Chapter 24 S Image charge method

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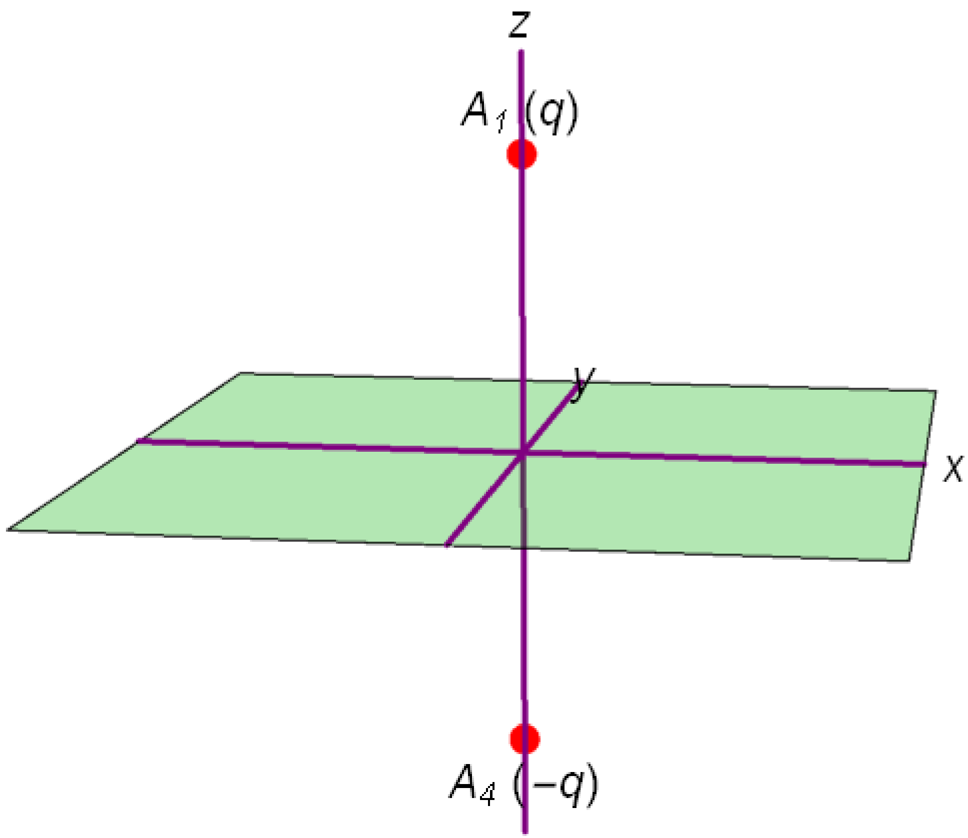
Part I Image charge method for grounded conductor planes

A simple example is a point charge located in front of an infinite plane conductor at zero potential. It is clear that this is equivalent to the problem of the original charge and opposite charge located at the mirror-image point behind the plane defined by the position of the conductor.

1. Image charge for one grounded conductor plane

We consider a point charge near a conducting plane. Suppose the *xy* plane is the surface of a conductor extending out to infinity. The electric potential of the conducting plane is zero. Now bring in a positive charge q and locate it a above the plane on the z axis. We expect the positive charge q to attract negative charge. Also, we remember that the electric field is always perpendicular to the surface of a conductor, at the conductor's surface. Very near the point charge q, the field lines must *start out* from q as if they were leaving a point charge radially.



To solve this problem, we use the image charge method. We assume that two equal and opposite point charges, *q at z* = *a* and -*q* at *z* = -*a*. The conducting plane bisects the line joining the two charges. The electric field is everywhere perpendicular to the plane. The field is perpendicular to the plane of the conductor, and in the neighborhood of *q* it approaches the field of a point charge.

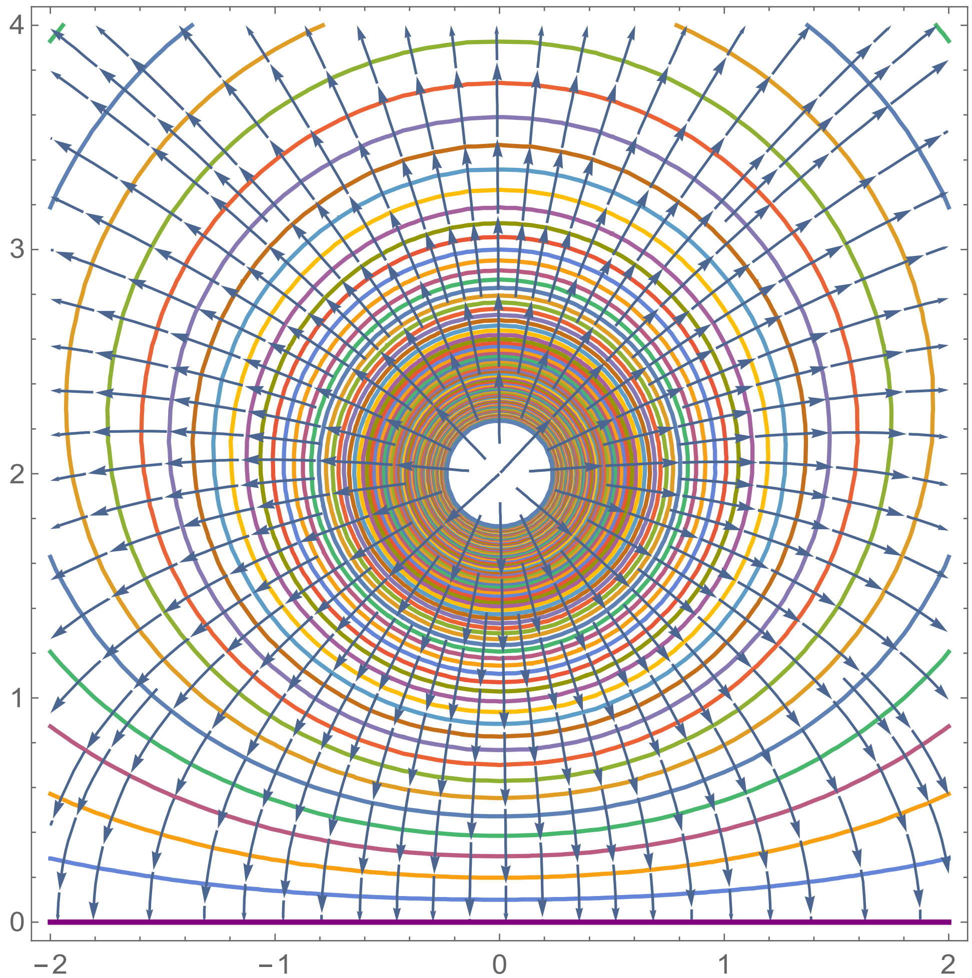


Fig. The electric field distribution on the *x*-*z* plane. $y = 0$. $a = 2$. $q = 1$.

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1} - \frac{1}{r_2}\right)$$

$$r_1 = \sqrt{x^2 + y^2 + (z-a)^2}\,, \qquad r_2 = \sqrt{x^2 + y^2 + (z+a)^2}$$

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_y = -\frac{\partial V}{\partial y}, \qquad E_z = -\frac{\partial V}{\partial z}$$

$$E_z(z=0) = \left(-\frac{\partial V}{\partial z}\right)(z=0)$$
$$= -\frac{aq}{2\pi\varepsilon_0 (a^2 + r^2)^{3/2}}$$

where

$$r = \sqrt{x^2 + y^2}$$

The surface charge density is

$$\sigma = \varepsilon_0 E_z = -\frac{qa}{2\pi(r^2 + a^2)^{3/2}} = -\frac{q}{2\pi a^2 (\frac{r^2}{a^2} + 1)^{3/2}} .$$

We make a plot of the surface charge density in the *x*-*y* plane.

$$f = -\frac{\sigma}{q}(2\pi a^2) = \frac{1}{(\frac{r^2}{a^2} + 1)^{3/2}} = \frac{1}{(\frac{x^2 + y^2}{a^q} + 1)^{3/2}}$$

with $r = \sqrt{x^2 + y^2}$.

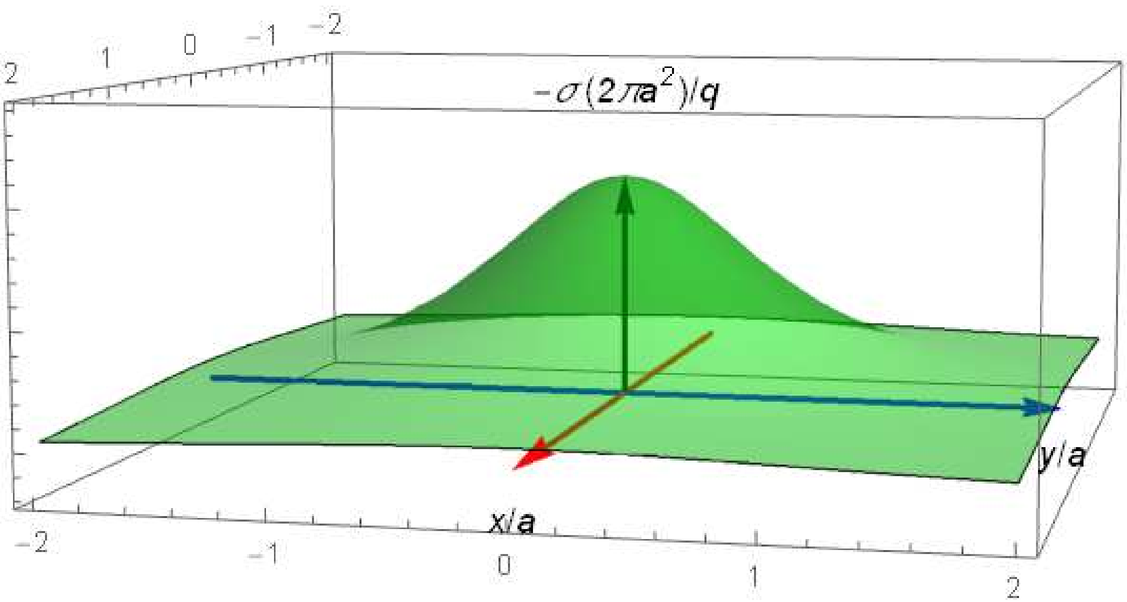


Fig. Surface charge density in the *x*-*y* plane with $z = 0$.

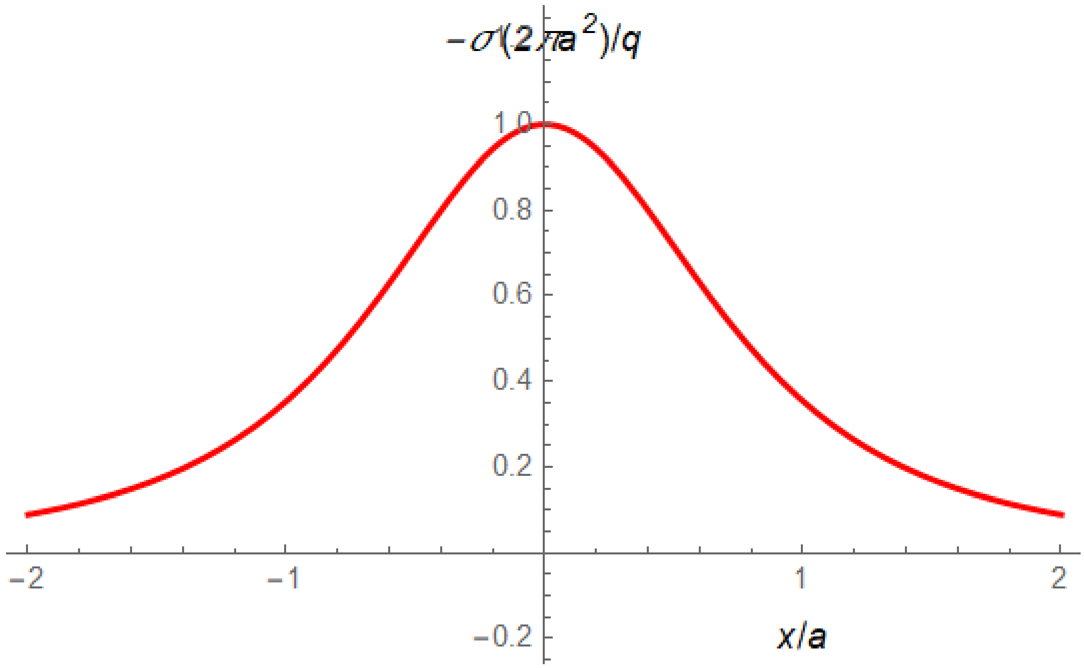


Fig. Surface charge density with $z = 0$ and $y = 0$.

