Electric field of dielectric medium

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Here we discuss the electric field near a dielectric (the polarization vector ***P***) for the two cases; (1) in the absence of an external electric field and in the presence of the external electric field.

1. Electric field in the vicinity of dielectric medium in the absence of external electric field

(a) Lorentz field (local field)

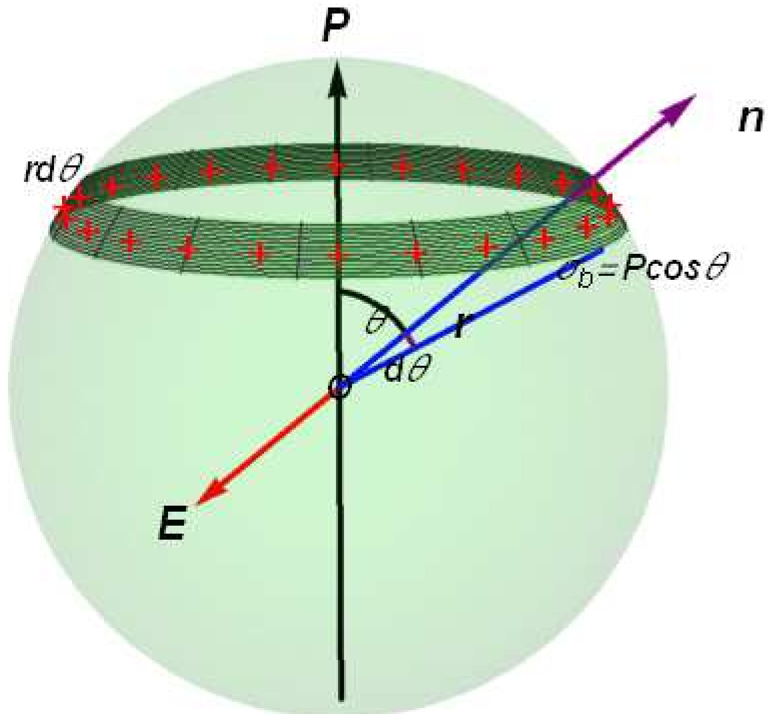
A field which acts on and polarizes an atom a **Lorentz field**. Here we discuss the electric field inside of a dielectric sphere in the absence of external electric field. This field is called a Lorentz field and expressed by

$$\boldsymbol{E} = -\frac{\boldsymbol{P}}{3\varepsilon_0},$$

where ***P*** is the polarization vector. The electric field outside the sphere is the same as one from an electric dipole moment

$$p = \frac{4\pi R^3}{3} P,$$

where R is the radius of dielectric sphere.



The Lorentz field due to the surface charge of the polarization vector can be calculated as follows. Let the direction of the polarization be the z axis and the angle measured from the z axis be θ. The charge density on the surface is $\boldsymbol{P}\cdot\boldsymbol{n} = P\cos\theta$. The surface area is $2\pi R\sin\theta(Rd\theta)$. So the total charge on this surface area is $[2\pi R\sin\theta Rd\theta]P\cos\theta$. This charge produces an electric field at the origin O. The electric field along the z axis is

$$\frac{1}{4\pi\varepsilon_0 R^2}[2\pi R\sin\theta Rd\theta]P\cos\theta(\cos\theta) = \frac{P}{2\varepsilon_0}\sin\theta\cos^2\theta d\theta$$

So the resultant electric field is obtained as

$$E_z = -\frac{P}{2\varepsilon_0}\int_0^{\pi}\cos^2\theta\sin\theta d\theta = -\frac{P}{2\varepsilon_0}\frac{2}{3} = -\frac{P}{3\varepsilon_0}$$

or

$$\boldsymbol{E} = -\frac{\boldsymbol{P}}{3\varepsilon_0}$$

((**Note**)) **Summary**

$$E_z = -\int_0^{\pi} \frac{(2\pi r\sin\theta)(rd\theta)P\cos\theta}{4\pi\varepsilon_0 r^2}\cos\theta$$
$$= -\frac{P}{2\varepsilon_0}\int_0^{\pi} \sin\theta\cos^2\theta d\theta$$
$$= -\frac{P}{3\varepsilon_0}$$

(b) Electric field outside the dielectrics

The electric potential of the electric dipole moment $(r > R)$ is given by

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2}$$

where

$$p = \frac{4\pi R^3}{3}P \qquad \text{(the electric dipole moment)}$$

where P is the magnitude of the polarization vector $\boldsymbol{P}$, and $z = r\cos\theta$. The electric field outside the sphere is obtained as

$$E_x = -\frac{\partial V}{\partial x} = \frac{3pzx}{4\pi\varepsilon_0 (z^2 + x^2)^{5/2}} = \frac{3p}{4\pi\varepsilon_0 r^3}\sin\theta\cos\theta\,,$$

$$E_z = -\frac{\partial V}{\partial z}$$
$$= \frac{p}{4\pi\varepsilon_0}\Big[\frac{3z^2}{(z^2 + x^2)^{5/2}} - \frac{1}{(z^2 + x^2)^{3/2}}\Big]$$
$$= \frac{p}{4\pi\varepsilon_0 r^3}(3\cos^2\theta - 1)$$

Note that in the polar coordinate

$$
\begin{aligned}
\boldsymbol{E} &= -\nabla V(r,\theta) \\
&= -\boldsymbol{e}_r \frac{\partial V}{\partial r} - \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} - \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi} \\
&= -\boldsymbol{e}_r \frac{\partial V}{\partial r} - \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta}
\end{aligned}
$$

leading to

$$
E_r = -\frac{\partial V}{\partial r} = \frac{2p\cos\theta}{4\pi\varepsilon_0 r^3}, \qquad E_\theta = -\frac{1}{r}\frac{\partial V}{\partial \theta} = \frac{p\sin\theta}{4\pi\varepsilon_0 r^3}
$$

since

$$
\frac{\partial V}{\partial \phi} = 0.
$$

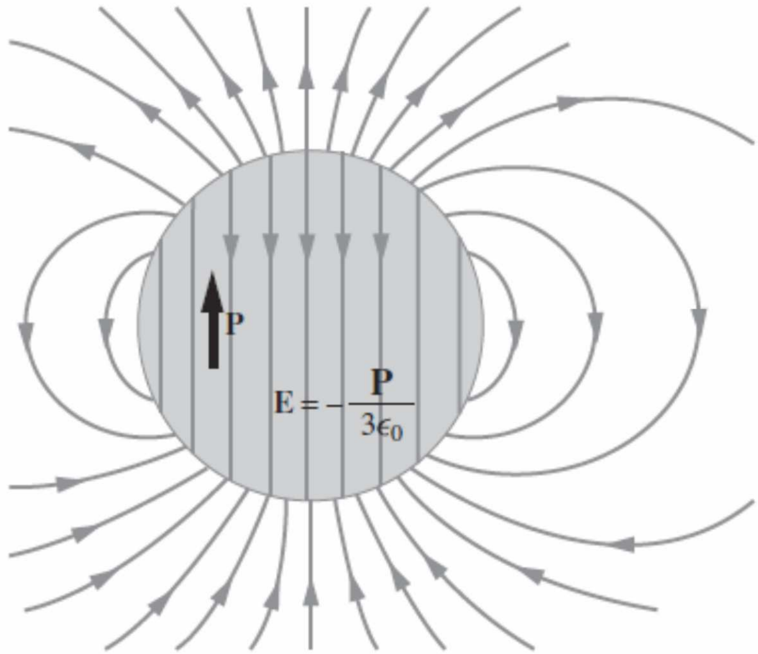
$$
\begin{aligned}
E_z &= E_r\cos\theta - E_\theta\sin\theta \\
&= \frac{2p\cos^2\theta}{4\pi\varepsilon_0 r^3} - \frac{p\sin^2\theta}{4\pi\varepsilon_0 r^3} \\
&= \frac{p(3\cos^2\theta - 1)}{4\pi\varepsilon_0 r^3} \\
&= \frac{p}{4\pi\varepsilon_0}\frac{(3z^2 - r^2)}{r^5}
\end{aligned}
$$

$$
\begin{aligned}
E_x &= E_r\sin\theta + E_\theta\cos\theta \\
&= \frac{3p\sin\theta\cos\theta}{4\pi\varepsilon_0 r^3} \\
&= \frac{p}{4\pi\varepsilon_0}\frac{3zx}{r^5}
\end{aligned}
$$

At the top of the dielectric sphere, ($r = R$, $\theta = 0$),

$$
E_x = 0,
$$

$$
E_z = \frac{2p}{4\pi\varepsilon_0 R^3} = \frac{2}{4\pi\varepsilon_0 R^3}\frac{4}{3}\pi R^3 P = \frac{2P}{3\varepsilon_0} \qquad \text{(outside at top, north pole)}
$$



