures $T_f = T_i$.