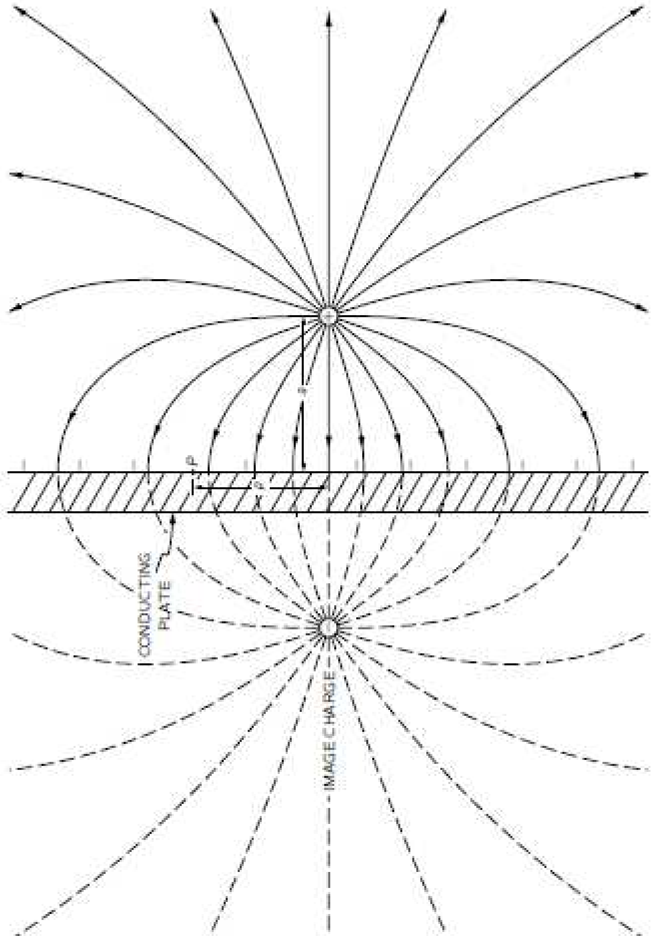
The total charge on the surface is

$$Q = \int_0^\infty 2\pi r \sigma dr$$
$$= -qa\int_0^\infty \frac{r}{(r^2 + a^2)^{3/2}} dr$$
$$= -q$$

where

$$\int_0^\infty \frac{r}{(r^2 + a^2)^{3/2}} dr = \frac{1}{a}$$

((Feynman))



2. Image charge method for two semi-infinite grounded conducting planes meeting at the right angle

Two semi-infinite grounded conducting planes meet at right angles. In the region between them, there is a point charge q, situate as shown in Fig. Set up the image configuration and calculate the potential in this region. What charges do you need, and where should they be located?

$$V(x,y,z)=\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{r_1}-\frac{1}{r_2}+\frac{1}{r_3}-\frac{1}{r_4}\right)$$

with

$$r_1=\sqrt{(x-a)^2+y^2+(z-b)^2}$$

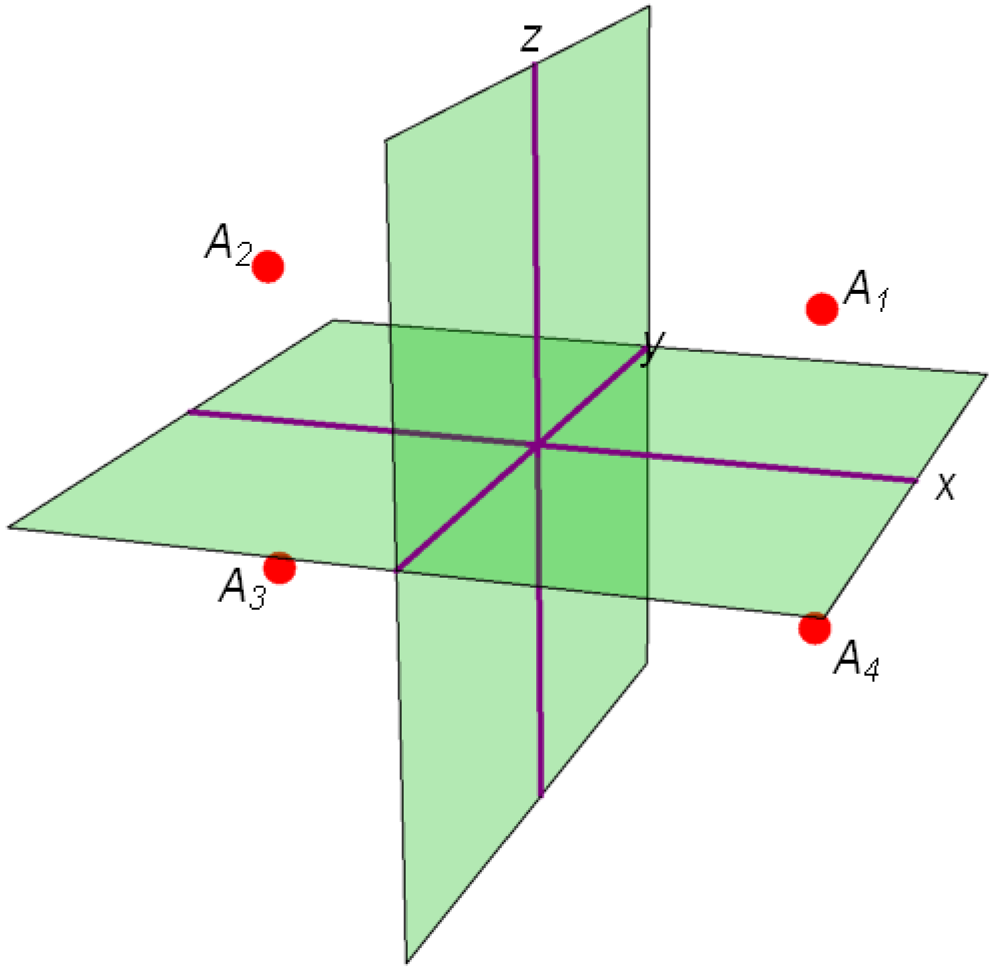
$$r_2 = \sqrt{(x+a)^2 + y^2 + (z-b)^2}$$

$$r_3 = \sqrt{(x+a)^2 + y^2 + (z+b)^2}$$

$$r_4 = \sqrt{(x-a)^2 + y^2 + (z+b)^2}$$

The charge (q) at the point $A_1 = (a, 0, b)$,
The point charge (-q) at the point $A_2 = (-a, 0, b)$,
The point charge (q) at the point $A_3 = (-a, 0, -b)$,
The point charge (-q) at the point $A_4 = (a, 0, -b)$.

where a = 3 and b = 1.7, q = 1.


z
y
x
A_1
A_2
A_3
A_4

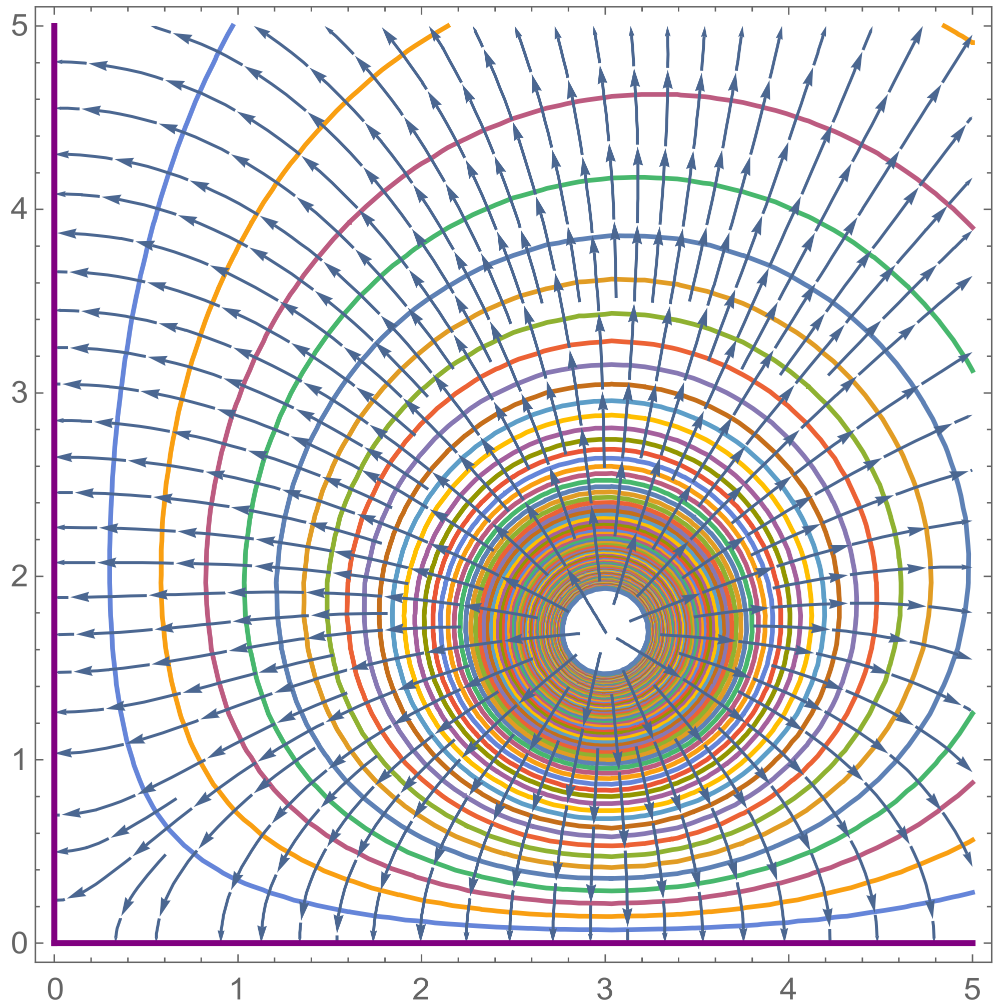
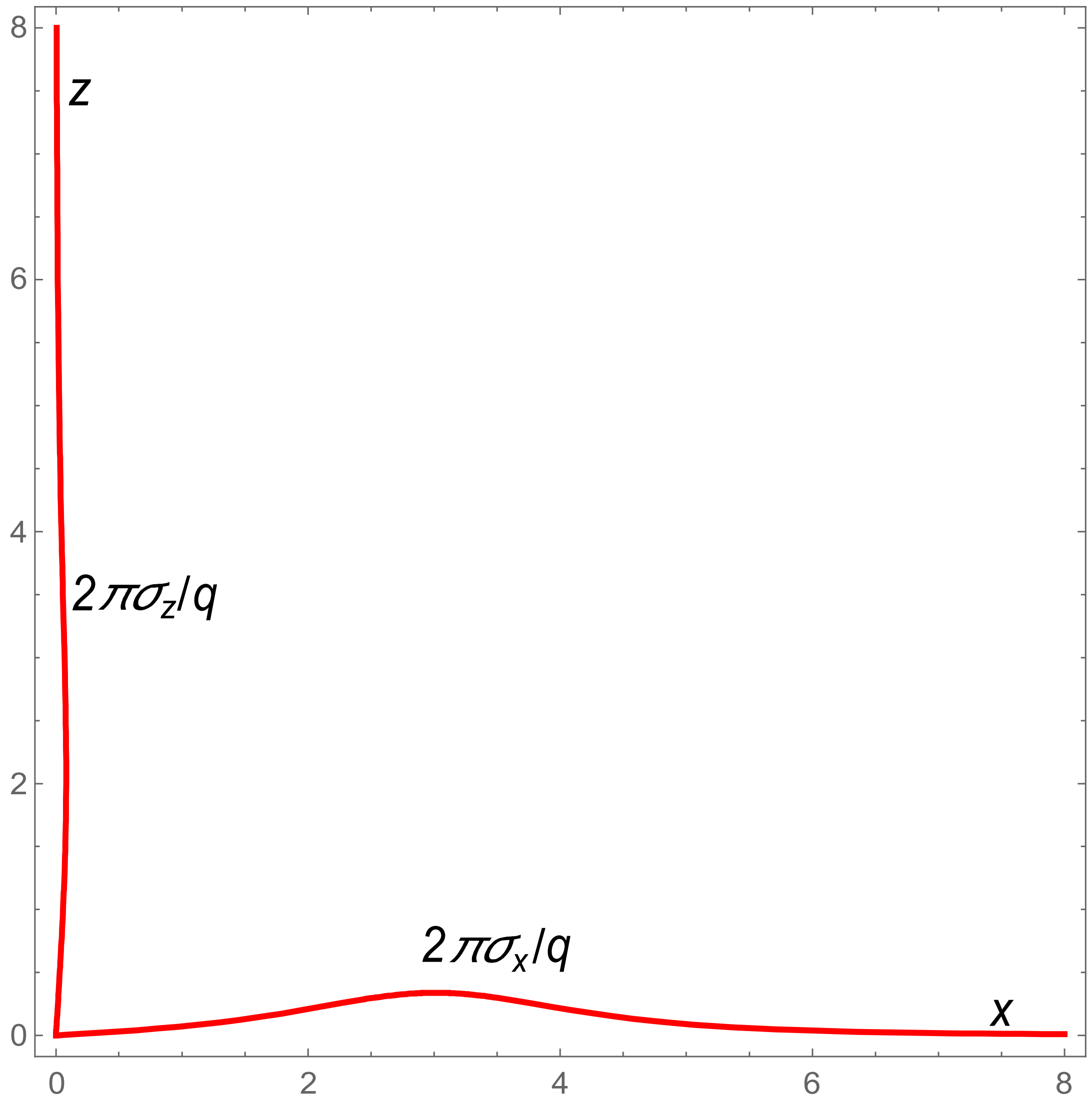