P
$\mathbf{E} = -\frac{\mathbf{P}}{3\epsilon_0}$

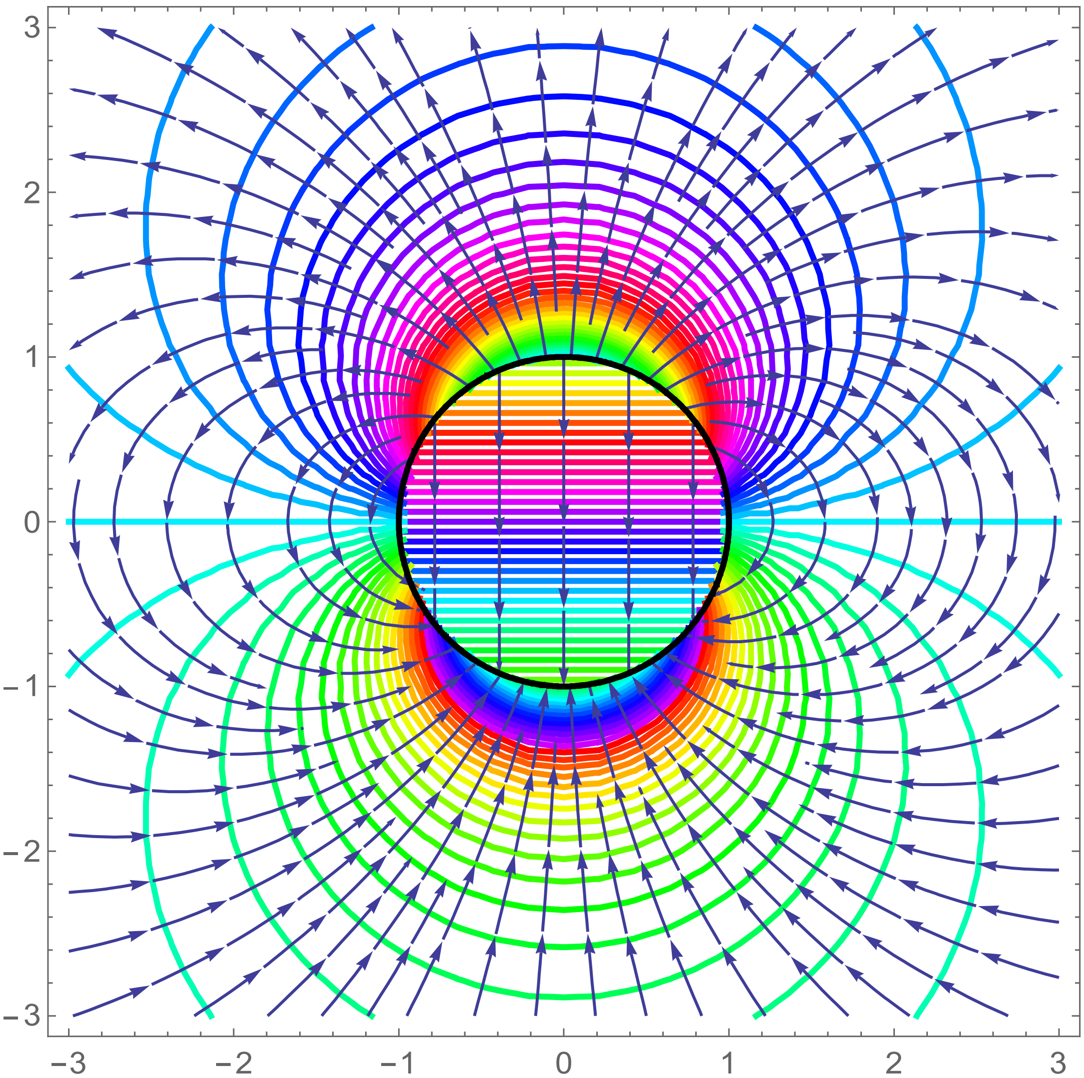
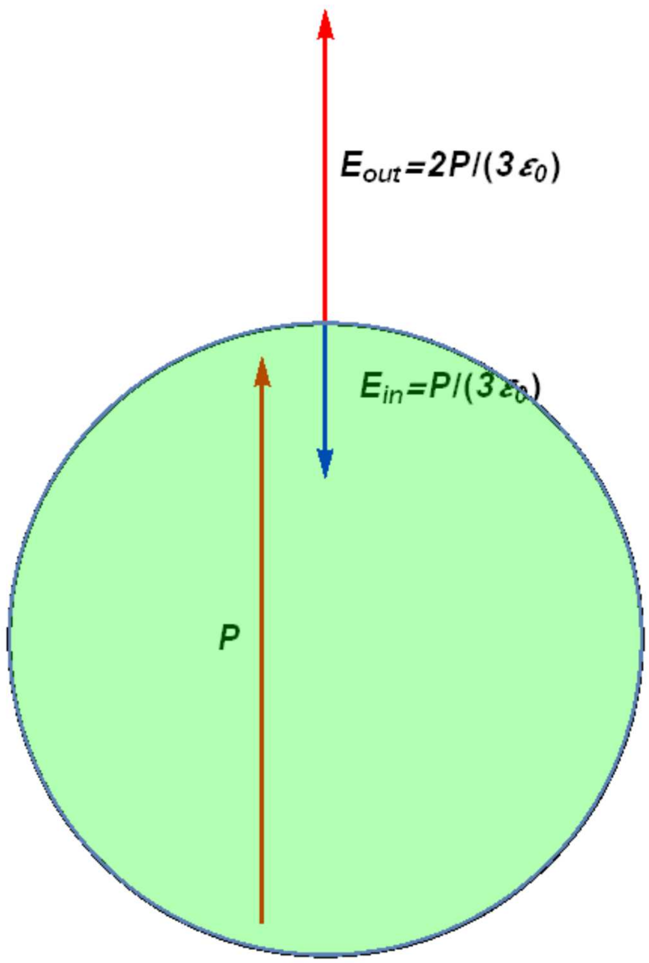


Fig. Electric field inside and outside the dielectric sphere, in the absence of an external electric field.

2. Boundary condition of the electric field and displacement vector at the top of sphere

The normal component of the electric field is discontinuous, while the tangential component of the electric field is continuous.



The normal component of the electric field is discontinuous on the boundary surface.

(a) Normal component of the electric field

$$E_{out,\perp} = \frac{2P}{3\varepsilon_0}, \qquad E_{in,\perp} = -\frac{P}{3\varepsilon_0},$$

leading to

$$\boxed{E_{out,\perp} - E_{in,\perp} = \frac{P}{\varepsilon_0}} \qquad \text{(discontinuity)}$$

(b) Normal component of the displacement vector

$$D_{out,\perp} = \varepsilon_0 E_{out,\perp} = \frac{2P}{3}, \qquad D_{in,\perp} = \varepsilon_0 E_{in,\perp} + P = -\frac{P}{3} + P = \frac{2}{3}P$$

leading to

$$D_{out,\perp} = D_{in,\perp} \qquad \text{(Continuity)}$$

(c) Tangential component of the electric field

$$E_{out,//} = 0\,, \qquad E_{in,//} = 0\,, \qquad \text{(continuous)}$$

This example illustrates the general rules for the behavior of the field components at the surface of a polarized medium. The normal component of the electric field ***E*** is discontinuous at the surface boundary, while the component of ***E*** parallel to the boundary surface remain continuous.

3. Analytical method from the Laplace equation with the boundary condition

Laplace equation

$$\boldsymbol{E} = -\nabla V\,, \qquad \nabla\cdot\boldsymbol{E} = -\nabla^2 V = 0 \qquad \text{for } r > R \text{ and } r < R$$

The solution of the Laplace equation is given as follows.

$$V(r,\theta) = \sum_{l=0}^{\infty}(A_l r^l + \frac{B_l}{r^{l+1}})P_l(\cos\theta)$$

or

$$V_{out}(r,\theta) = \sum_{l=0}^{\infty}\frac{B_l}{r^{l+1}}P_l(\cos\theta)\,, \qquad V_{in}(r,\theta) = \sum_{l=0}^{\infty}A_l r^l P_l(\cos\theta)$$

$V(r,\theta)$ is continuous at $r = R$

$$V(r = R,\theta) = V_0(\theta) = \sum_{l=0}^{\infty}A_l R^l P_l(\cos\theta) = \sum_{l=0}^{\infty}\frac{B_l}{R^{l+1}}P_l(\cos\theta)$$

$$A_l = \frac{2l+1}{2R^l}\int_0^{\pi}V_0(\theta)P_l(\cos\theta)\sin\theta d\theta$$

$$A_l R^l = \frac{B_l}{R^{l+1}}\,, \qquad B_l = A_l R^{2l+1}$$

$$B_l = \left(\frac{2l+1}{2}\right) R^{l+1} \int_0^{\pi} V_0(\theta) P_l(\cos\theta) \sin\theta d\theta$$

((**Boundary condition**))

$$(-\frac{\partial V_{out}}{\partial r} + \frac{\partial V_{in}}{\partial r})_{r=R} = \frac{1}{\varepsilon_0}\sigma_0(\theta) = \frac{1}{\varepsilon_0}\boldsymbol{P}\cdot\boldsymbol{n} = P\cos\theta$$

since the normal component of ***E*** on the surface is discontinuous.

$$\sum_{l=0}^{\infty}(2l+1)A_l R^{l-1} P_l(\cos\theta) = \frac{1}{\varepsilon_0}\sigma_0(\theta)$$

$$A_l = \frac{1}{2\varepsilon_0 R^{l-1}} \int_0^{\pi} \sigma_0(\theta) P_l(\cos\theta) \sin\theta d\theta$$

When $\sigma_0(\theta) = P\cos\theta = P\ P_1(\cos\theta)$

$$A_1 = \frac{P}{2\varepsilon_0}\int_0^{\pi}[P_1(\cos\theta)]^2 \sin\theta d\theta = \frac{P}{3\varepsilon_0}$$

$A_l = 0$ except for $l = 1$

$$B_1 = R^3 A_1 = R^3 \frac{P}{3\varepsilon_0}$$

For $r < R$,

$$V = \frac{P}{3\varepsilon_0} r\cos\theta$$

For $r > R$

$$V = \frac{B_1}{r^2}\cos\theta = \frac{R^3 A_1}{r^2}\cos\theta = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta$$

The electric field is given by

$$\boldsymbol{E} = -\nabla V = -\boldsymbol{e}_r \frac{\partial V}{\partial r} - \boldsymbol{e}_\theta \frac{1}{r}\frac{\partial V}{\partial \theta} - \boldsymbol{e}_\phi \frac{1}{r\sin\theta}\frac{\partial V}{\partial \phi}$$

For $r < R$,

$$V = \frac{P}{3\varepsilon_0} r\cos\theta$$

and

$$\boldsymbol{E} = \frac{P}{3\varepsilon_0}(-\boldsymbol{e}_r \cos\theta + \boldsymbol{e}_\theta \sin\theta) = -\frac{P}{3\varepsilon_0}\boldsymbol{e}_z$$

where

$$\boldsymbol{e}_z = -\boldsymbol{e}_r \cos\theta + \boldsymbol{e}_\theta \sin\theta$$

For $r > R$

$$V = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta$$

Since

$$p = \frac{4\pi R^3}{3} P$$

$$V = \frac{p\cos\theta}{4\pi\varepsilon_0 r^2} = \frac{\boldsymbol{p}\cdot r}{4\pi\varepsilon_0 r^2}$$

where $\boldsymbol{p}$ is the electric dipole moment.

((**Mathematica**))

```
Clear["Global`*"];

V1[r_, θ_] := P/(3 ϵ0) r Cos[θ] ;

r2xRule =
 Thread[{r, θ, ϕ} → CoordinatesFromCartesian[{x, y, z}, Spherical]]
```

$\left\{r \to \sqrt{x^2 + y^2 + z^2},\ \theta \to \text{ArcCos}\left[\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right],\ \phi \to \text{ArcTan}[x, y]\right\}$

```
V2D1 = V1[r, θ] /. r2xRule /. y → 0 // Simplify;

rule1 = {R → 1, P → 1, ϵ0 → 1};

V2D11 = V2D1 /. rule1
```

$\frac{z}{3}$

```
Ex1 = -D[V2D11, x] // Simplify;
Ez1 = -D[V2D11, z] // Simplify;
```

```
g1 = ContourPlot[Evaluate[Table[V2D11 == α, {α, -0.5, 0.5, 0.02}]],
    {x, -3, 3}, {z, -3, 3},
    ContourStyle → Table[{Thick, Hue[0.03 i]}, {i, 0, 60}],
    RegionFunction → Function[{x, z}, x^2 + z^2 < 1]];

g2 = StreamPlot[Evaluate[{Ex1, Ez1}], {x, -3, 3}, {z, -3, 3},
    RegionFunction → Function[{x, z}, x^2 + z^2 < 1]];
```

V2[r_, θ_] := $\frac{P}{3\,\epsilon 0}\,\frac{R^3}{r^2}$ Cos[θ] ;