Fig. $a = 3$. $b = 1.7$. $q = 1$. The electric field distribution and the equipotential line.

The induced surface charge density is

$$\sigma_x = \varepsilon_0 E_z(z=0) = \frac{q}{2\pi}\{-\frac{b}{[b^2+(x-a)^2]^{3/2}} + \frac{b}{[b^2+(x+a)^2]^{3/2}}\}$$

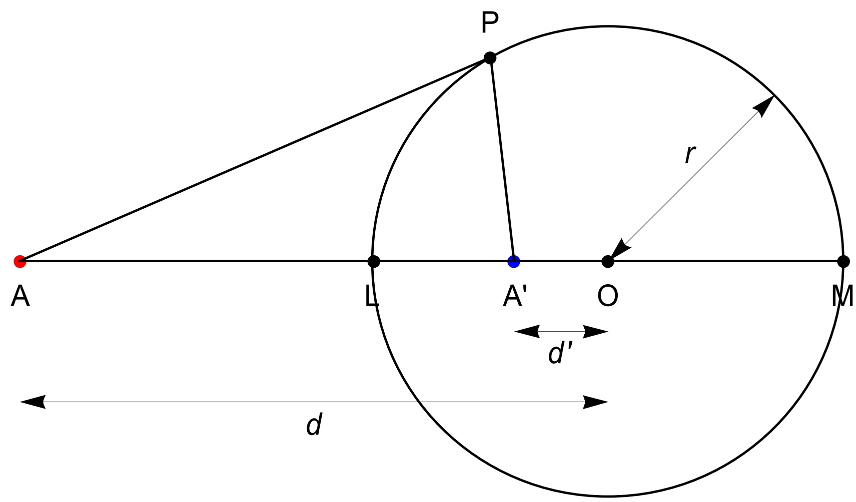
$$\sigma_z = \varepsilon_0 E_x(x=0) = \frac{q}{2\pi}\{-\frac{a}{[a^2+(z-b)^2]^{3/2}} + \frac{a}{[a^2+(z+b)^2]^{3/2}}\}$$



Part II Image charge method for grounded spherical conductor

Here we discuss the electric field and electric potential of a point charge near a grounded spherical conductor. There are two cases where a point charge is either outside or inside of the spherical conductor. The electrical potential at the surface of the sphere is zero. This problem can be solved by using the image charge method.

1. Apollonius circle



When this ratio is constant, the point P lies on a so-called Apollonius circle. The rotation of this circle around the axis passing through two points A and A' leads to a sphere where the electric potential is equal to zero.

The electric potential

$$V = \frac{1}{4\pi\varepsilon_0}\left(\frac{q}{\overline{AP}} - \frac{q'}{\overline{A'P}}\right)$$

When $V = 0$,

$$\frac{q}{q'} = \frac{\overline{AP}}{\overline{A'P}}$$

This relation is valid when P moves to the point L and the point M.

$$\overline{OA} = d, \qquad \overline{OA'} = d'$$

$$\frac{q}{q'} = \frac{\overline{AL}}{\overline{A'L}} = \frac{\overline{AM}}{\overline{A'M}}$$

or

$$\frac{r - d'}{d - r} = \frac{r + d'}{r + d} = \frac{q'}{q}$$

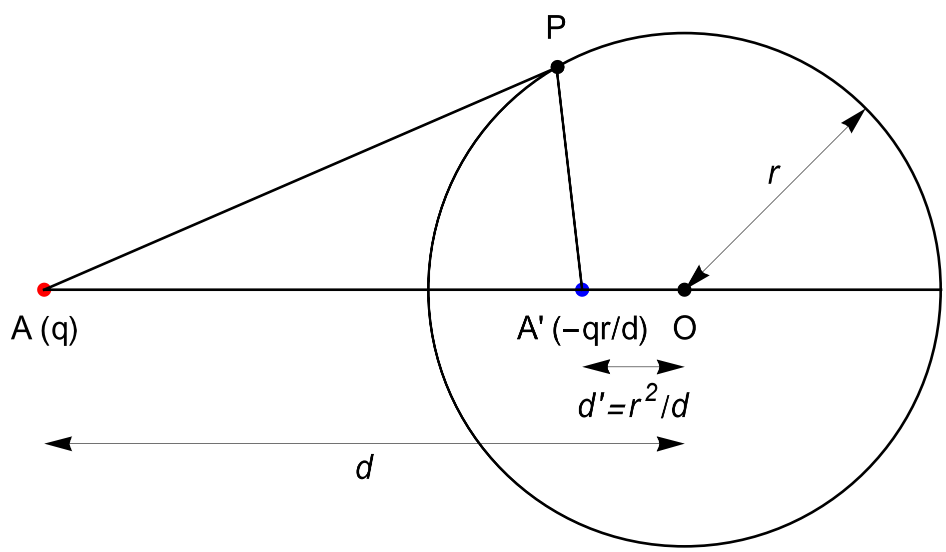
leading to

$$d' = \frac{r^2}{d}, \qquad q' = \frac{r}{d} q .$$

2. Electric field due to a point charge *q* outside the spherical conductor

We assume that a point charge q is located outside the spherical conductor. The electric potential outside the spherical conductor is given by

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-d)^2 + y^2 + z^2}} - \frac{\frac{r}{d}}{\sqrt{(x-\frac{r^2}{d})^2 + y^2 + z^2}}\right)$$



From the symmetry of the system, we consider the simple case with $y = 0$.

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-d)^2 + z^2}} - \frac{\frac{r}{d}}{\sqrt{(x-\frac{r^2}{d})^2 + z^2}}\right).$$

The corresponding electric field is expressed by

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_z = -\frac{\partial V}{\partial z}$$

The electric potential on the surface of the conductor is zero.

((**Mathematica**))

ContourPlot and StreamPlot. The electric field outside the conducting sphere. $d = 1.5$. $r = 1$. $q = 1$.

```
Clear["Global`*"]; d1 = 1.5; r1 = 1; q1 = 1;
```

$$V1 = \frac{q1}{\sqrt{(x - d1)^2 + y^2}} - \frac{\frac{q1\,r1}{d1}}{\sqrt{\left(x - \frac{r1^2}{d1}\right)^2 + y^2}};$$

```
Ex = -D[V1, x] // Simplify;
Ey = -D[V1, y] // Simplify;

f1 = ContourPlot[
   Evaluate[Table[V1 == α, {α, -2, 2, 0.05}]],
   {x, -3, 3}, {y, -3, 3},
   RegionFunction → Function[{x, y}, x^2 + y^2 > r1^2]];

f2 = StreamPlot[{Ex, Ey}, {x, -3, 3}, {y, -3, 3},
   RegionFunction → Function[{x, y}, x^2 + y^2 > r1^2]];

f3 = Graphics[{Red, PointSize[0.02], Point[{r1, 0}]}];
f4 = Graphics[{Black, Thick, Circle[{0, 0}, 1]}];
Show[f1, f2, f3, f4]
```

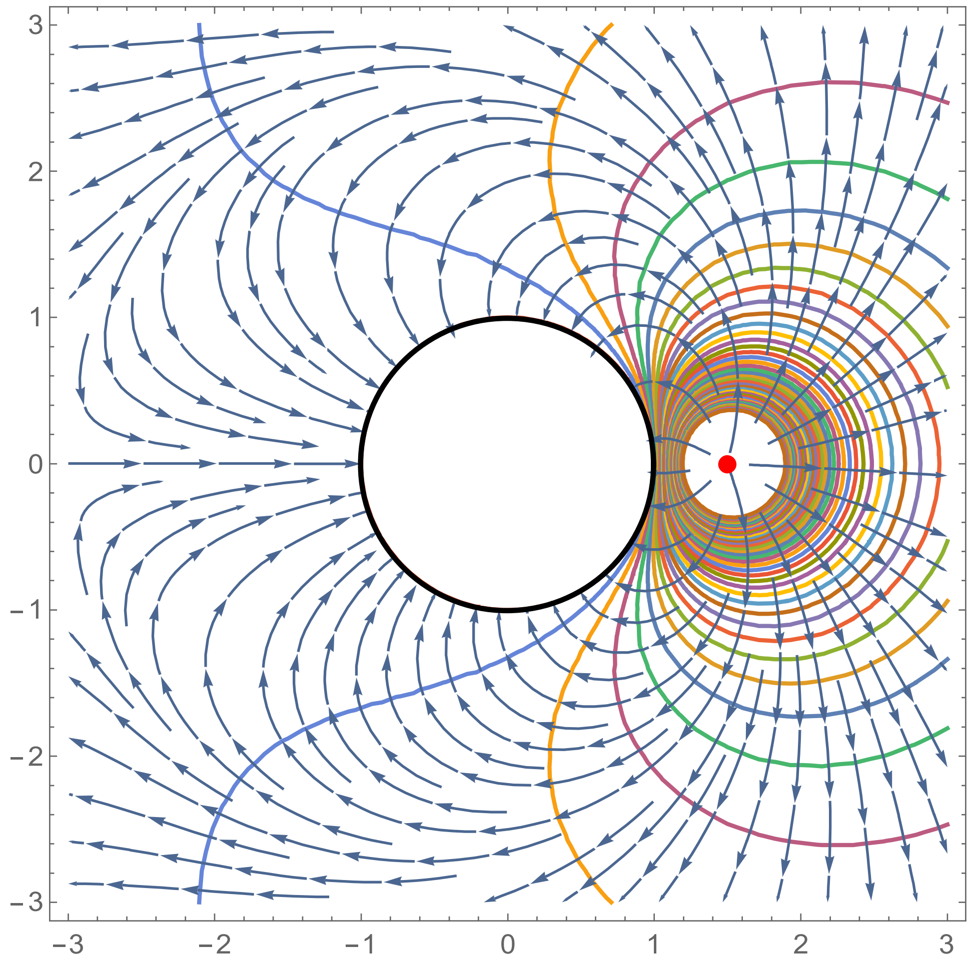
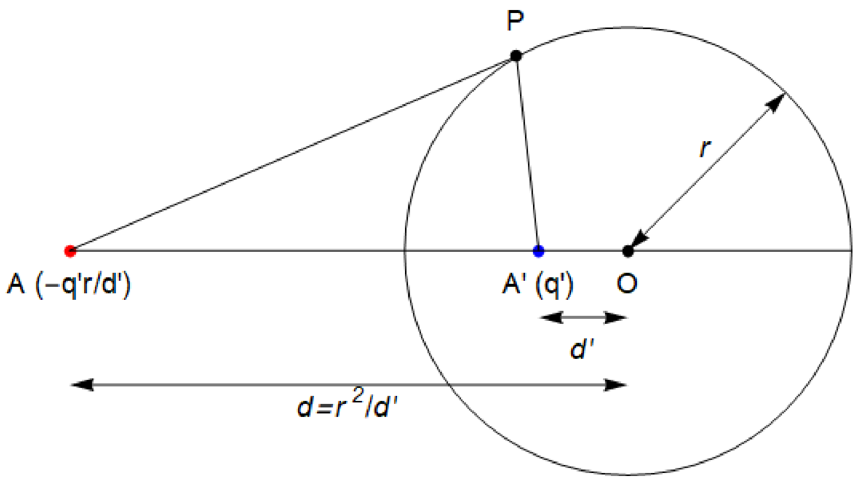


Fig. Electric potential and electric field due to a charge outside the conducting sphere. $E = 0$ inside the conductor.

3. Electric field due to a point charge inside the spherical conductor

We assume that a point charge q is located inside the spherical conductor. The electric potential in the spherical conductor is given by

$$V = \frac{q'}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-d')^2+y^2+z^2}} - \frac{\frac{r}{d'}}{\sqrt{(x-\frac{r^2}{d'})^2+y^2+z^2}}\right)$$



From the symmetry of the system, we consider the simple case with $y = 0$.

$$V = \frac{q'}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x-d')^2+z^2}} - \frac{\frac{r}{d'}}{\sqrt{(x-\frac{r^2}{d'})^2+z^2}}\right).$$

The corresponding electric field is expressed by

$$E_x = -\frac{\partial V}{\partial x}, \qquad E_z = -\frac{\partial V}{\partial z}$$

The electric potential on the surface of the conductor is zero.

((**Example**))

The electric field distribution when a point charge inside the conductor shifts from the center to places near the surface of conductor.

((**Mathematica**))

```
Clear["Global`*"];
a1 = 0.85;
r1 = 1;
q1 = 1;
V1 = q1/Sqrt[(x + a1)^2 + y^2] - (q1 r1/a1)/Sqrt[(x + r1^2/a1)^2 + y^2];
V2 = q1/Sqrt[(x)^2 + y^2];
Ex1 = -D[V1, x] // Simplify; Ey1 = -D[V1, y] // Simplify;
Ex2 = -D[V2, x] // Simplify; Ey2 = -D[V2, y] // Simplify;
f1 = ContourPlot[Evaluate[Table[V1 == α, {α, -2, 2, 0.05}]],
   {x, -1.5, 1.5}, {y, -1.5, 1.5},
   RegionFunction → Function[{x, y}, x^2 + y^2 < r1^2]];
f2 = StreamPlot[{Ex1, Ey1}, {x, -1.5, 1.5}, {y, -1.5, 1.5},
   RegionFunction → Function[{x, y}, x^2 + y^2 < r1^2]];
f3 = Graphics[{Blue, PointSize[0.02], Point[{-a1, 0}]}];
f4 = Graphics[{Black, Thickness[0.02], Circle[{0, 0}, 1]}];
f5 = ContourPlot[Evaluate[Table[V2 == α, {α, -2, 2, 0.05}]],
  {x, -1.5, 1.5}, {y, -1.5, 1.5},
  RegionFunction → Function[{x, y}, x^2 + y^2 > r1^2]];
f6 = StreamPlot[{Ex2, Ey2}, {x, -1.5, 1.5}, {y, -1.5, 1.5},
  RegionFunction → Function[{x, y}, x^2 + y^2 > r1^2]];
f7 =
 Graphics[
  {Text[Style["Spherical metal shell", Italic, Black, 15], {0, 1.15}],
   Text[Style["a=" <> ToString[a1], Black, 18], {1.15, 1.15}]}];
Show[f1, f2, f3, f4, f5, f6, f7]
```

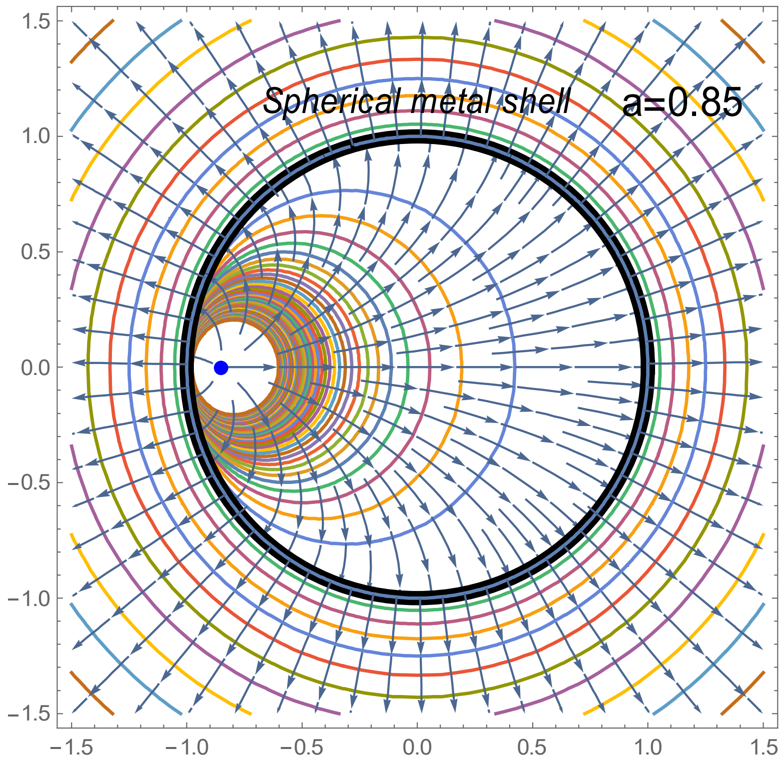


Fig. Electric potential and electric field due to a charge inside the conducting sphere shell. E = 0 on the spherical conductor shell. A point charge is at $x = -a$ with $a = 0.85$. The electric field outside the conducting sphere shell is that generated by a electric field due to the point charge at the center.

4. A spherical conductor in the presence of uniform electric field

There are two charges (-q, and q) located on the x axis, which are separated by large distance (2d). Around the origin, these two charges produce a uniform electric field E_0 along the x axis. Suppose that we put a conducting sphere with a radius r. Using the image charge method, we have image charges at point A' (-q') and B'(q') inside the conductor. These image charges produce an electric dipole moment p along the x axis.

$$E_0 = \frac{2q}{4\pi\varepsilon_0 d^2}, \qquad p = 2d'q' = 2\frac{r^2}{d}\frac{qr}{d} = \frac{2qr^3}{d^2}$$

where

$$d' = \frac{r^2}{d}, \qquad q' = \frac{qr}{d}$$

In this case, the electric dipole moment can be expressed by

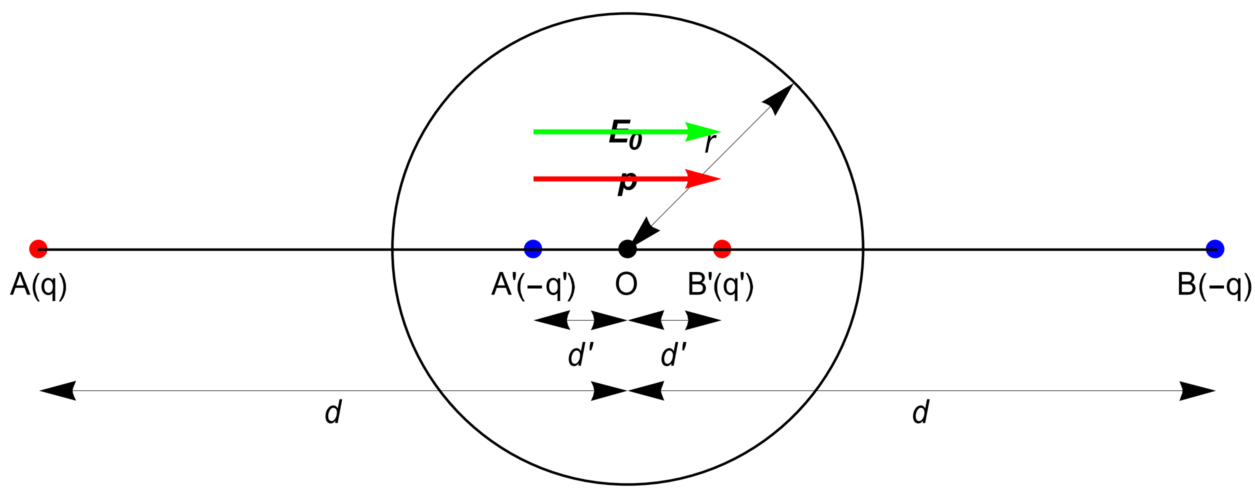
$$p = \frac{2qr^3}{d^2} = \alpha E_0$$

where α is the electric polarizability,

$$\frac{2qr^3}{d^2} = \alpha \frac{2q}{4\pi\varepsilon_0 d^2}$$

or

$$\alpha = 4\pi\varepsilon_0 r^3$$



The electric potential outside the spherical conductor is

$$V = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{(x+d)^2+z^2}} - \frac{1}{\sqrt{(x-d)^2+z^2}} - \frac{\frac{r}{d}}{\sqrt{(x+\frac{r^2}{d})^2+z^2}} + \frac{\frac{r}{d}}{\sqrt{(x-\frac{r^2}{d})^2+z^2}}\right)$$