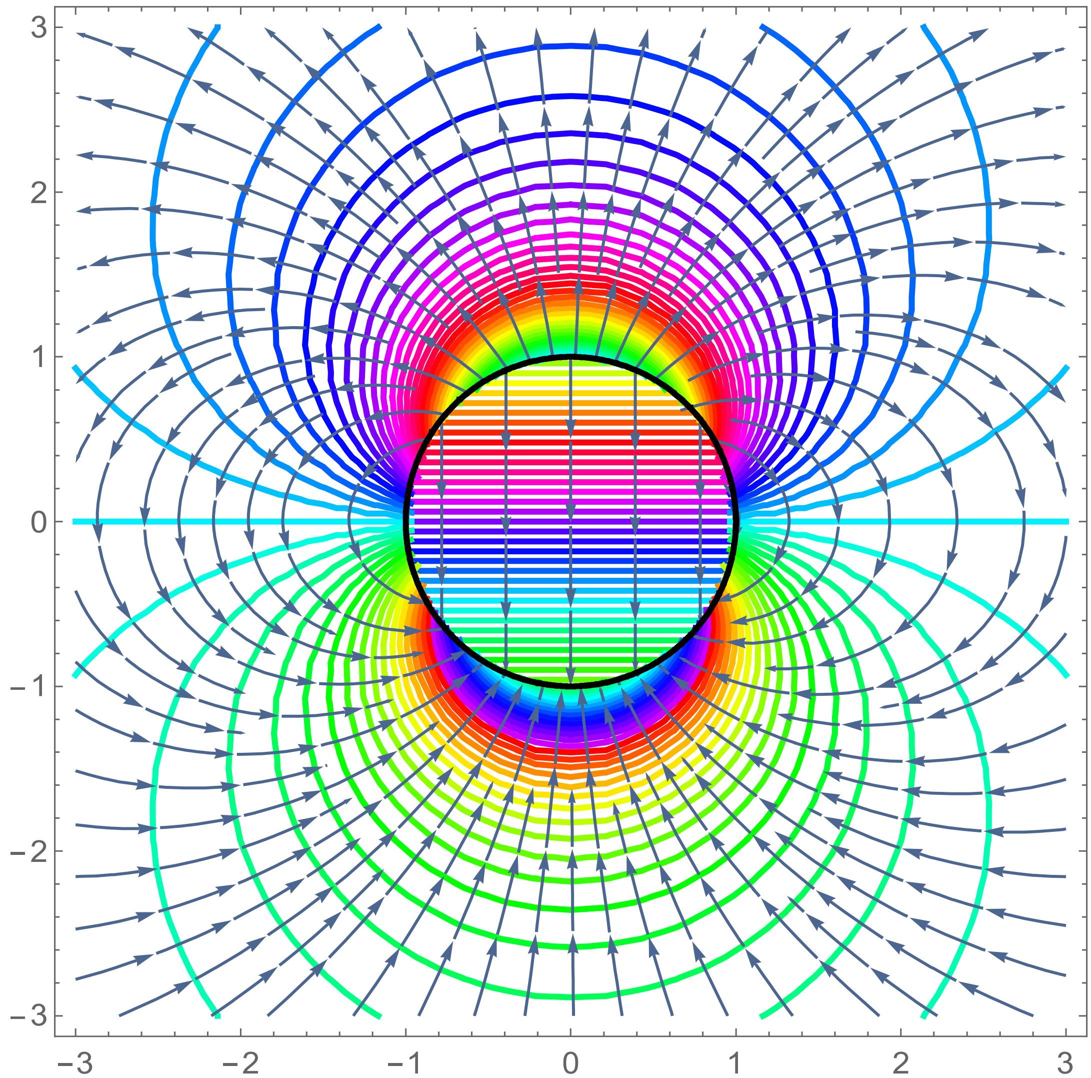
```
V2D2 = V2[r, θ] /. r2xRule /. y → 0 // Simplify
```

$$\frac{P\,R^3\,z}{3\,\left(x^2+z^2\right)^{3/2}\,\epsilon 0}$$

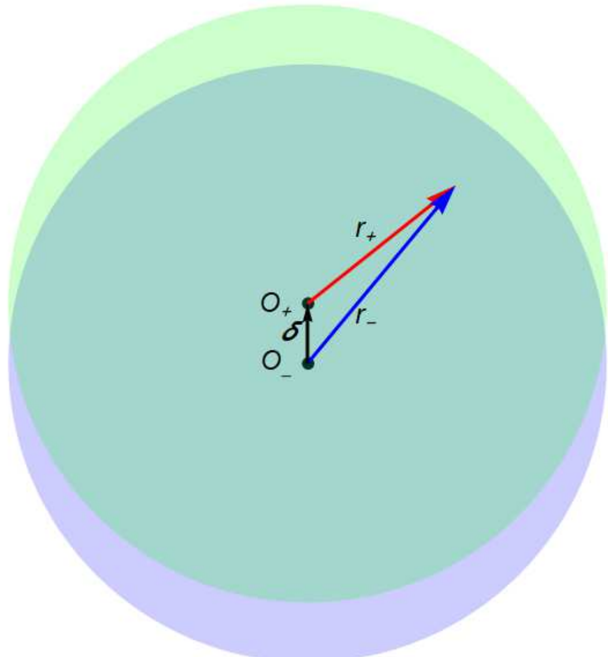
```
rule1 = {R → 1, P → 1, ϵ0 → 1};

V2D22 = V2D2 /. rule1
```

$$\frac{z}{3\,\left(x^2+z^2\right)^{3/2}}$$

```
Ex2 = -D[V2D22, x] // Simplify;
Ez2 = -D[V2D22, z] // Simplify;

g3 = ContourPlot[Evaluate[Table[V2D22 == α, {α, -2, 2, 0.01}]],
   {x, -3, 3}, {z, -3, 3},
   ContourStyle → Table[{Thick, Hue[0.03 i]}, {i, 0, 60}],
   RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];

g4 = StreamPlot[Evaluate[{Ex2, Ez2}], {x, -3, 3}, {z, -3, 3},
   RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];

g5 = ParametricPlot[{Cos[θ], Sin[θ]}, {θ, 0, 2 π},
   PlotStyle → {Thick, Black}];

Show[g1, g2, g3, g4, g5, PlotRange → All]
```



4. Alternative way of deriving the electric field of a polarized sphere



There are two spheres of charge: a positive sphere and a negative sphere. Without polarization the two are superimposed and cancel completely. But when the material is uniformly polarized, all the plus charges move slightly upward (the *z* direction), and all the minus charges more slightly downward. The two spheres no longer overlap perfectly: at the top there is a cap of leftover positive charge and at the bottom a cap of negative charge. This leftover charge is precisely the bound surface charge σ_b.

$$E_+(4\pi r_+^2) = \frac{1}{\varepsilon_0}\frac{4\pi}{3}\rho r_+^3 \qquad \boldsymbol{E}_+ = \frac{1}{3\varepsilon_0}\rho(\boldsymbol{r}_+ - \boldsymbol{r}_-) = -\frac{1}{3\varepsilon_0}\rho\boldsymbol{\delta}$$

$$E_-(4\pi r_-^2) = \frac{1}{\varepsilon_0}\frac{4\pi}{3}(-\rho r_-^3), \qquad \boldsymbol{E}_- = -\frac{1}{\varepsilon_0}\frac{1}{3}\rho\boldsymbol{r}_-$$

where ρ is the charge density. Thus the resulting electric field is given by

$$\boldsymbol{E}_{tot} = \boldsymbol{E}_+ + \boldsymbol{E}_- = \frac{1}{3\varepsilon_0}\rho(\boldsymbol{r}_+ - \boldsymbol{r}_-) = -\frac{1}{3\varepsilon_0}\rho\boldsymbol{\delta}$$

where

$$\boldsymbol{r}_+ - \boldsymbol{r}_- = -\boldsymbol{\delta}$$

The electric dipole moment p is defined by

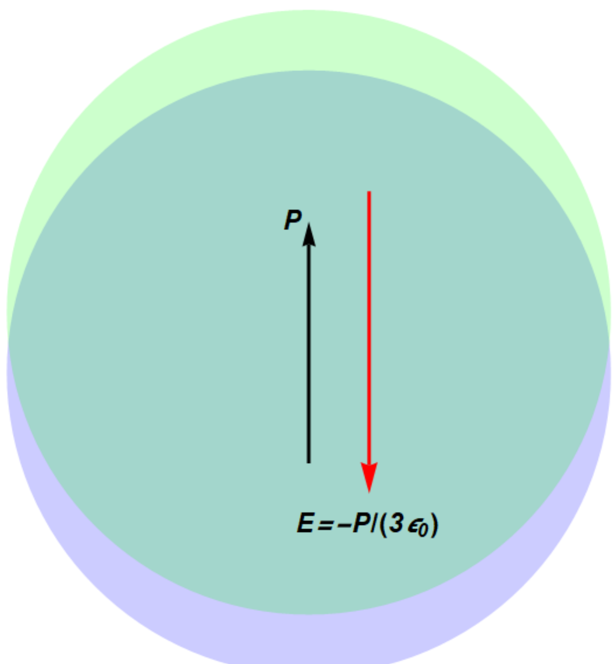
$$Np = Ne\delta$$

where N is the total number of electric dipole moments. The electric polarization is

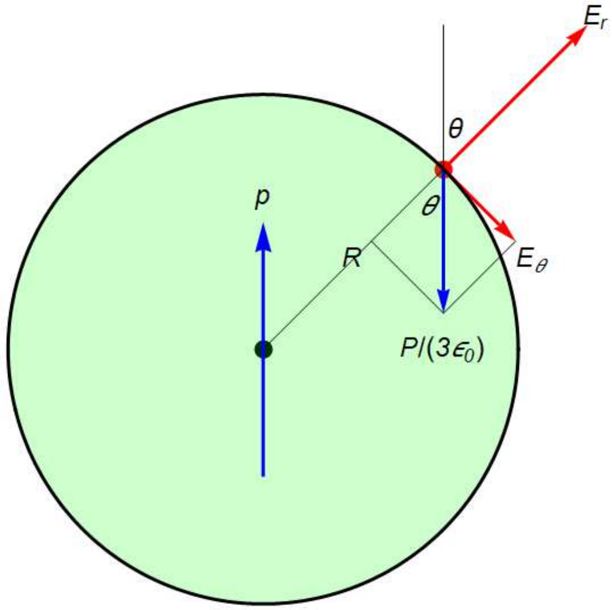
$$P = \frac{Np}{V} = np = ne\delta = \rho\delta\,, \qquad \text{or} \qquad \rho\boldsymbol{\delta} = e\boldsymbol{P}$$

Using this expression of $\boldsymbol{P}$, we have

$$\boxed{\boldsymbol{E}_{tot} = -\frac{\boldsymbol{P}}{3\varepsilon_0}}$$



Boundary condition



Outside of the sphere (electric field of electric dipole)

$$(E_{out})_r = \frac{2p\cos\theta}{4\pi\varepsilon_0 R^3}, \qquad (E_{out})_\theta = \frac{p\sin\theta}{4\pi\varepsilon_0 R^3}$$

$$P = \frac{p}{\frac{4\pi}{3}R^3}$$

Inside of the sphere

$$\boldsymbol{E}_{in} = -\frac{\boldsymbol{P}}{3\varepsilon_0}$$

$$(E_{in})_r = -\frac{P}{3\varepsilon_0}\cos\theta = -\frac{\frac{3p}{4\pi R^3}}{3\varepsilon_0}\cos\theta = -\frac{p}{4\pi\varepsilon_0 R^3}\cos\theta$$

$$(E_{in})_\theta = \frac{P}{3\varepsilon_0}\sin\theta = -\frac{\frac{3p}{4\pi R^3}}{3\varepsilon_0}\sin\theta = -\frac{p}{4\pi\varepsilon_0 R^3}\sin\theta$$

(a) Boundary condition for *E*

Normal component:

$$\left(E_{out}\right)_r - \left(E_{in}\right)_r = \frac{3p}{4\pi\varepsilon_0 R^3}\cos\theta = \frac{P\cos\theta}{\varepsilon_0} = \frac{\boldsymbol{P}.\boldsymbol{n}}{\varepsilon_0}$$

Tangential component:

$$\left(E_{out}\right)_\theta - \left(E_{in}\right)_\theta = 0$$

(b) Boundary condition for *D*

$$\begin{aligned}\left(D_{out}\right)_r - \left(D_{in}\right)_r &= \varepsilon_0\left(E_{out}\right)_r - [\varepsilon_0\left(E_{in}\right)_r + P_r] \\ &= \varepsilon_0[\left(E_{out}\right)_r - \left(E_{in}\right)_r - P\cos\theta] \\ &= 0\end{aligned}$$