((**Mathematica**))

```
Clear["Global`*"];

V[x_, z_] :=
```

$$\frac{q}{4\pi\,\epsilon 0}\left(\frac{1}{\sqrt{(z+d)^2+x^2}}-\frac{1}{\sqrt{(z-d)^2+x^2}}+\frac{r/d}{\sqrt{\left(z-\frac{r^2}{d}\right)^2+x^2}}-\frac{r/d}{\sqrt{\left(z+\frac{r^2}{d}\right)^2+x^2}}\right);$$

```
V2D = V[x, z];

rule1 = {r → 1, E0 → 1, q → 1, d → 20, ϵ0 → 1};

V2D1 = V2D /. rule1;
```

```
Ex = -D[V2D1, x] // Simplify; Ez = -D[V2D1, z] // Simplify;

g1 = ContourPlot[Evaluate[Table[V2D1 == α/10000, {α, -10, 10, 1}]],
    {x, -3, 3}, {z, -3, 3},
    ContourStyle → Table[{Thick, Hue[0.03 i]}, {i, 0, 60}],
    RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];

g2 = StreamPlot[Evaluate[{Ex, Ez}], {x, -3, 3}, {z, -3, 3},
   RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];
g3 = Graphics[{Black, Thick, Circle[{0, 0}, 1]}];

Show[g1, g2, g3, PlotRange → All]
```

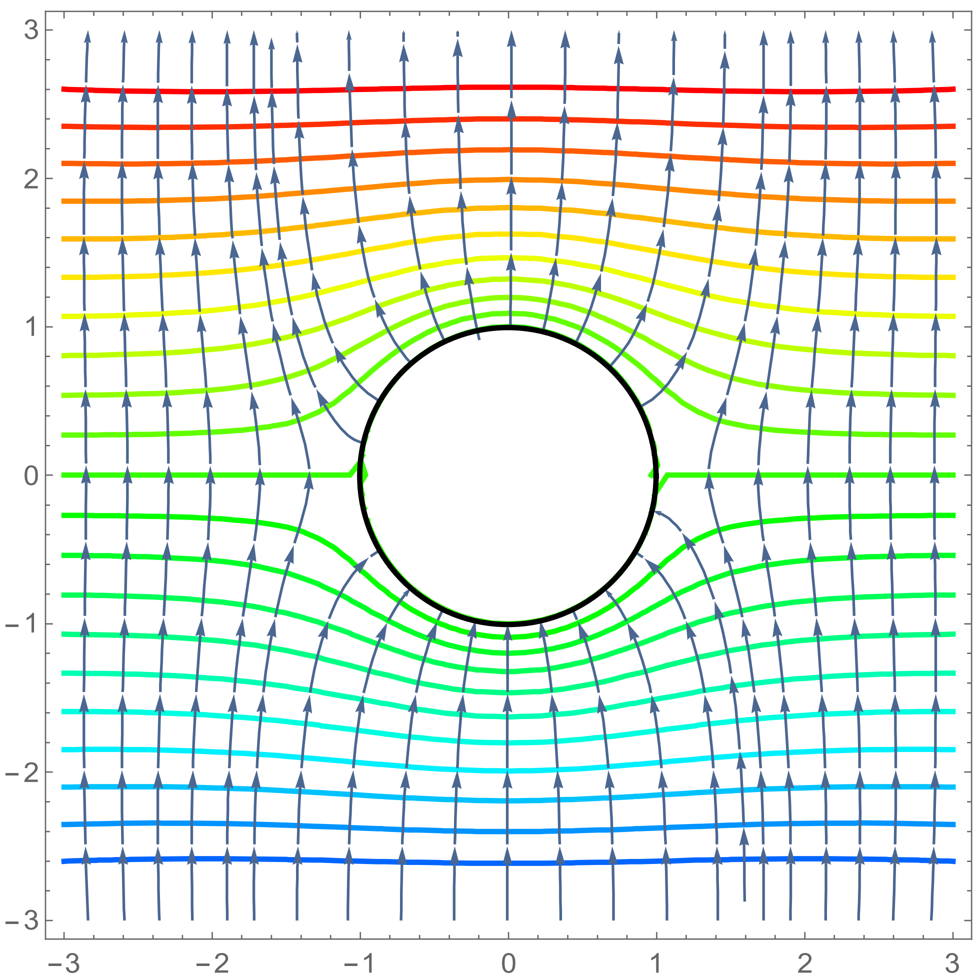


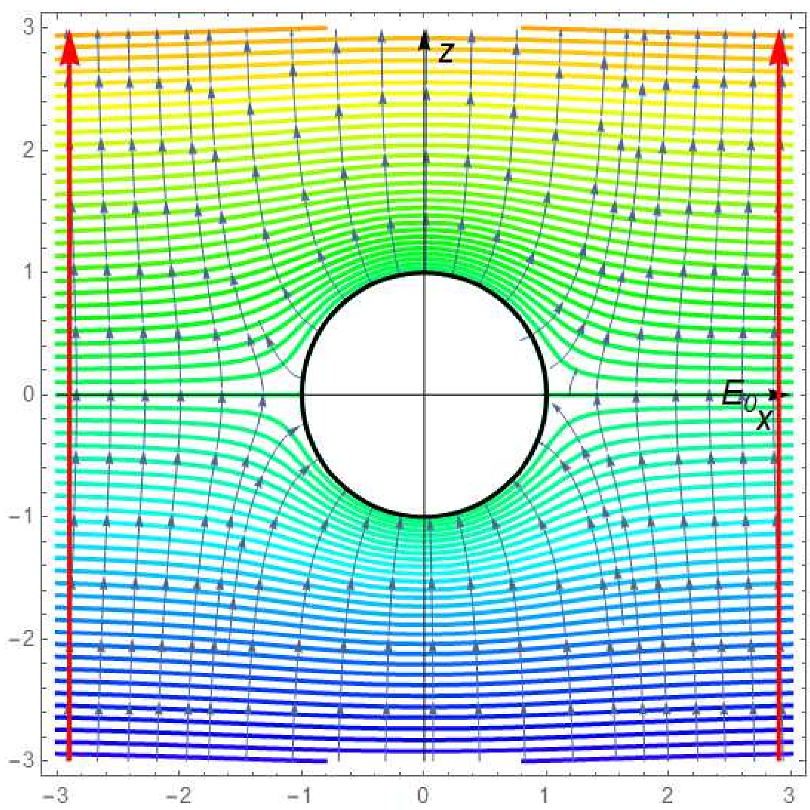
Fig. Charge image method: Electric field around a spherical conductor (centered at the origin) in the presence of electric field arising from two charges (q and $-q$) which are separated by large distance d. When d is very large, the electric field produced by two charges produces an electric field along the z axis.

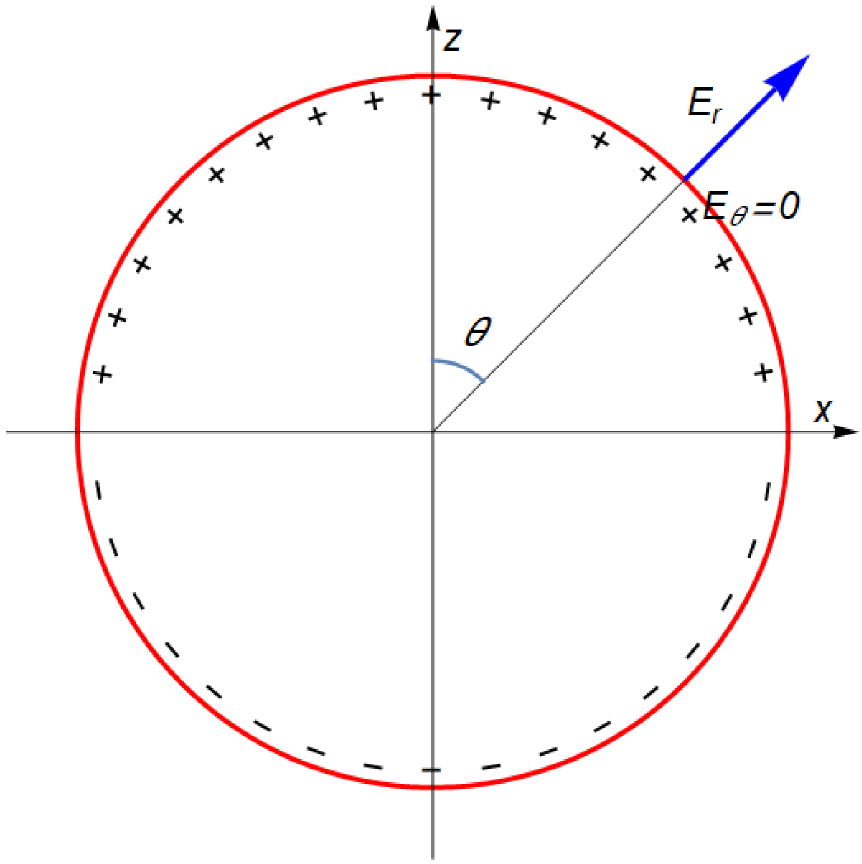
5. Alternative way (solution of Laplace's equation)

D.J. Griffiths, Introduction to Electrodynamics, 4th edition (Peason, 2013).
Example-8, p.141

((**Example**))

An uncharged metal sphere of radius R is placed in an otherwise uniform electric field $E_0\boldsymbol{e}_z$. The field will push positive charge to the northern surface of the sphere, leaving a negative charge on the southern surface. This induced charge, in turn, distort the field in the neighborhood of the sphere. Find the potential in the region outside the sphere.





The sphere is an equipotential – we may as well set it to zero. Then by symmetry the entire *xy* plane is at potential zero. This time, however, *V* does not go to zero at large *z*. In fact, far from the sphere the field is $E_0\boldsymbol{e}_z$, and hence

$$V \to -E_0 z + C$$

Since $V = 0$, in the equatorial plane, the constant *C* must be zero. Accordingly, the boundary conditions for this problem are

(i) $V = 0$ at $r = R$
(ii) $V = -E_0 r\cos\theta$ for $r \gg R$

We must fit these boundary conditions with a function of the form,

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta)$$

which is the general solution of the Laplace's equation; $\nabla^2 V = 0$. The first condition yields

$$A_l R^l + \frac{B_l}{R^{l+1}} = 0$$

or

$$B_l = -A_l R^{2l+1}$$

So

$$V(r,\theta) = \sum_{l=0}^{\infty} A_l \left(r^l - \frac{R^{2l+1}}{r^{l+1}}\right) P_l(\cos\theta)$$

For $r \gg R$, the second term in parentheses is negligible, and therefore condition (ii) requires that

$$\sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta) = -E_0 r \cos\theta$$

Evidently only one term is present: $l = 1$. In fact, since $P_1(\cos\theta) = \cos\theta$, we can read off immediately

$$A_1 = -E_0 \qquad \text{all other } A_l\text{'s zero}$$

Conclusion:

$$V(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

The first term $-E_0 r\cos\theta$ is due to the external field. The electric field ***E*** is obtained as

$$E_r = -\frac{\partial V(r,\theta)}{\partial r} = E_0\left(1 + \frac{2R^3}{r^3}\right)\cos\theta$$

$$E_\theta = -\frac{1}{r}\frac{\partial V(r,\theta)}{\partial \theta} = E_0\left(1 - \frac{R^3}{r^3}\right)\sin\theta$$

$$E_\phi = 0$$

At $r = R$

$$E_r = 3E_0 \cos\theta, \qquad E_\theta = 0$$

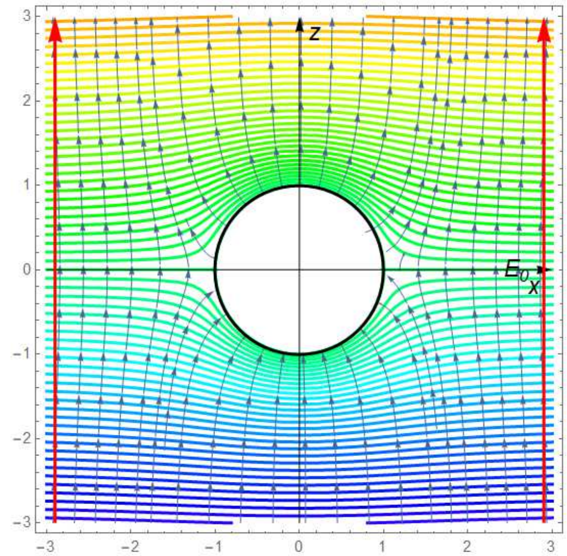
The electric field is normal to the surface of the sphere metal. The induced surface charge density is

$$\sigma(\theta) = \varepsilon_0 E_r \,|_{r=R} = 3\varepsilon_0 E_0 \cos\theta$$

As expected, it is positive in the northern hemisphere $(0 \le \theta \le \frac{\pi}{2})$ and negative in the southern $(\frac{\pi}{2} \le \theta \le \pi)$. Note that $\boldsymbol{E} = 0$ inside the metal sphere. So the tangential component is equal to zero for the inside and outside on the boundary, as is expected from the continuity of tangential components. The normal component of the electric field is not continuous.

((**Mathematica**))

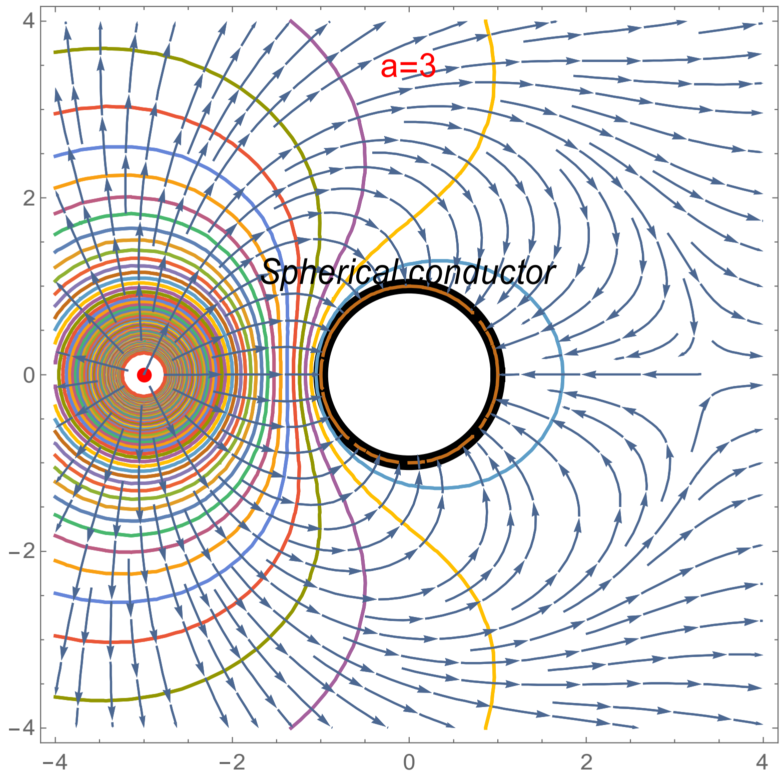
```
Clear["Global`*"];
r2xRule = {r → Sqrt[x^2 + y^2 + z^2], θ → ArcCos[z/Sqrt[x^2 + y^2 + z^2]],
   ϕ → ArcTan[x, y]};
V1 = -E0 (r - R^3/r^2) Cos[θ];
rule1 = {R → 1, E0 → 1};
V11 = V1 /. rule1 /. r2xRule /. y → 0 // Simplify;
Ex1 = -D[V11, x];
Ez1 = -D[V11, z];
g1 = ContourPlot[Evaluate[Table[V11 == α, {α, -4, 4, 0.1}]],
   {x, -3, 3}, {z, -3, 3},
   ContourStyle → Table[{Hue[0.01 i], Thick}, {i, 0, 80}],
   Epilog → {Black, Thick, Circle[{0, 0}, 1]},
   RegionFunction → Function[{x, z}, x^2 + z^2 > 1] ];
 g2 = StreamPlot[{Ex1, Ez1}, {x, -3, 3}, {z, -3, 3},
    StreamPoints → 80,
    RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];
 h1 = Show[g1, g2, PlotRange → All];
 h2 = Graphics[{Black, Thin, Arrowheads[0.03],
     Arrow[{{-3, 0}, {3, 0}}], Arrow[{{0, -3}, {0, 3}}],
     Text[Style["x", Black, 15, Italic], {2.8, -0.2}],
     Text[Style["z", Black, 15, Italic], {0.2, 2.8}], Red,
     Thick, Arrowheads[0.05], Arrow[{{-2.9, -3}, {-2.9, 3}}],
     Arrow[{{2.9, -3}, {2.9, 3}}],
     Text[Style["E₀", Black, 15, Italic], {2.6, 0}]}];
 Show[h1, h2]
```

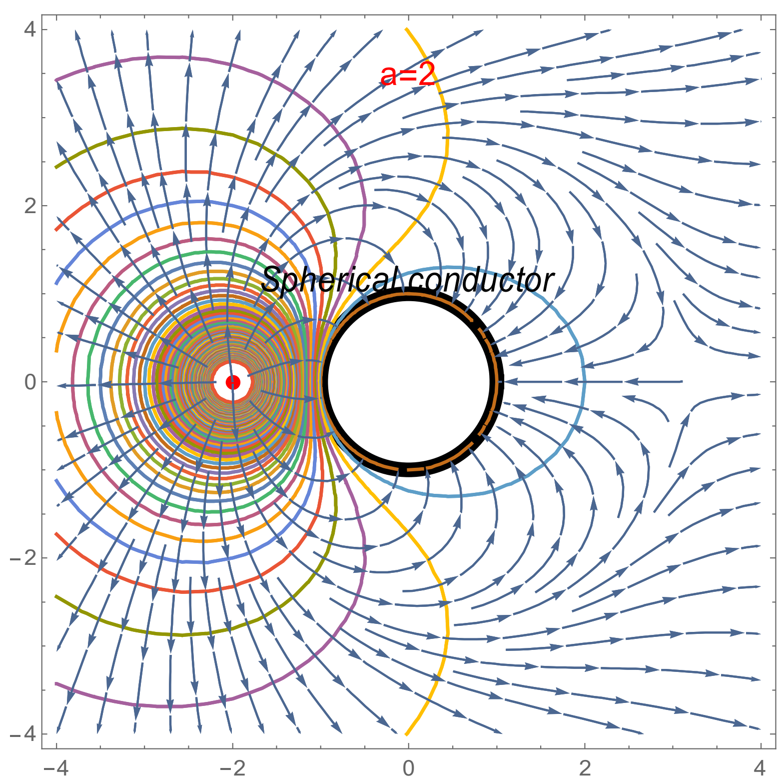


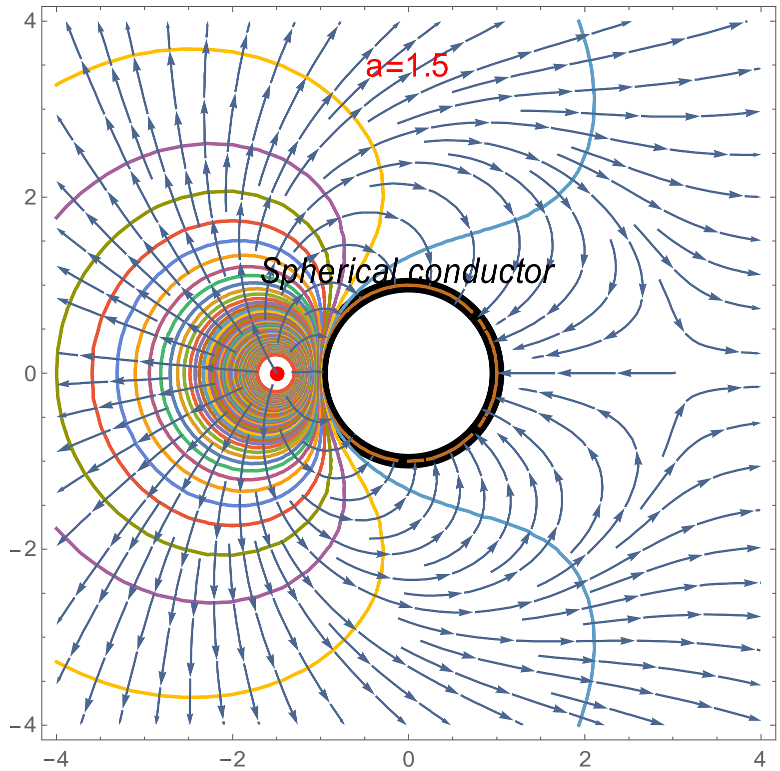
6. Electrical field distribution for a point charge inside (or outside) of the spherical metal (conducting) shell

(a) A charge outside the spherical conducting shell

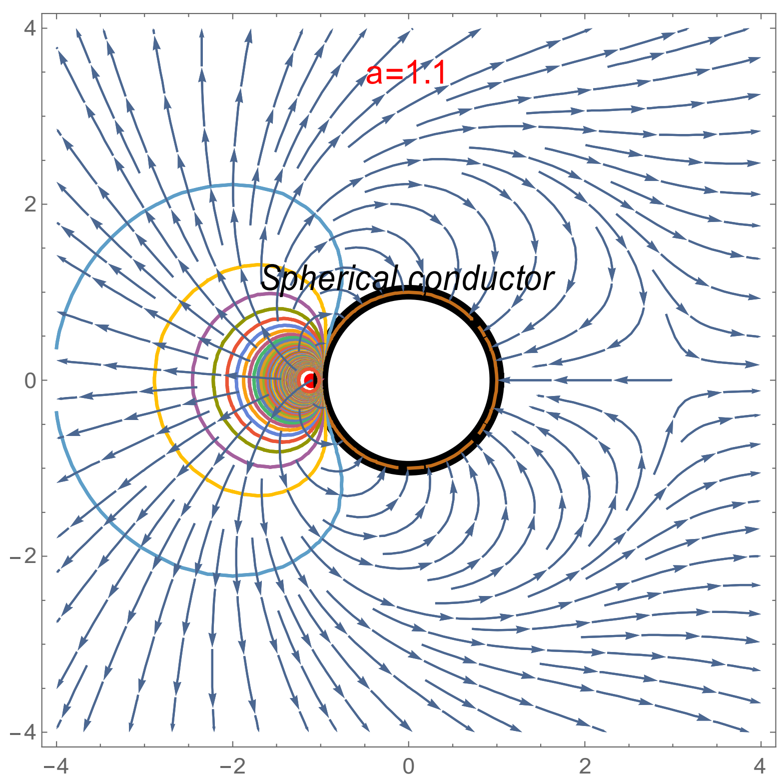
A point charge at (0,-*a*) outside the spherical metal shell. The electric field inside the spherical metal shell is equal to zero. The electric potential at the surface is equal to zero. The electric field is normal to the surface of spherical metal shell.