(c) General case

In general, we have the following boundary conditions for ***E*** and ***D***

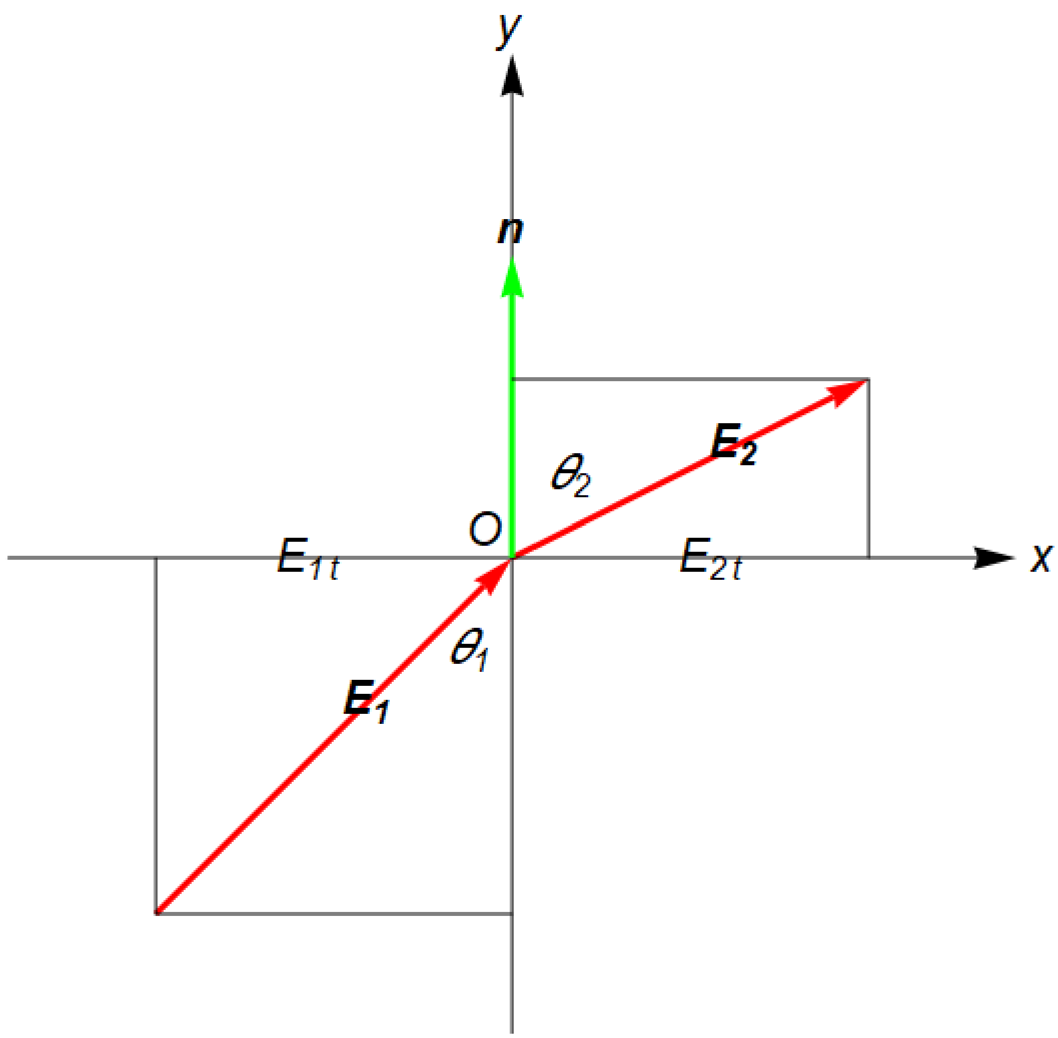


Fig. $\varepsilon_1 = 1$. $\varepsilon_2 = 2$. and $\theta_1 = \dfrac{\pi}{4}$

$$\boldsymbol{E}_1 \times \boldsymbol{n} = \boldsymbol{E}_2 \times \boldsymbol{n}$$

$E_{1t} = E_{2t}$ (tangential component)

where $\boldsymbol{n}$ is the unit vector normal to the boundary surface.

$$E_1 \sin\theta_1 = E_2 \sin\theta_2 \qquad (1)$$

or

$$E_2 = \frac{\sin\theta_1}{\sin\theta_2} E_1$$

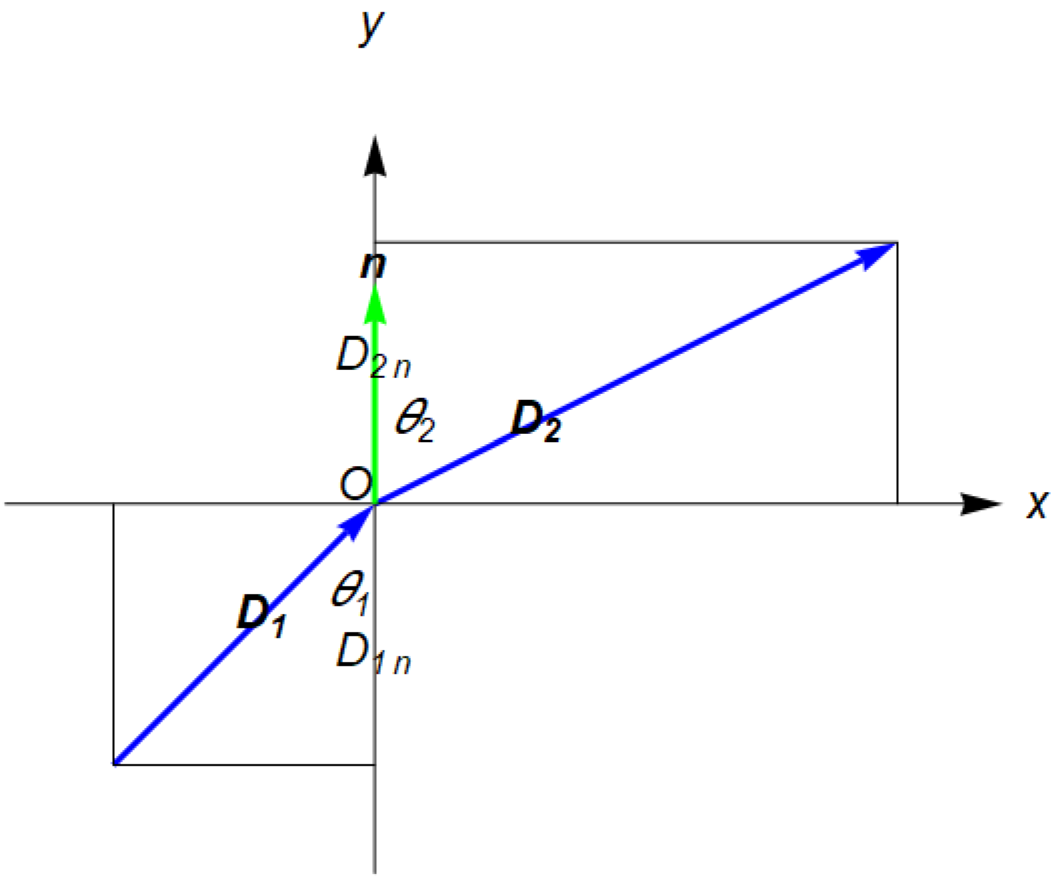


Fig. $\varepsilon_1 = 1$. $\varepsilon_2 = 2$. and $\theta_1 = \frac{\pi}{4}$.

$$\boldsymbol{D}_1 \cdot \boldsymbol{n} = \boldsymbol{D}_2 \cdot \boldsymbol{n}$$

$$D_{1n} = D_{2n} \qquad \text{(normal component)}$$

$$D_1 \cos\theta_1 = D_2 \sin\theta_2$$

Since $D_1 = \varepsilon_1 E_1$, and $D_2 = \varepsilon_2 E_2$, we have

$$\varepsilon_1 E_1 \cos\theta_1 = \varepsilon E_2 \cos\theta_2 \qquad (2)$$

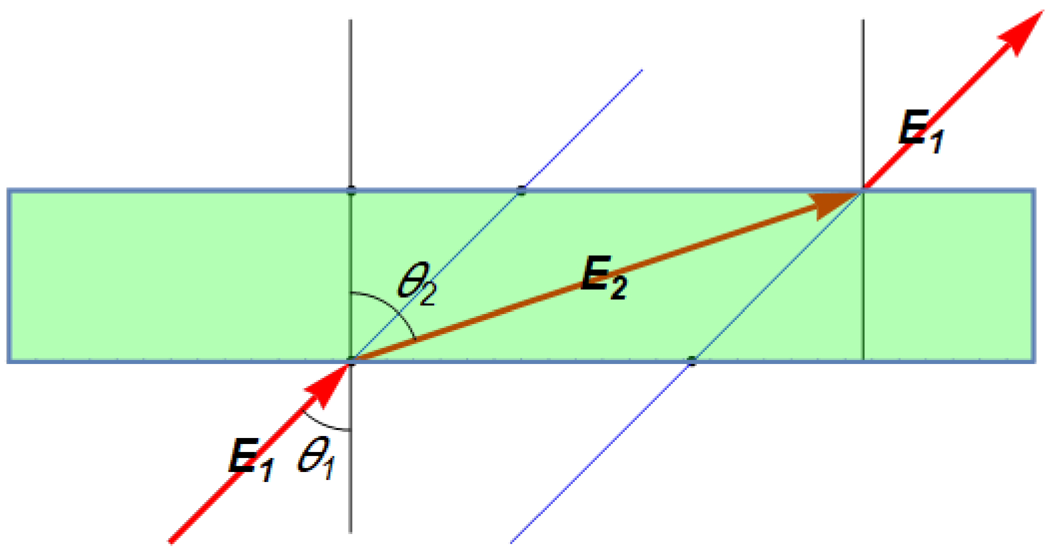
From Eqs.(1) and (2), we have

$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

5. Example-1

(a) Method using the boundary condition

We put a plane of dielectric with an infinitely large area and a finite thickness in the presence of a uniform electric field $\boldsymbol{E}_1$. The angle between the electric field $\boldsymbol{E}_1$ and the normal vector $\boldsymbol{n}$ is θ_1. We discuss the detail of the electric field inside the dielectric. The dielectric constants of vacuum and dielectric are ε_0 and ε, respectively.



We note that the tangential component of the electric field is continuous on the boundary surface.

$$E_1 \sin\theta_1 = E_2 \sin\theta_2$$

The normal component of the displacement vector is continuous on the boundary surface.

$$D_1 \cos\theta_1 = D_2 \cos\theta_2 \qquad \text{or} \qquad \varepsilon_0 E_1 \cos\theta_1 = \varepsilon E_2 \cos\theta_2$$

where

$$D_1 = \varepsilon_0 E_1, \qquad D_2 = \varepsilon E_2$$

These equations lead to the relation

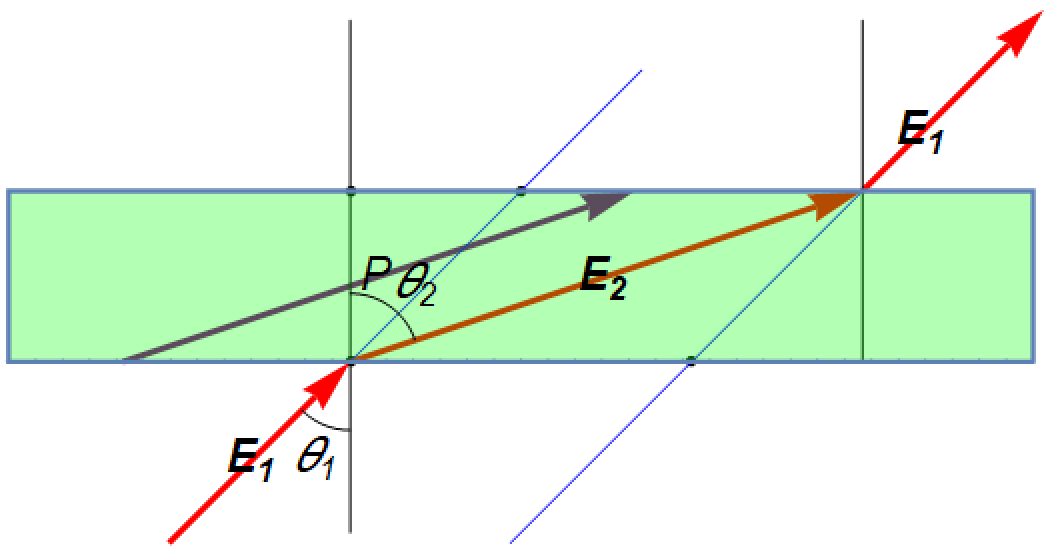
$$\tan\theta_2 = \frac{\varepsilon}{\varepsilon_0}\tan\theta_1, \qquad \theta_2 = \arctan[\frac{\varepsilon}{\varepsilon_0}\tan\theta_1]$$

with

$$\kappa = \frac{\varepsilon}{\varepsilon_0}$$

(b) Alternative way using the polarization vector

We discuss the above problem in alternative way.



A dielectric sphere in a uniform external electric field

$$\boldsymbol{E}_d = -\frac{\sigma_b}{\varepsilon_0}\boldsymbol{e}_y .$$

wit

$$\sigma_b = P\cos\theta_2$$

$$\boldsymbol{E}_2 = \boldsymbol{E}_1 + \boldsymbol{E}_d$$

The normal component:

$$E_2 \cos\theta_2 = E_1 \cos\theta_1 - \frac{P\cos\theta_2}{\varepsilon_0} = E_1 \cos\theta_1 - \frac{\varepsilon_0 \chi_e E_2 \cos\theta_2}{\varepsilon_0}$$

$$E_2 (1+\chi_e) \cos\theta_2 = E_1 \cos\theta_1$$

or

$$E_2 (1+\chi_e) \cos\theta_2 = E_1 \cos\theta_1$$

or

$$E_2 \kappa \cos\theta_2 = E_1 \cos\theta_1$$

or

$$\varepsilon E_2 \cos\theta_2 = \varepsilon_0 E_1 \cos\theta_1$$

Note that $\chi_e = \frac{\varepsilon}{\varepsilon_0} - 1 = \kappa - 1$

$$\boldsymbol{D}_2 = \varepsilon_0 \boldsymbol{E}_2 + \boldsymbol{P}, \qquad \boldsymbol{P} = \varepsilon_0 \chi_e \boldsymbol{E}_2$$

or

$$\varepsilon \boldsymbol{E}_2 = \varepsilon_0 \boldsymbol{E}_2 + \varepsilon_0 \chi_e \boldsymbol{E}_2$$

$$\chi_e = \frac{\varepsilon}{\varepsilon_0} - 1 = \kappa - 1$$

6. Example-2

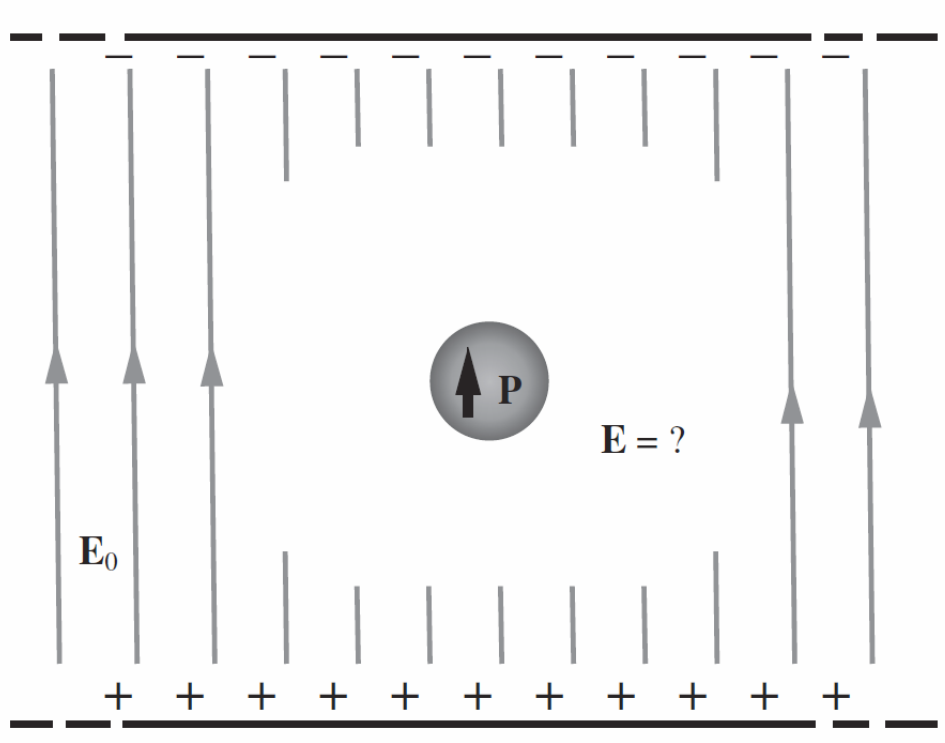


Fig. The sources of the field $\boldsymbol{E}_0$ remain fixed. The dielectric sphere develops some polarization $\boldsymbol{P}$. The total field $\boldsymbol{E}$ is the superposition of $\boldsymbol{E}_0$ and the field of this polarized sphere.

(a) Method-1

$$P_1 = \varepsilon_0 \chi_e E_0$$

where $\chi_e = \kappa - 1$

$$E_1 = E_0 - \frac{P_1}{3\varepsilon_0} = E_0 - \frac{1}{3\varepsilon_0}\varepsilon_0 \chi_e E_0 = (1 - \frac{\chi_e}{3})E_0$$

$$P_2 = \varepsilon_0 \chi_e E_1 = \varepsilon_0 \chi_e (1 - \frac{\chi_e}{3})E_0$$

$$E_2 = E_0 - \frac{P_2}{3\varepsilon_0}$$

$$= E_0 - \frac{1}{3\varepsilon_0}\varepsilon_0\chi_e(1-\frac{\chi_e}{3})E_0$$

$$= [1-\frac{1}{3}\chi_e(1-\frac{\chi_e}{3})]E_0$$

$$P_3 = \varepsilon_0\chi_e E_2 = \varepsilon_0\chi_e[1-\frac{1}{3}\chi_e(1-\frac{\chi_e}{3})]E_0$$

$$E_3 = E_0 - \frac{P_3}{3\varepsilon_0}$$

$$= E_0 - \frac{1}{3\varepsilon_0}\varepsilon_0\chi_e[1-\frac{1}{3}\chi_e(1-\frac{\chi_e}{3})]E_0$$

$$= E_0[1-\frac{1}{3}\chi_e + (-\frac{1}{3}\chi_e)^2 + (-\frac{\chi_e}{3})^3]E_0$$

Repeatedly we get the final form

$$E = \frac{E_0}{1+\frac{1}{3}\chi_e} = \frac{E_0}{1+\frac{1}{3}(\kappa-1)} = \frac{3}{\kappa+2}E_0$$

(b) Method-2

$$P = \varepsilon_0\chi_e E$$

where $\chi_e = \kappa - 1$

$$E = E_0 - \frac{P}{3\varepsilon_0} = E_0 - \frac{1}{3\varepsilon_0}\varepsilon_0\chi_e E$$

$$E(1+\frac{\chi_e}{3}) = E_0$$

$$E = \frac{E_0}{1+\frac{1}{3}\chi_e} = \frac{E_0}{1+\frac{1}{3}(\kappa-1)} = \frac{3}{\kappa+2}E_0$$

7. Electric field around a dielectric sphere in the presence of external electric field

We consider a dielectric sphere in the presence of a uniform external electric field. Suppose that a uniform electric field ($E"$) is produced inside the dielectric sphere. The electric potential is expressed by

$$V_{in} = -E" r\cos\theta .$$

The electric field outside the sphere is sum of (i) a uniform external electric field $\boldsymbol{E}_0$ along the z axis and (ii) the electric field produced by electric dipole moment arising from dielectric sphere. The electric potential is expressed by

$$\begin{aligned} V_{out} &= -E_0 r\cos\theta + E'\frac{R^3}{r^2}\cos\theta \\ &= (-E_0 r + E'\frac{R^3}{r^2})\cos\theta \end{aligned}$$

Note that V_{in} and V_{out} satisfy the Laplace equation. In the limit of $r \to \infty$,

$$V_{out} = -E_0 r\cos\theta .$$

((**Boundary condition**))

(a) for $r = R$, $V_{in} = V_{out}$

$$-E" R = -E_0 R + E'\frac{R^3}{R^2},$$

leading to

$$E_0 = E' + E" .$$

(b) The normal component of ***D*** is continuous on the boundary surface,

$$\varepsilon_0 \frac{\partial V_{out}}{\partial r} = \varepsilon \frac{\partial V_{in}}{\partial r},$$

leading to

$$\varepsilon_0(E_0 + 2E') = \varepsilon E''$$

Using the above equations, we get

$$E' = \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)E_0, \qquad E'' = \left(\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)E_0$$

So we have

$$V_{in} = -\left(\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)E_0 r\cos\theta \qquad (r < R)$$

$$V_{out} = E_0[-r\cos\theta + \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\frac{R^3}{r^2}\cos\theta] \qquad (r > R)$$

The electric field inside the sphere is obtained as

$$\boldsymbol{E}_{in} = -\boldsymbol{e}_z\frac{\partial V_{in}}{\partial z} = \boldsymbol{e}_z\left(\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\boldsymbol{E}_0$$

The electric field outside the sphere

$$\begin{aligned}\boldsymbol{E}_{out} &= -\boldsymbol{e}_r\frac{\partial V_{out}}{\partial r} - -\boldsymbol{e}_\theta\frac{1}{r}\frac{\partial V_{out}}{\partial \theta} \\ &= \boldsymbol{e}_r E_0\cos\theta[1 + 2\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\frac{R^3}{r^3}] \\ &\quad + \boldsymbol{e}_\theta E_0\sin\theta[-r + \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\frac{R^3}{r^2}]\end{aligned}$$

At the top of the sphere,

$$\begin{aligned}(\boldsymbol{E}_{out} - \boldsymbol{E}_{in})_\perp &= E_0[1 + 2\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right) - \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}] \\ &= 3E_0\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\end{aligned}$$

We note that the normal component of the displacement vector $\boldsymbol{D}$ is continuous on the top of the sphere,

$$\left(\boldsymbol{D}_{out}\right)_{\perp} = \left(\boldsymbol{D}_{in}\right)_{\perp}$$

with

$$\boldsymbol{D}_{out} = \varepsilon_0 \boldsymbol{E}_{out}, \qquad \boldsymbol{D}_{in} = \varepsilon_0 \boldsymbol{E}_{in} + \boldsymbol{P}$$

Thus we get

$$\left(\varepsilon_0 \boldsymbol{E}_{out}\right)_{\perp} = (\varepsilon_0 \boldsymbol{E}_{in} + \boldsymbol{P})_{\perp}$$

or

$$\boldsymbol{P}_{\perp} = \varepsilon_0 \left(\boldsymbol{E}_{out} - \boldsymbol{E}_{in}\right)_{\perp} = 3\varepsilon_0 E_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)$$

The polarization vector is

$$\boldsymbol{P} = 3\left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\varepsilon_0 \boldsymbol{E}_0 = 3\left(\frac{\kappa - 1}{\kappa + 2}\right)\varepsilon_0 \boldsymbol{E}_0$$

where $\kappa = \dfrac{\varepsilon}{\varepsilon_0}$.