a=3
Spherical conductor

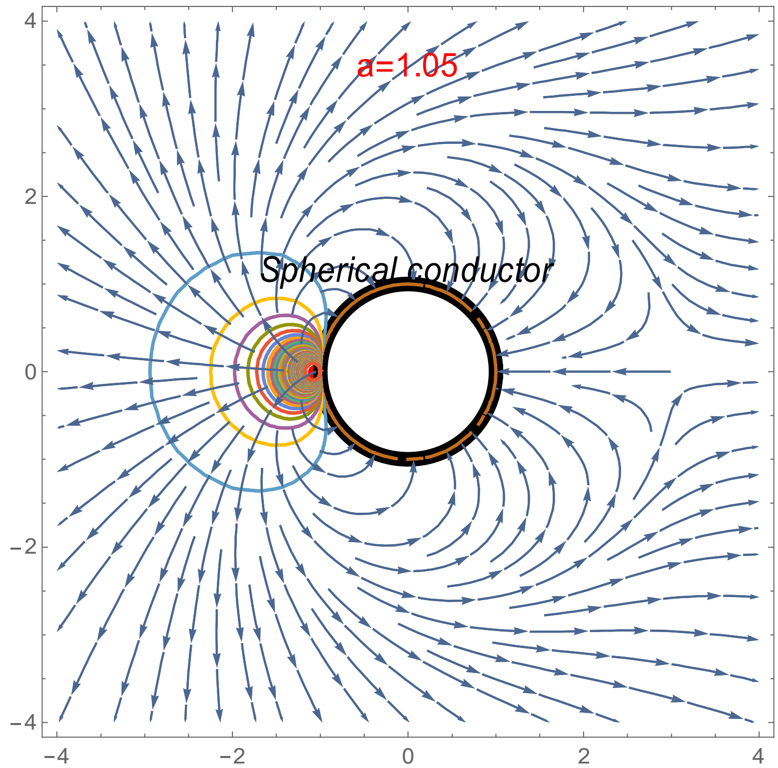

a=2
Spherical conductor

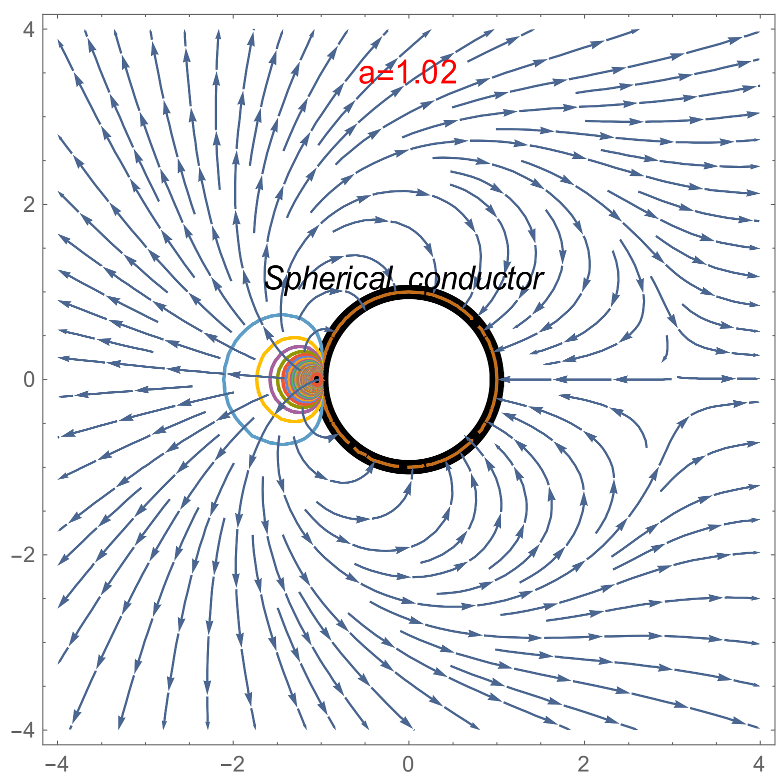


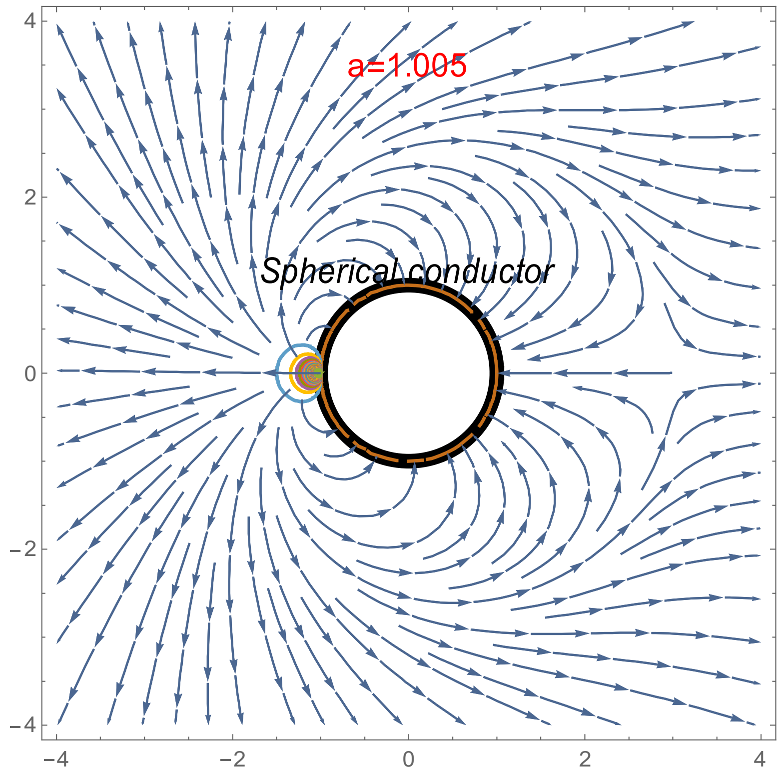
a=1.5
Spherical conductor
4
2
0
−2
−4
−4
−2
0
2
4



a=1.1
Spherical conductor
4
2
0
−2
−4
−4
−2
0
2
4

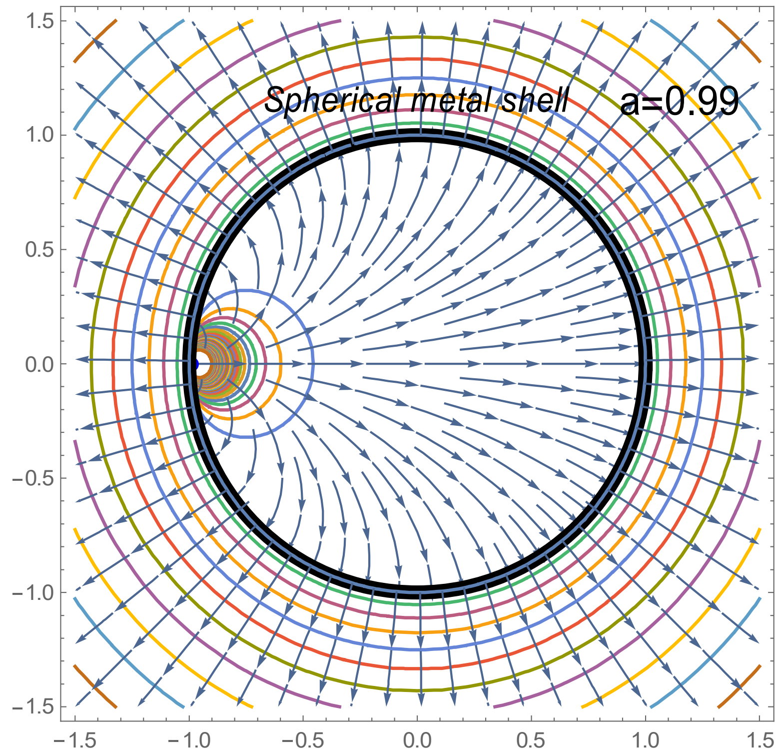

a=1.05
Spherical conductor
4
2
0
−2
−4
−4
−2
0
2
4

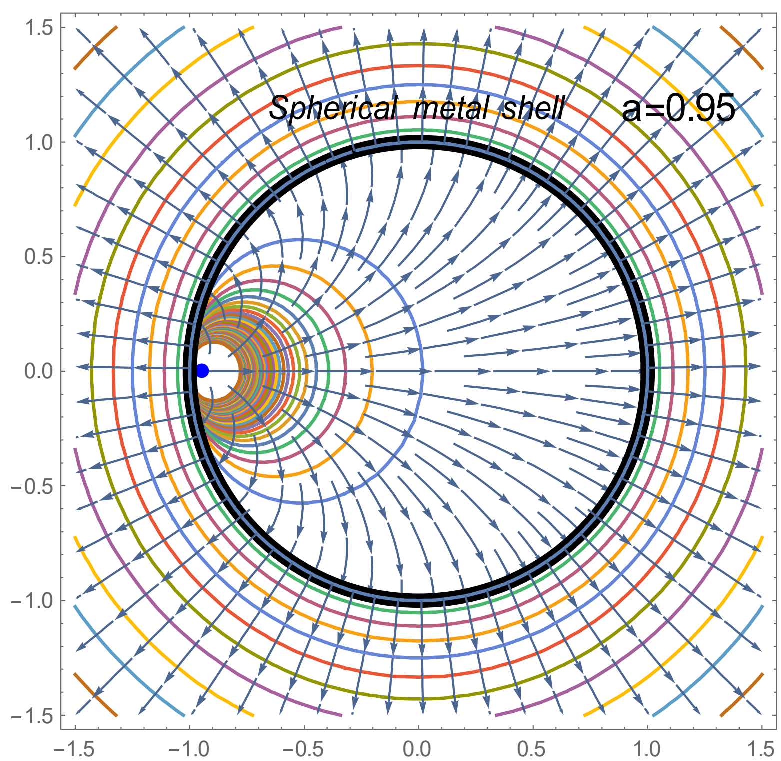

a=1.02
Spherical conductor
4
2
0
−2
−4
−4
−2
0
2
4

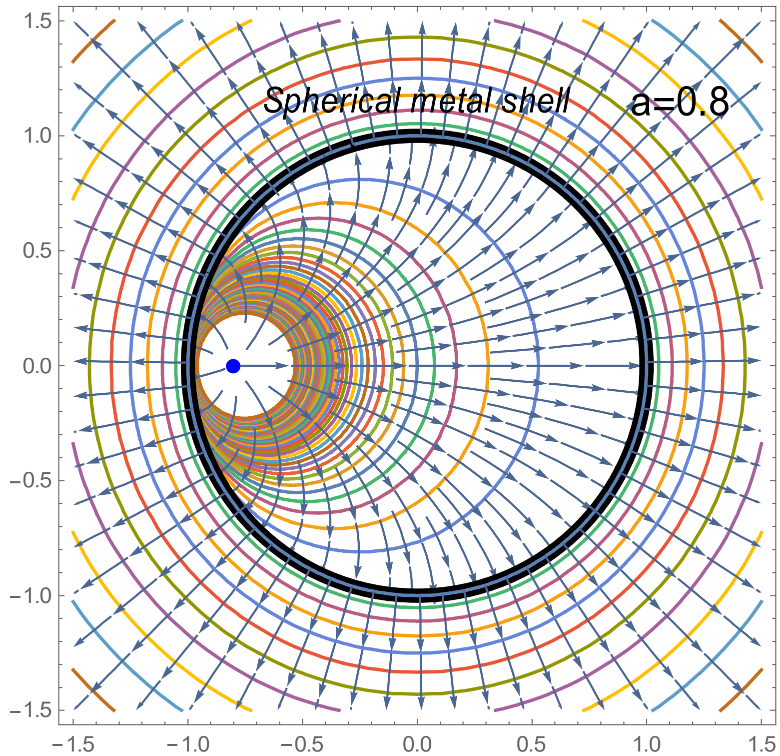


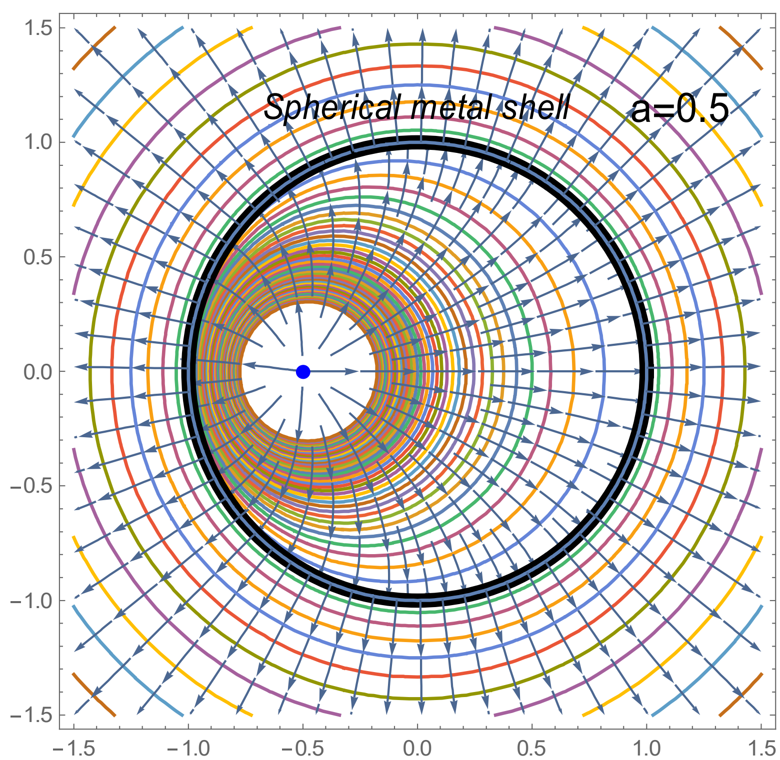
(b) A positive charge inside the spherical conducting shell