The polarization vector $\boldsymbol{P}$ is rewritten as

$$\boldsymbol{P} = (\kappa - 1)\left(\frac{3}{\kappa + 2}\right)\varepsilon_0 \boldsymbol{E}_0 = (\kappa - 1)\varepsilon_0 \boldsymbol{E}_{in}$$

where

$$\boldsymbol{E}_{in} = \left(\frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}\right)\boldsymbol{E}_0 = \left(\frac{3}{\kappa + 2}\right)\boldsymbol{E}_0$$

The electric dipole moment p is

$$\begin{aligned}
p &= \frac{4\pi R^3}{3} P \\
&= \frac{4\pi R^3}{3} 3\varepsilon_0 E_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \right) \\
&= 4\pi\varepsilon_0 R^3 E_0 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \right) \\
&= \alpha E_0
\end{aligned}$$

where α is the polarizability

$$\alpha = 4\pi\varepsilon_0 R^3 \left(\frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \right) = 4\pi\varepsilon_0 R^3 \left(\frac{\kappa - 1}{\kappa + 2} \right)$$

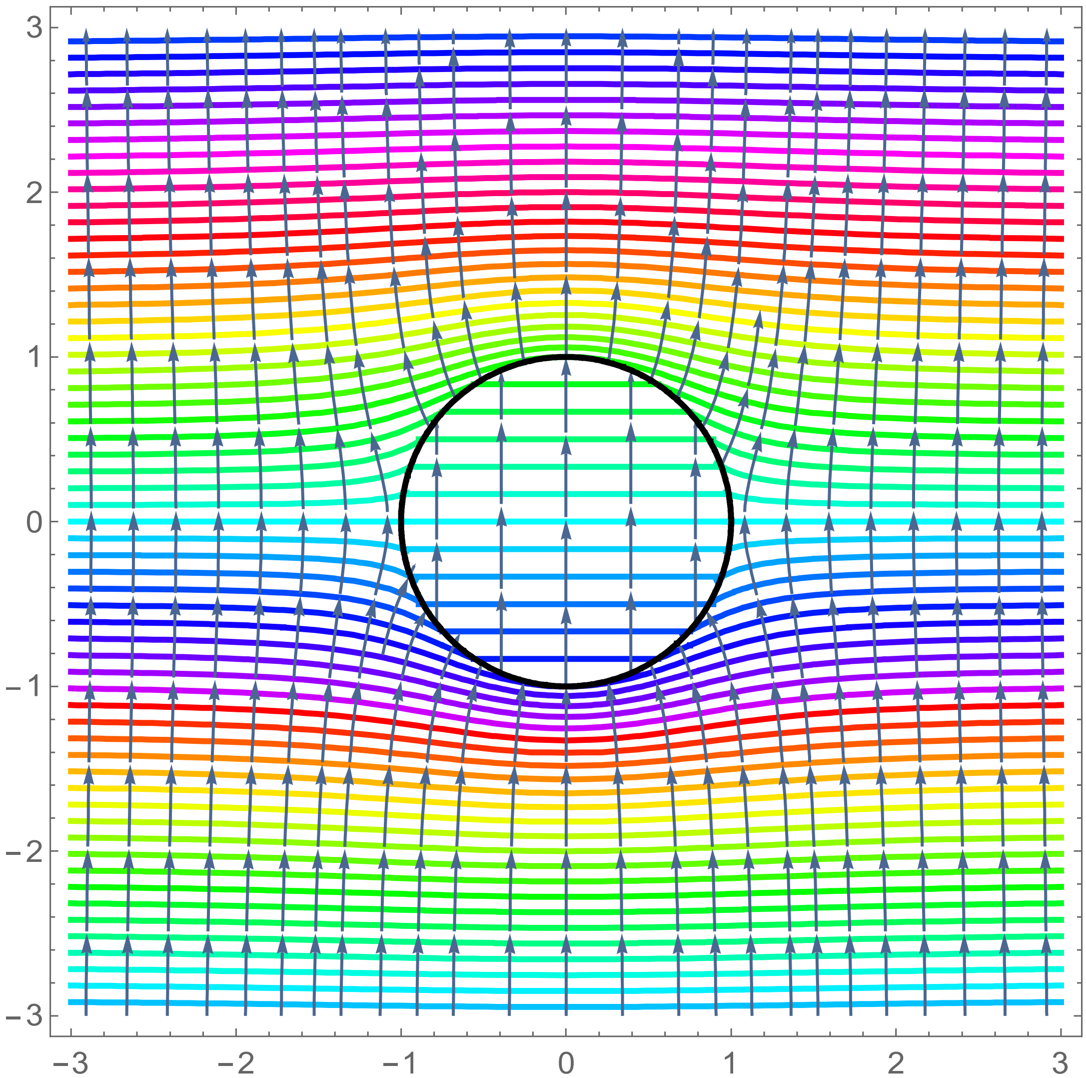


Fig. Electrical field around a dielectric sphere in the presence of an external electric field along the z axis. $R = 1$. $E_0 = 1$. $\kappa = 3$.

((**Mathematica**))

Electric field produced by a uniformly polarized sphere of radius R

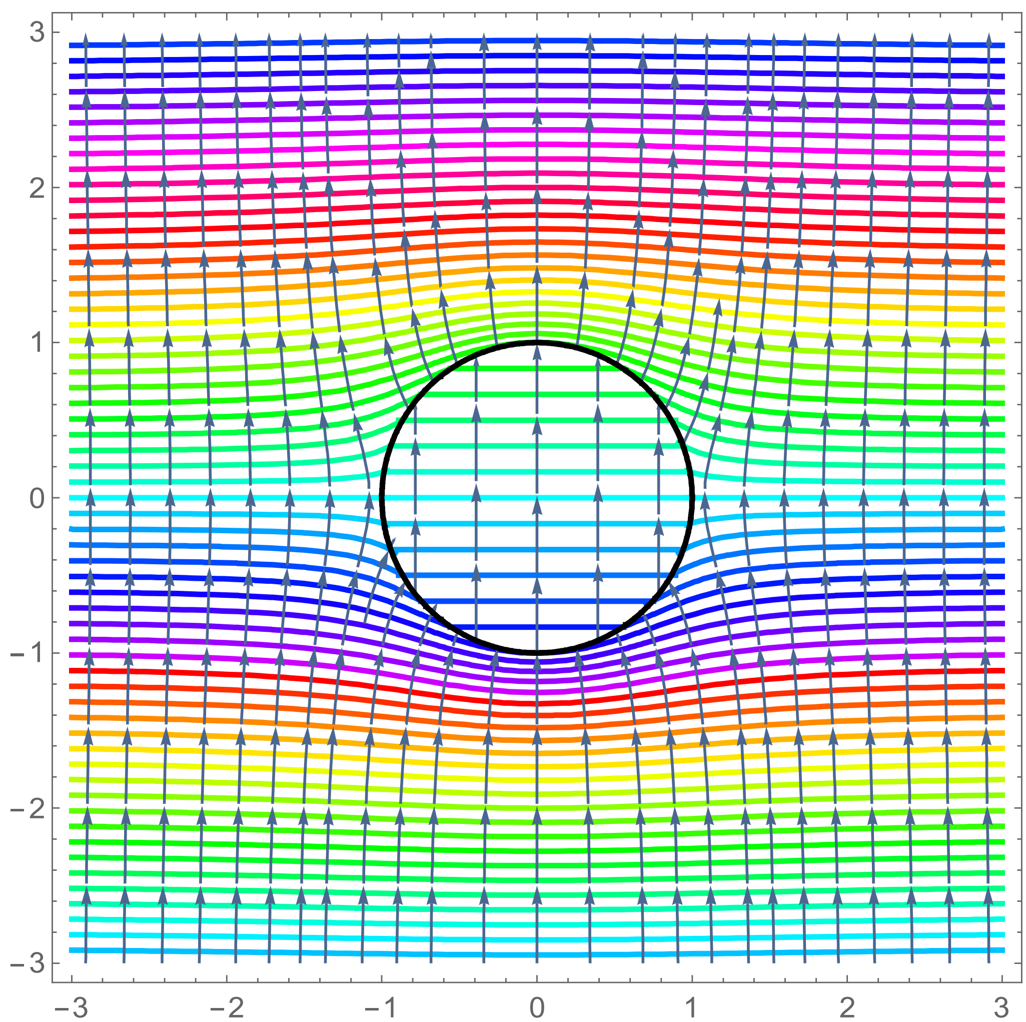
```
Clear["Global`*"];
rule1 = {R → 1, κ → 3, E0 → 1};
V1[r_, θ_] := -3/(κ + 2) r E0 Cos[θ];
r2xRule = {r → Sqrt[x^2 + y^2 + z^2], θ → ArcCos[z/Sqrt[x^2 + y^2 + z^2]],
   ϕ → ArcTan[x, y]};
V2D1 = V1[r, θ] /. r2xRule /. y → 0 // Simplify;
V2D11 = V2D1 /. rule1;
Ex1 = -D[V2D11, x] // Simplify;
Ez1 = -D[V2D11, z] // Simplify;
```

```
g1 = ContourPlot[Evaluate[Table[V2D11 == α, {α, -5, 5, 0.1}]],
    {x, -3, 3}, {z, -3, 3},
    ContourStyle → Table[{Thick, Hue[0.03 i]}, {i, 0, 60}],
    RegionFunction → Function[{x, z}, x^2 + z^2 < 1]];

g2 = StreamPlot[Evaluate[{Ex1, Ez1}], {x, -3, 3}, {z, -3, 3},
    RegionFunction → Function[{x, z}, x^2 + z^2 < 1]];
```

$$V2[r\_,\ \theta\_] := E0 \left(-r \operatorname{Cos}[\theta] + \left(\frac{\kappa - 1}{\kappa + 2}\right) \frac{R^3}{r^2} \operatorname{Cos}[\theta]\right);$$

```
V2D2 = V2[r, θ] /. r2xRule /. y → 0 // Simplify;

V2D22 = V2D2 /. rule1;

Ex2 = -D[V2D22, x] // Simplify;
Ez2 = -D[V2D22, z] // Simplify;
```

```
g3 = ContourPlot[Evaluate[Table[V2D22 == α, {α, -5, 5, 0.1}]],
   {x, -3, 3}, {z, -3, 3},
   ContourStyle → Table[{Thick, Hue[0.03 i]}, {i, 0, 60}],
   RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];

g4 = StreamPlot[Evaluate[{Ex2, Ez2}], {x, -3, 3}, {z, -3, 3},
   RegionFunction → Function[{x, z}, x^2 + z^2 > 1]];

g5 = ParametricPlot[{Cos[θ], Sin[θ]}, {θ, 0, 2 π},
   PlotStyle → {Thick, Black}];

Show[g1, g2, g3, g4, g5, PlotRange → All]
```



8. Electric fields in cavities (atomic sites) of a dielectric

Microscopic view of the dielectrics (Feynman)

Earlier in the history of physics, when it was supposed to be very important to define every quantity by direct experiment, people were delighted to discover that they could define what they meant by $\boldsymbol{E}$ and $\boldsymbol{D}$ in a dielectric without having to crawl around between the atoms. The average field $\boldsymbol{E}$ is numerically equal to the field $\boldsymbol{E}_0$ that would be measured in a slot cut parallel to the field. And the field $\boldsymbol{D}$ could be measured by finding $\boldsymbol{E}_0$ in a slot cut normal to the field. But nobody ever measures them that way anyway, so it was just one of those philosophical things.

We consider the capacitor consisting of two parallel plates. Suppose that the space between two parallel plates is filled with a dielectric. What is the electric field $\boldsymbol{E}$ inside the dielectric?

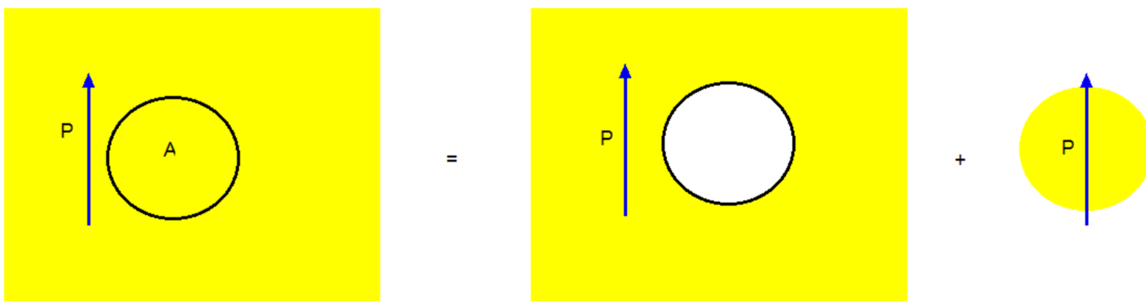


Fig. The field at any point A in a dielectric can be considered as the sum of the field in a spherical hole plus the field due to a spherical plug.

The electric field inside the dielectric, before the sphere is removed, is the sum of the field from all charges outside the spherical volume plus the fields from charges within the polarized sphere. That is, if we call the field in the uniform dielectric, we can write

$$\boldsymbol{E} = \boldsymbol{E}_{hole} + \boldsymbol{E}_{plug}$$

where $\boldsymbol{E}_{hole}$ is the field in the hole and $\boldsymbol{E}_{plug}$ is the field inside a sphere which is uniformly polarized (see Fig.). The fields due to a uniformly polarized sphere are shown in **Fig**. The electric field inside the sphere is uniform, and its value is

$$E_{plug} = -\frac{P}{3\varepsilon_0} \qquad (2)$$

Using (1), we get

$$E_{hole} = E - E_{plug} = E + \frac{P}{3\varepsilon_0}. \qquad (3)$$

The field in a spherical cavity is greater than the average field by the amount $\frac{P}{3\varepsilon_0}$. (The spherical hole gives a field 1/3 of the way between a slot parallel to the field and a slot perpendicular to the field.)

9. Microscopic theory: Clausius-Mossoti relation

In a liquid we expect that the field which will polarize an individual atom is more like E_{hole} than just E. If we use the E_{hole} for the polarizing field, then P becomes

$$P = \varepsilon_0 N\alpha E_{hole} = \varepsilon_0 N\alpha (E + \frac{P}{3\varepsilon_0})$$

or

$$P = \frac{\varepsilon_0 N\alpha E}{1 - \frac{1}{3}N\alpha}$$

Remembering that $\kappa - 1 = \frac{P}{\varepsilon_0 E}$, we have

$$\kappa - 1 = \frac{N\alpha}{1 - \frac{1}{3}N\alpha}$$

which gives us the dielectric constant of a liquid in terms of α, the atomic polarizability. This is called the *Clausius-Mossotti* equation. Whenever $N\alpha$ is very small, as it is for a gas (because the density N is small), then the term $N\alpha / 3$ can be neglected compared with 1, and we get our old result,

$$\kappa - 1 = N\alpha$$

Let's compare the value of κ with some experimental results. It is first necessary to look at gases for which, using the measurement of κ, we can find α. For instance, for carbon disulfide at zero degrees centigrade the dielectric constant is 1.0029, so $N\alpha$ is 0.0029. Now the density of the gas is easily worked out and the density of the liquid can be found in handbooks. At 20 °C, the density

of liquid CS_2 is 381 times higher than the density of the gas at 0 °C. This $N\alpha/3$ means that N is 381 times higher in the liquid than it is in the gas so, that—if we make the approximation that the basic atomic polarizability of the carbon disulfide doesn't change when it is condensed into a liquid $N\alpha$ in the liquid is equal to 381 times 0.0029, or 1.11. Notice that the $N\alpha/3$ term amounts to almost 0.4, so it is quite significant. With these numbers we predict a dielectric constant of 2.76, which agrees reasonably well with the observed value of 2.64.
works very well. The dipole moment induced in each molecule is

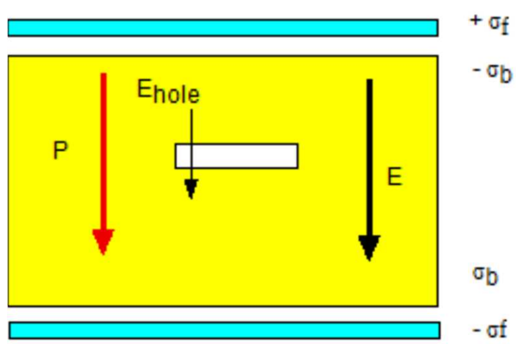
$$\boldsymbol{p} = \alpha \boldsymbol{E}$$

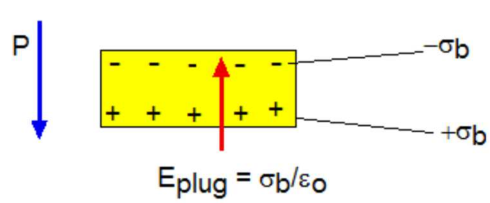
where α is the polarizability of every molecule. The resulting polarization of the medium $\boldsymbol{P}$ is

$$\boldsymbol{P} = N\boldsymbol{p} = N\alpha\boldsymbol{E}$$

10. Electric field at the disk-like hole

We cut a hole (thin slab) inside the dielectric. What is the electric field inside the hole? There is a surface charge (due to the polarization) around the hole, leading to the electric field given by P/ε_0 ($=\sigma_b/\varepsilon_0$) parallel to the external electric field $\boldsymbol{E}_0$ ($= \sigma_f/\varepsilon_0$).





From the principle of superposition, the electric field in the hole is given by

$$E_{hole} = E - E_{plug} = E - (-\frac{P}{\varepsilon_0}) = E + \frac{P}{\varepsilon_0}$$
$$= \frac{\sigma_f - \sigma_b}{\varepsilon_0} + \frac{\sigma_b}{\varepsilon_0} = \frac{\sigma_f}{\varepsilon_0} = \frac{D}{\varepsilon_0}$$

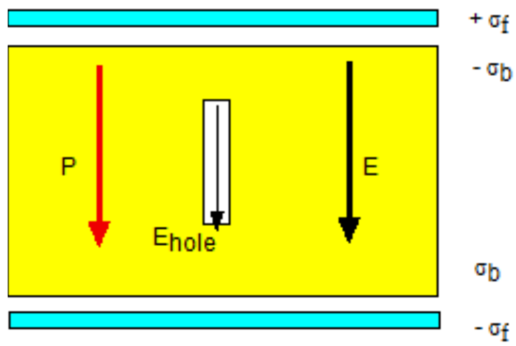
or

$$E_{hole} = \frac{D}{\varepsilon_0}$$

Here ***D*** is called electric displacement vector (or electric flux density). $D = \sigma_f$.

11. Electric field at the cylindrical hole

The measured electric field ***E*** is obtained in a usual way by cutting a needle-like (cylindrical) hole parallel to the external field ($\boldsymbol{E}_0$) and placing a test charge in the hole



Since there is no surface charge around the hole because of the needle-like shape, the electric field inside the hole is

$$E_{hole} = E - E_{plug} = E = \frac{\sigma_f - \sigma_b}{\varepsilon_0}.$$

since

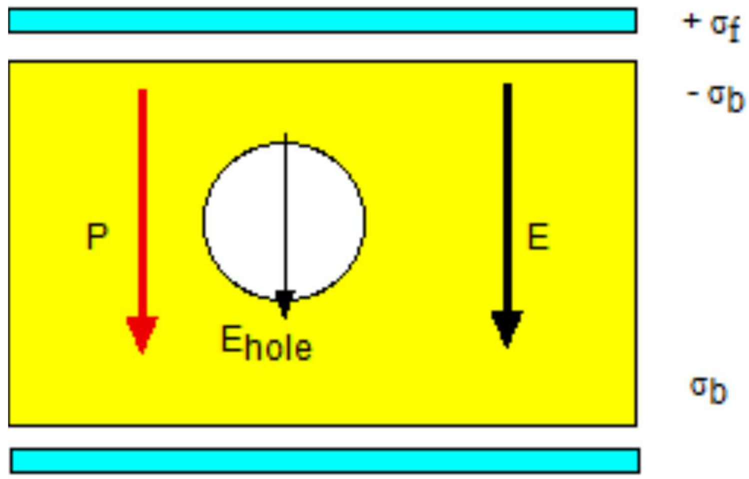
$$E_{plug} = 0$$

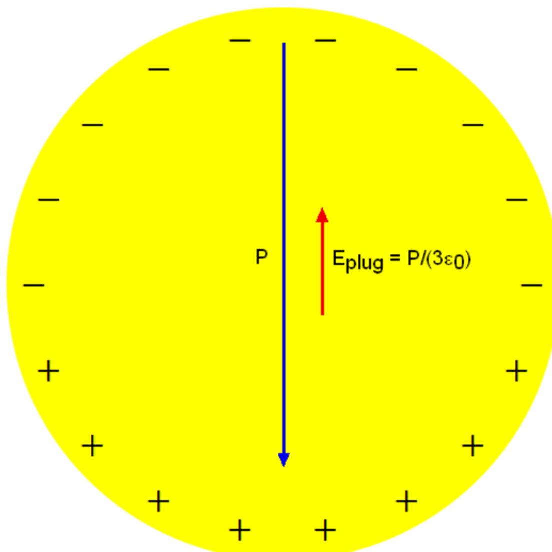
9. Electric field at the spherical hole

For a spherical hole, the electric field is is given by

$$E_{hole} = E - E_{plug} = E - (-\frac{P}{3\varepsilon_0}) = E + \frac{P}{3\varepsilon_0}$$

This is the Lorentz relation.





((**Note**))

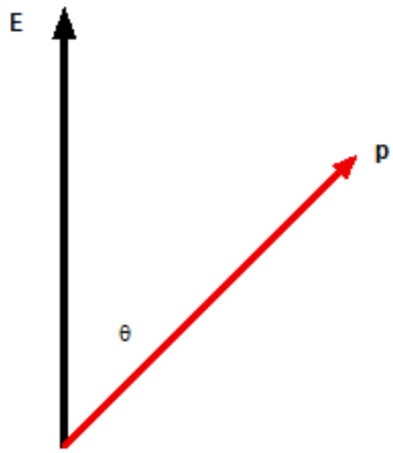
The electric field ($\boldsymbol{E}_{\text{hole}}$) acting at an atom in a cubic site (in the hole) is the macroscopic field $\boldsymbol{E}$ plus $\boldsymbol{P}/(3\varepsilon_0)$ from the polarization of the other atoms in the system.