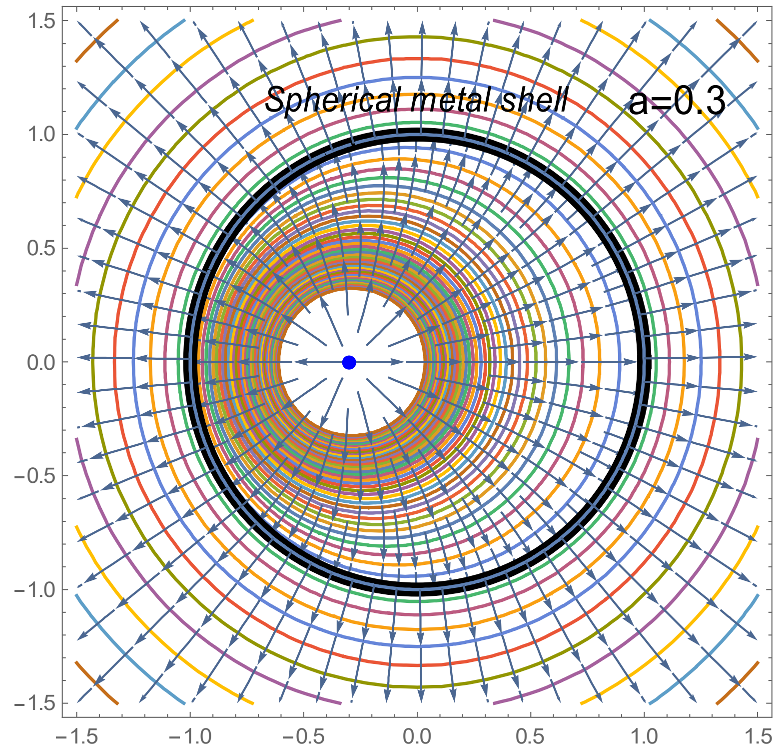
A point charge (positive) at (0,-*a*) inside the spherical metal shell with $0 < a < 1.0$. The electric field inside the spherical metal shell is not zero. The electric potential at the surface is equal to zero.

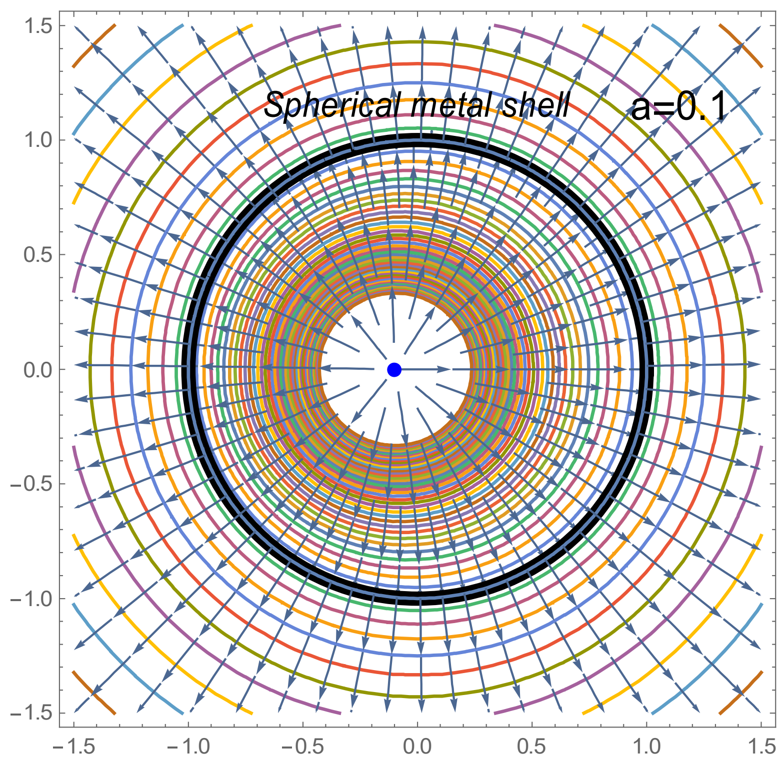

Spherical metal shell
a=0.99
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

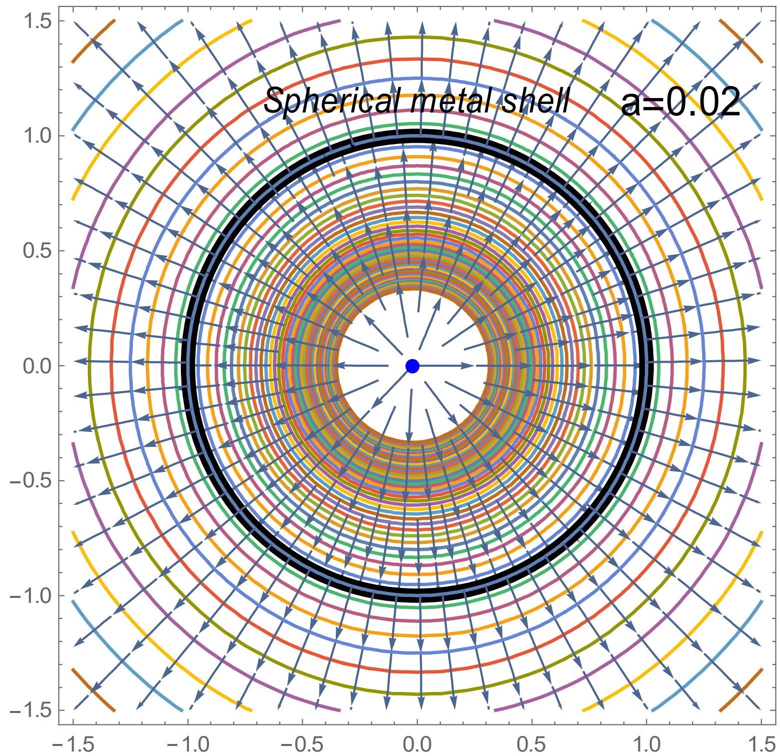

Spherical metal shell
a=0.95
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5


Spherical metal shell
a=0.8


Spherical metal shell
a=0.5

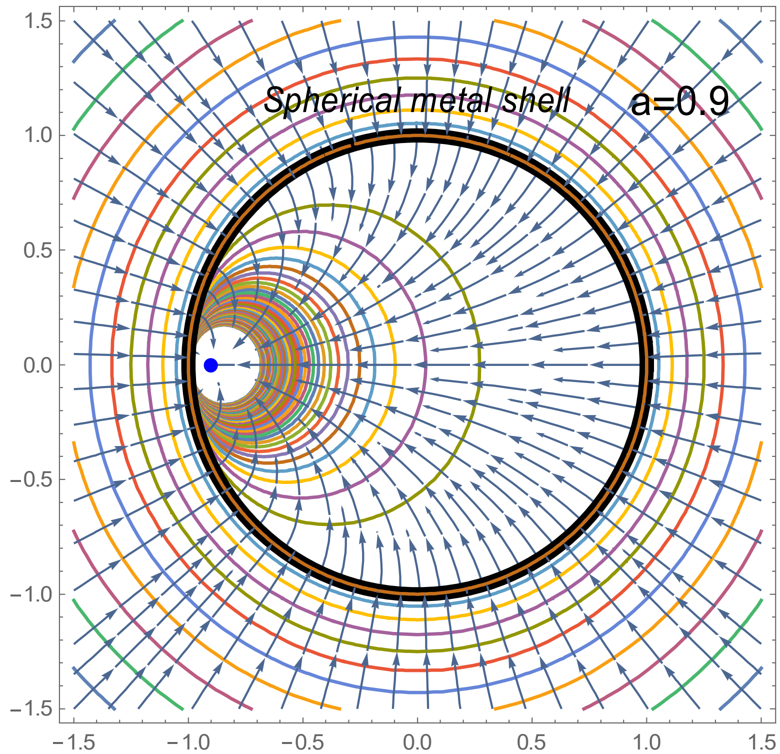

Spherical metal shell
a=0.3
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5

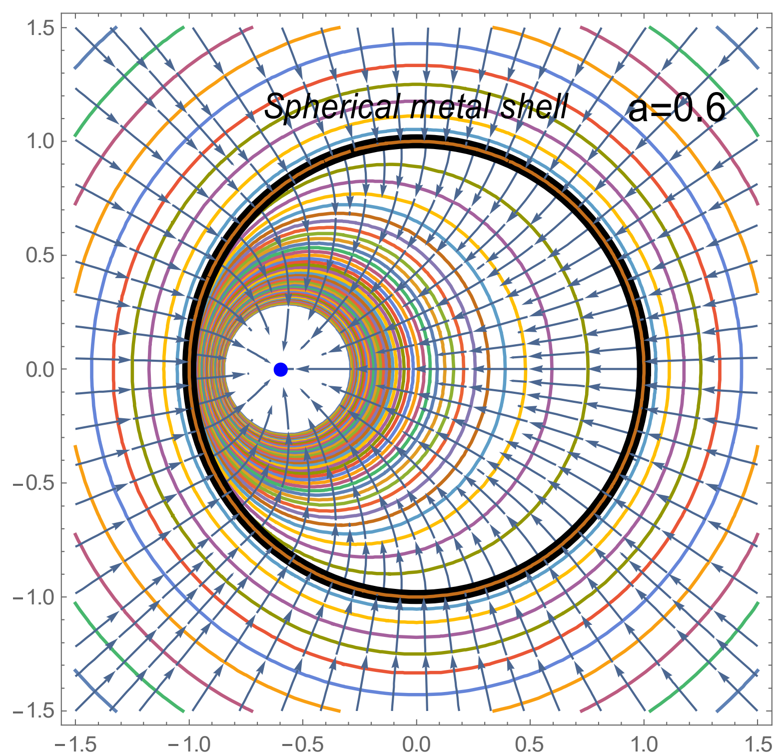

Spherical metal shell
a=0.1
1.5
1.0
0.5
0.0
−0.5
−1.0
−1.5
−1.5
−1.0
−0.5
0.0
0.5
1.0
1.5



(c) A negative charge inside the spherical conducting shell

A point charge (negative) at (0,-*a*) inside the spherical metal shell with $0 < a < 1.0$. The electric field inside the spherical metal shell is not zero. The electric potential at the surface is equal to zero.


Spherical metal shell
a=0.9


Spherical metal shell
a=0.6