12. Langevin-Debye formula

We assume that an electric dipole moment p of each molecule in the presence of an electric field. The potential energy is given by

$$U = -\boldsymbol{p}\cdot\boldsymbol{E} = -pE\cos\theta$$

N is the number of molecules per unit volume and θ is the angle between $\boldsymbol{p}$ and $\boldsymbol{E}$. The polarization



The polarization P is given by

$$P = Np\langle\cos\theta\rangle$$

where

$$\langle\cos\theta\rangle = \frac{\int e^{-\frac{U}{k_BT}}\cos\theta d\Omega}{\int e^{-\frac{U}{k_BT}}d\Omega} = \frac{\int_0^\pi e^{\frac{pE\cos\theta}{k_BT}}\cos\theta(2\pi\sin\theta d\theta)}{\int_0^\pi e^{\frac{pE\cos\theta}{k_BT}}(2\pi\sin\theta d\theta)}$$

and k_B is the Boltzmann constant.

For simplicity we put $x = \frac{pE}{k_BT}$ and $s = \cos\theta$. Then we have

$$\langle\cos\theta\rangle = \frac{\int_{-1}^{1} e^{sx}sds}{\int_{-1}^{1} e^{sx}ds} = \coth x - \frac{1}{x} = L(x)$$

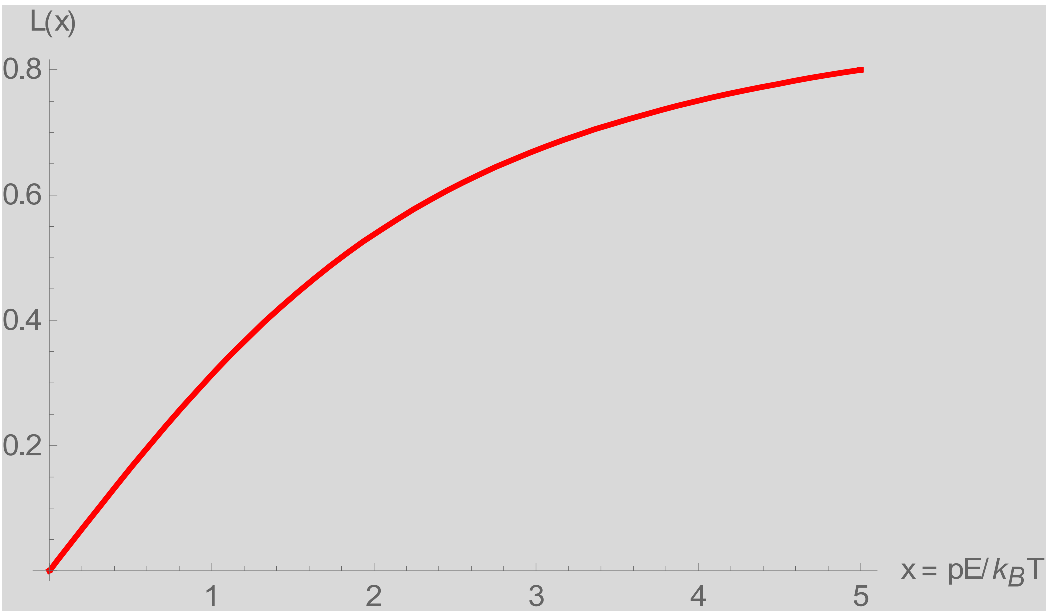
where $L(x)$ is the Langevin function.

((**Mathematica**)) Derivation of the Langevin-Debye formula

```
f1 = (∫_{-1}^{1} Exp[s x] s ⅆs) / (∫_{-1}^{1} Exp[s x] ⅆs) // Simplify
```

$$-\frac{1}{x} + \mathrm{Coth}[x]$$

```
Plot[f1, {x, 0, 5}, AxesLabel → {"x = pE/kBT", " L(x)"},
 PlotStyle → {Red, Thick}, Background → LightGray]
```




```
Series[f1, {x, 0, 10}]
```

$$\frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \frac{2x^9}{93\,555} + O[x]^{11}$$

For x<<1, the Langevin function is approximated as

$$L(x) = \frac{x}{3} - \frac{x^3}{45} + ... \approx \frac{x}{3}$$

and the derivative d$L(x)$/dx at $x = 0$ is equal to 1/3. Using this we have a Langevin-Debye formula,

$$P = Np\frac{x}{3} = \frac{Np^2 E}{3k_B T} = N\alpha E$$

where $\alpha = \dfrac{p^2}{3k_B T}$ is called the polarizability.

13. Clausius-Mossotti relation

Suppose that there are electric dipole moments in the hole. The electric field in the hole is given by E_{hole},

$$E_{hole} = E + \frac{P}{3\varepsilon_0}$$

Then the polarization P in the hole is given by

$$P = N\alpha E_{hole} = N\alpha(E + \frac{P}{3\varepsilon_0}),$$

leading to

$$P = \frac{N\alpha}{1 - \dfrac{N\alpha}{3\varepsilon_0}} E = \varepsilon_0 \chi E = \varepsilon_0(\kappa - 1)E$$

From this relation, we have the Clausius-Mossotti relation

$$\frac{3N\alpha}{3\varepsilon_0 - N\alpha} = \kappa - 1$$

or

$$\left(\frac{\kappa - 1}{\kappa + 2}\right) = \frac{N\alpha}{3\varepsilon_0}$$

14. Ferroelectric

Suppose that there are electric dipole moments in the hole. The electric field in the hole is given by E_{hole},

$$E_{hole} = E + \lambda P$$

where λ is dependent on the shape of the hole.

We use the Langevin-Debye formula

$$y = \frac{P}{P_{sat}} = L(x) \qquad \text{(a)}$$

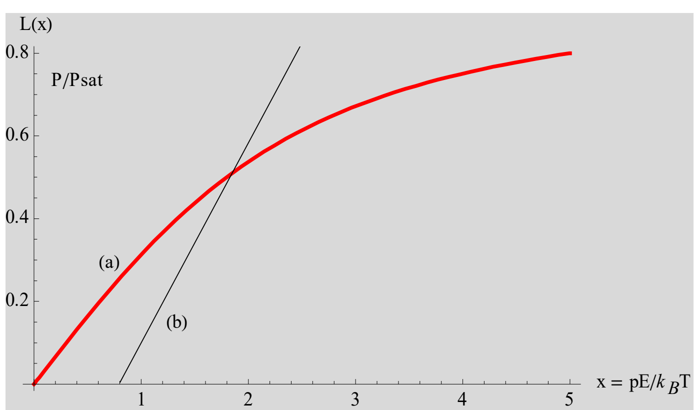
where $P_{sat} = Np$ is the saturation polarization and x is given by

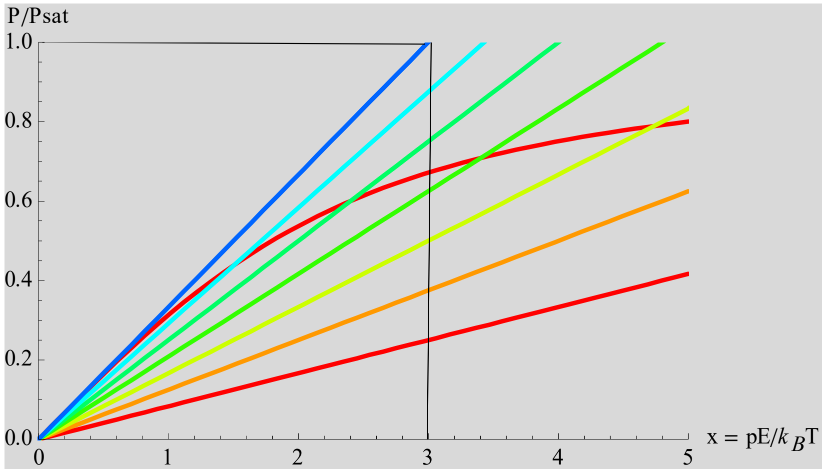
$$x = \frac{pE_{hole}}{k_B T} = \frac{p}{k_B T}(E + \lambda P) = \frac{p}{k_B T}E + \frac{\lambda N p^2}{k_B T}\frac{P}{Np}$$
$$= \frac{p}{k_B T}E + \frac{\lambda N p^2}{k_B T} y$$

or

$$y = \frac{k_B T}{\lambda N p^2}\left(x - \frac{p}{k_B T}E\right) \qquad \text{(b)}$$

where $\lambda = 1/(3\varepsilon_0)$ for the sphere. For any particular E, this is a straight-line relationship between $y = P/P_{sat}$ and x. The x intercept is at $\frac{p}{k_B T}E$, and the slope is $\frac{k_B T}{\lambda N p^2}$. The intersection of two curves denoted by (a) and (b) gives us the solution for P/P_{sat}.





We now look at how the solutions will go for various circumstances. We assume that $E = 0$. The slope of the line (b) is proportional to T. When the slope is larger than 1/3, there is no solution for finite value of P. For $T>T_c$, we have a solution $P/P_{sat} = 0$. On the other hand, for $T<T_c$, we have a solution for the finite value of P/P_{sat}. The characteristic temperature Tc is called a Curie temperature, and is given by

$$T_c = \frac{3\lambda N p^2}{k_B}$$

Then a dielectric material should polarize itself spontaneously below T_c.

15. Spontaneous polarization

When $E = 0$. y is given by

$$y = \frac{3T}{T_c}x = 3tx$$

with $t = T/T_c$. The reduced temperature (t) dependence of the spontaneous polarization is a solution of $y = \frac{3T}{T_c}x$ and $y = L(x)$.

REFERENCES

E.M. Purcell and D.J. Morin, Electricity and Magnetism, 3rd edition (Cambridge, 2013).
R. Kubo and T. Nagamiya, Solid State Physics (McGraw-Hill, 1969).
C. Kittel, Introduction to Solid State Physics 4-th edition (John Wiley & Sons, 1971).

APPENDIX

APPENDIX-I Depolarization factor for the $\boldsymbol{E}_{\mathbf{plug}}$

If P_x, P_y, and P_z are the components of the polarization $\boldsymbol{P}$ referred to the principal axes of an ellipsoid, then the components of the depolarization field are written as

$$E_x = -N_x P_x$$
$$E_y = -N_y P_y$$
$$E_z = -N_z P_z$$

Here N_x, N_y, N_z are the depolarization factors; their values depend on the ratios of the principal axes of the ellipsoid. The N's are positive and satisfy the sum rule $N_x + N_y + N_z = 1$.

Shape	Axis	N (CGS)	N (SI)
Sphere	any	$4\pi/3$	1/3
Thin slab	normal	4π	1
Thin slab	in plane	0	0
Long circular cylinder	longitudinal	0	0
Long circular cylinder	transverse	2π	1/2

If $\boldsymbol{E}_0$ is uniform and parallel to a principal axis of the ellipsoid, then we have

$$E = E_0 - N\frac{P}{\varepsilon_0}$$

Then the polarization P is given by

$$P = \varepsilon_0 \chi E = \varepsilon_0 \chi (\varepsilon_0 E_0 - NP)$$

or

$$P = \frac{\varepsilon_0 \chi}{1 + N\chi} E_0$$

APPENDIX II Units

$\boldsymbol{\tau} = \boldsymbol{p} \times \boldsymbol{E}$

$$[\tau] = [N \cdot m]$$

$$[E] = [\frac{F}{q}] = [\frac{N}{C}] = [\frac{V}{m}]$$

$$[p] = [C \cdot m]$$

$$U = -\boldsymbol{p} \cdot \boldsymbol{E}$$

$$[U] = [J] = [N \cdot m]$$

$$[pE] = [qLE] = [N \cdot m] = [J]$$

$$P = Np$$

$$[P] = [Np] = [\frac{C \cdot m}{m^3}] = [\frac{C}{m^2}]$$